Nonlinear Functional Analysis and Applications
Vol. 22, No. 2 (2017), pp. 243-257
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

EXISTENCE AND MULTIPLICITY OF HOMOCLINIC SOLUTIONS FOR DAMPED VIBRATION SYSTEMS WITH SUB-QUADRATIC POTENTIAL

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Abstract. In this paper, the existence result for homoclinic solutions of damped vibration systems with sub-quadratic potential is obtained by linking theorem. The conditions in this case are simple and more relaxed. Especially, no symmetry and no AR type conditions imposed on the potential in proving the existence result are the novelties in the paper. Recent results in the literature are generalized and significantly improved.

1. Introduction and main results

In this paper we mainly prove existence of homoclinic solutions for generic damped vibration systems involving sub-linearities without symmetry. More

[0]Received May 23, 2016. Revised July 22, 2016.
[0]2010 Mathematics Subject Classification: 34C25, 58E20, 47H04.
[0]Keywords: AR type conditions, symmetry, compactness, sub-quadratic function.
[0]This work was supported partly by National Natural Science Foundation of China (11401111) and Project of Scientific and Technical Programme of Guangzhou (201607010218).

precisely, we consider:

$$\ddot{u}(t) + B\dot{u}(t) - L(t)u(t) + W_u(t, u(t)) = 0, \qquad \forall\ t \in \mathbb{R}, \tag{DS}$$

where B is an antisymmetric $N \times N$ constant matrix, $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix valued function and $W_u(t, u)$ denotes the gradient of $W(t, u)$ with respect to u.

The damped vibration system is an extension of the following second order Hamiltonian system:

$$\ddot{u}(t) - L(t)u(t) + W_u(t, u(t)) = 0, \qquad \forall\ t \in \mathbb{R}, \tag{HS}$$

this is a classical equation which can describe many mechanical systems, such as a pendulum. With the aid of variational methods, in the last twenty more years, the existence and multiplicity of homoclinic solutions of (HS) have been intensively studied, see for instance [2, 3, 6-10, 12, 14, 17-20] and references therein. Indeed the existence of homoclinic solutions for Hamiltonian systems and their importance in the research of dynamical systems has been recognized from Poincaré. According to the growth of $W(t, u)$ near infinity with respect to u, the existing literature usually distinguished between two situations for (HS): the super-quadratic cases and the sub-quadratic cases. Compared with the case where $W(t, u)$ is super-quadratic, there is few literature available for the case where $W(t, u)$ is sub-quadratic. To the best of our knowledge, there is only paper [6] dealt with the existence of homoclinic solutions for (HS) under the condition that $W(t, u)$ is of sub-quadratic growth as $|u| \to \infty$ without symmetry.

Recently, some of the existence and multiplicity results were obtained for (DS) in [4, 5, 13, 16, 21, 22, 23]. In these papers, [13, 16, 23] dealt with the cases where $W(t, u)$ are super-quadratic, and the papers [4, 5, 21, 22] dealt with the cases where $W(t, u)$ are sub-quadratic. In [4], the authors proposed assumptions for the potential function $W(t, u)$ as following:

(L) $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix valued function and there exist constants $\alpha > 1, \beta \geq 0$ such that

$$\text{meas}\{t \in \mathbb{R} : |t|^{-\alpha} L(t) < bI_N\} < \infty, \qquad \forall\ b > 0,$$

and

$$l(t) = \inf_{|u|=1} (L(t)u, u) \geq -\beta, \qquad \forall\ t \in \mathbb{R}.$$

(AH_1) $W(t, u) \geq 0,\ \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N$ and there exist constants $\mu \in (0, 2)$ and $R_1 > 0$ such that

$$(W_u(t, u), u) \leq \mu W(t, u),\ \ \forall\ t \in R \quad \text{and} \quad |u| \geq R_1$$

and

$$(W_u(t, u), u) \leq 2W(t, u),\ \ \forall\ t \in R \quad \text{and} \quad |u| \leq R_1.$$

(AH_2) $\liminf_{|u|\to 0} \frac{W(t,u)}{|u|^2} = \infty$ uniformly for $t \in R$ and there exists one constant $c_1 > 0$ such that

$$W(t,u) \le c_1|u|, \ \forall\, t \in \mathbb{R} \quad \text{and} \quad |u| \le R_1,$$

where R_1 is the constant in (AH_1).

(AH_3) $\liminf_{|u|\to\infty} \frac{W(t,u)}{|u|} \ge d$ uniformly for $t \in R$, where $d > 0$ is a constant.

By using the variant fountain theorem established in [24], they proved the following:

Theorem A. Suppose that (L), $(AH_1)-(AH_3)$ and $W(t,u)$ is even in u hold, then (DS) possesses infinitely many nontrivial homoclinic orbits.

We emphasize that in order to use the variant fountain theorem to ensure the existence of infinitely many homoclinic solutions, the symmetry and AR type conditions have to be required as in Theorem A. If no symmetry, the situation is very different and becomes very difficult, the reason is that it seems to be hard to find path leading to the (PS) condition.

To the best of our knowledge, only in the two papers [5], [21], the authors concern with the existence problem. In paper [21], the authors merely consider a special sub-quadratic case where $W(t,u) = a(t)|u|^\gamma$, $1 < \gamma < 2$. They obtained the existence of a nontrivial homoclinic solution for (DS) by using a standard minimizing argument. We point out that the potential function $W(t,u)$, as the authors remarked, is even and satisfies $(W_u(t,u),u) \le \gamma W(t,u)$, the so-called Ambrosetti-Rabinowitz type conditions in sub-quadratic case. In order to meet the minimizing principle, the authors impose a constraint condition on the damped term. They presented the following hypotheses:

- (L_1) $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there is a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) > 0$ for all $t \in \mathbb{R}$ and $(L(t)u,u) \ge \alpha(t)|u|^2$ and $\alpha(t) \to +\infty$ as $|t| \to +\infty$; and this condition implies that there is a constant $\beta > 0$ such that $(L(t)u,u) \ge \beta|u|^2$ for all $t \in \mathbb{R}$ and $u \in \mathbb{R}^N$;
- (H_1) $W(t,u) = a(t)|u|^\gamma$, *i.e.*, $V(u) = |u|^\gamma$, where $a : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $a(t_0) > 0$ for some $t_0 \in \mathbb{R}$ and $a \in L^{\frac{2}{2-\gamma}}(\mathbb{R},\mathbb{R})$, $1 < \gamma < 2$ is a constant;
- (H_2) B is an antisymmetric $N \times N$ constant matrix such that $\|B\| < \sqrt{\beta}$, β is defined in (H_1).

Under conditions $(L_1), (H_1), (H_2)$, they proved that system (DS) possesses at least one nontrivial homoclinic solution.

In paper [5], the authors investigated the existence of the so called fast homoclinic solutions for the following damped vibration problems:

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + W_u(t, u(t)) = 0, \qquad \forall\ t \in \mathbb{R},$$

by using the standard minimizing argument under condition that $L(t)$ satisfies (L_2), that is $L(t)$ is positive definite.

In the present paper, motivated by above mentioned papers [4, 5, 13, 16, 21, 22, 23], the main goal of the paper is to deal with the case where $W(t,u)$ is generic sub-quadratic and $L(t)$ is not positive definite. More precisely, we present the following assumptions:

(W) $W(t,u) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, there exist constants $c_1 > 0$, $1 \le \gamma \le \mu < 2$ and continuous functions $a(t)$, $b(t)$ such that

$$W(t,u) \ge c_1|u|^{\mu}, \quad W(t,0) \equiv 0, \tag{1.1}$$

$$|W_u(t,u)| \le a(t)|u|^{\gamma-1} + b(t)|u|^{\mu-1},\ \ \forall\ (t,u) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.2}$$

where $a(t), b(t) : \mathbb{R} \to \mathbb{R}^+$ satisfy $a \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R}^+), b \in L^{\frac{2}{2-\mu}}(\mathbb{R}, \mathbb{R}^+)$.

Now, we state our main result as follows.

Theorem 1.1. *Suppose that* (L) *and* (W) *are satisfied. Then* (DS) *possesses at least one nontrivial homoclinic solution.*

Compared with the conditions on $W(t,u)$ in [4, 5, 21], our conditions are simple and more relaxed. There is no symmetry imposed on $W(t,u)$ in Theorem 1.1, there are no any AR type conditions in sub-quadratic case in condition (W). Compared with the conditions in [4], [5], [21], we consider the more generic cases for $W(t,u)$ and $L(t)$. It is worth mentioning that we do not impose any constraint on damped term's coefficient matrix B (see (H_2) in [21]). Therefore we extend the results in [4], [5], [21].

Theorem 1.1 is a new one even when $B = 0$ (referring [6], [20] and references therein). In this paper we devote to get our solution in Theorem 1.1. We shall apply the linking theorem introduced in [11] to get the result in Theorem 1.1. Since there is no symmetry and the potential is sub-quadratic, and the associated energy functional in Theorem 1.1 is indefinite, *i.e.*, unbounded from below and from above, the main problem here is the handling of the Palais-Smale sequences. When variational methods are used in studying dynamical systems, properties (especially, boundedness and compactness) of Palais-Smale sequences or Cerami sequences of the associated energy functional play always a central role. The novelty of our paper is that we prove the boundedness of the Palais-Smale sequences without any AR type conditions, which are crucial for those in [4, 13, 16, 21, 22, 23]. We emphasize that the proof methods we use are different than those adopted in [13, 16, 23], since the sub-quadratic case and the indefiniteness of the functional make the proof of boundedness of a PS sequence fairly delicate to treat and new strategies must be found. The proving method is very different from that used in [5].

The remainder of this paper is organized as follows. In section 2, some preliminary results for the working space and variational setting associated with (DS) are presented. In section 3, we give the proof of Theorem 1.1.

Throughout the paper, we denote by c or c_i various positive constants which may vary from line to line and are not essential to the problem.

2. Preliminary results

In this section, we describe some properties of the working space E and the variational setting associated with (DS). Letting

$$H = H^1(\mathbb{R}, \mathbb{R}^N) = \{u \in L^2(\mathbb{R}, \mathbb{R}^N) : \dot{u} \in L^2(\mathbb{R}, \mathbb{R}^N)\},$$

then H is a Hilbert space with the inner product

$$\langle u, v\rangle_H = \int_{\mathbb{R}} [(\dot{u}(t), \dot{v}(t)) + (u(t), v(t))]dt$$

and the corresponding norm $\|u\|_H^2 = \langle u, v\rangle_H$, where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$. Note that the embedding

$$H \hookrightarrow L^p(\mathbb{R}, \mathbb{R}^N)$$

is continuous for any $p \in [2, +\infty)$. Define an operator $K : H \to H$ by

$$\langle Ku, v\rangle_H = \int_{\mathbb{R}} (Bu, \dot{v})dt$$

for all $u, v \in H$. Then K is self-adjoint on H since $B = [b_{ij}]$ is an antisymmetric $N\times N$ constant matrix. Moreover, we denote by $\mathcal{A}$ the self-adjoint extension of the operator $-d^2/dt^2 + K + L(t)$ with domain $\mathfrak{D}(\mathcal{A}) \subset L^2 \equiv L^2(\mathbb{R}, \mathbb{R}^N)$. Let $E = \mathfrak{D}(|\mathcal{A}|^{1/2})$, and define on E the inner product and norm by

$$\langle u, v\rangle_E = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_2 + (u, v)_2, \quad \|u\|_E^2 = \langle u, u\rangle_E,$$

where $(\cdot,\cdot)_2$ as usual denotes the inner product on L^2. Then E is a Hilbert space. From the fact that $C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ is dense in E, then it is also easy to verify that E is continuously embedded in $H^1(\mathbb{R}, \mathbb{R}^N)$. Moreover, Using the proofs similar to Lemma 3.1 in [14] (see also [4], [13]), we can prove the following important lemma.

Lemma 2.1. *If $L(t)$ satisfies (L), then E is compactly embedded in $L^p \equiv L^p(\mathbb{R}, \mathbb{R}^N)$ for all $1 \le p \le +\infty$.*

By Lemma 2.1, we see that $\mathcal{A}$ possesses a compact resolvent. Therefore, the spectrum $\sigma(\mathcal{A})$ consists of only eigenvalues numbered in $\lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$ (counted with multiplicity) and a corresponding system of eigenfunctions $\{e_j : j \in \mathbb{N}\}(\mathcal{A}e_j = \lambda_j e_j)$ forms an orthogonal basis in L^2. Let

$$n^- = \sharp\{j|\lambda_j < 0\}, \;\; n^0 = \sharp\{j|\lambda_j = 0\}, \;\; \overline{n} = n^- + n^0$$

and

$$E^- = \text{span}\{e_1, \cdots, e_{n^-}\}, \; E^0 = \text{span}\{e_{n^-+1}, \cdots, e_{\overline{n}}\}, \; E^+ = \overline{\text{span}\{e_{\overline{n}+1}, \cdots\}}.$$

Then $E = E^- \oplus E^0 \oplus E^+$. We introduce on E the following inner product

$$\langle u, v\rangle = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_2 + (u^0, v^0)_2$$

and norm

$$\|u\|^2 = \langle u, u\rangle,$$

where $u, v \in E = E^- \oplus E^0 \oplus E^+$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$ correspondingly. Clearly the norms $\|\cdot\|$ and $\|\cdot\|_E$ are equivalent. Consequently, for each $p \in [1, +\infty]$, there exists $\eta_p > 0$ such that

$$\|u\|_p \leq \eta_p \|u\|, \qquad \forall\, u \in E, \tag{2.1}$$

where $\|\cdot\|_p$ denotes the usual norm in L^p.

By (1.2), there exists constant c_1 such that

$$W(t, u) \leq c_1(a(t)|u|^\gamma + b(t)|u|^\mu), \quad \forall\, (t, u) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.2}$$

Define the functional Φ on E by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}} [|\dot{u}(t)|^2 + (Bu(t), \dot{u}(t)) + (L(t)u(t), u(t))]dt - \int_{\mathbb{R}} W(t, u(t))dt.$$

Then

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}} W(t, u(t))dt, \tag{2.3}$$

where $u = u^- + u^0 + u^+ \in E$. Furthermore, define

$$\Psi(u) = \int_{\mathbb{R}} W(t, u(t))dt.$$

By (2.2), we know that Φ and Ψ are both well defined.

Lemma 2.2. *Let (L) and (W) be satisfied. Then $\Psi \in C^1(E, \mathbb{R})$ and $\Psi' : E \to E^*$ is compact. Hence $\Phi \in C^1(E, \mathbb{R})$. Moreover,*

$$\Psi'(u)v = \int_{\mathbb{R}} (W_u(t, u), v)dt, \tag{2.4}$$

$$\begin{aligned} \Phi'(u)v &= \int_{\mathbb{R}} [(\dot{u}, \dot{v}) + (Bu, \dot{v}) + (L(t)u, v) - (W_u(t, u), v)]dt \\ &= \langle u^+, v^+ \rangle - \langle u^-, v^- \rangle - \int_{\mathbb{R}} (W_u(t, u), v)dt \\ &= \langle u^+, v^+ \rangle - \langle u^-, v^- \rangle - \Psi'(u)v \end{aligned} \tag{2.5}$$

for all $u, v \in E = E^- \oplus E^0 \oplus E^+$ *with* $u = u^- + u^0 + u^+$ *and* $v = v^- + v^0 + v^+$ *correspondingly and critical points of* Φ *on* E *are homoclinic solutions of* (DS).

Proof. First, we show that Ψ is Gâteaux differentiable. For any given $u \in E$, let us define linear functional $J(u)$ on E as follows:

$$J(u)v = \int_{\mathbb{R}} (W_u(t, u(t)), v(t))dt, \qquad \forall\, v \in E.$$

We show that $J(u)$ is bounded. For $u \in E$ is given, for any $v \in E$, by (1.2) and (2.1) we have

$$\begin{aligned} J(u)v = \int_{\mathbb{R}} (W_u(t, u), v)dt &\le \int_{\mathbb{R}} (a(t)|u|^{\gamma-1} + b(t)|u|^{\mu-1})|v|dt \\ &\le \|a\|_{\frac{2}{2-\gamma}} \|u\|_2^{\gamma-1} \|v\|_2 + \|b\|_{\frac{2}{2-\mu}} \|u\|_2^{\mu-1} \|v\|_2 \\ &\le c_1[\|u\|^{\gamma-1} + \|u\|^{\mu-1}]\|v\| = c\|v\|. \end{aligned}$$

Hence $J(u)$ is bounded. By virtue of (1.2), for any $\zeta \in [0, 1]$, it is easy to check that

$$\begin{aligned} &|(W_u(t, u + \zeta h), h)| \\ &\le |W_u(t, u + \zeta h)|\ |h| \le [a(t)|u + \zeta h|^{\gamma-1} + b(t)|u + \zeta h|^{\mu-1}]|h| \\ &\le [a(t)|u|^{\gamma-1} + a(t)\zeta^{\gamma-1}|h|^{\gamma-1} + b(t)|u|^{\mu-1} + b(t)\zeta^{\mu-1}|h|^{\mu-1}]|h| \\ &\le [a(t)|u|^{\gamma-1}|h| + a(t)|h|^{\gamma} + b(t)|u|^{\mu-1}|h| + b(t)|h|^{\mu}] \end{aligned}$$

for all $t \in \mathbb{R},\ u, h \in \mathbb{R}^N$. Therefore, for any $u, h \in E$, by the Mean Value Theorem and Lebesgue Dominated Convergence Theorem, for any $\theta(t) \in [0, 1]$, we have

$$\begin{aligned} &\lim_{s \to 0} (\Psi(u + s\ h) - \Psi(u))/s \\ &= \lim_{s \to 0} \int_{\mathbb{R}} (W_u(t, u(t) + \theta(t)\ s\ h(t)), h(t))dt \\ &= \int_{\mathbb{R}} (W_u(t, u(t)), h(t))dt \\ &= J(u)h. \end{aligned}$$

Then by definition, $D\Psi(u) = J(u)$ is the Gâteaux derivative of Ψ at u.

Next, we verify that $D\Psi(u)$ is weakly continuous. Let $u_k \rightharpoonup u$ in E, by (1.2), there holds

$$|W_u(t,u_k) - W_u(t,u)| \le [a(t)(|u_k|^{\gamma-1} + |u|^{\gamma-1}) + b(t)(|u_k|^{\mu-1} + |u|^{\mu-1})],$$

which yields that

$$\begin{aligned}
&|(W_u(t,u_k) - W_u(t,u), v)| \\
&\le [a(t)(|u_k|^{\gamma-1} + |u|^{\gamma-1})|v| + b(t)(|u_k|^{\mu-1} + |u|^{\mu-1})|v|] \\
&\le [a(t)(|u_k - u|^{\gamma-1}|v| + 2|u|^{\gamma-1}|v|) + b(t)(|u_k - u|^{\mu-1}|v| + 2|u|^{\mu-1}|v|)].
\end{aligned}$$

Since, by Lemma 2.1, $u_k \to u$ in $L^2(\mathbb{R},\mathbb{R}^N)$, passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^{\infty} \|u_k - u\|_2 < +\infty,$$

this implies that $u_k(t) \to u(t)$ for almost every $t \in \mathbb{R}$ and

$$\textstyle\sum_{k=1}^{\infty} |u_k - u| = \omega(t) \in L^2(\mathbb{R},\mathbb{R}).$$

Therefore, we obtain

$$\begin{aligned}
&|(W_u(t,u_k) - W_u(t,u), v)| \\
&\le c_1[a(t)(|\omega(t)|^{\gamma-1}|v| + |u|^{\gamma-1}|v|) + b(t)(|\omega(t)|^{\mu-1}|v| + |u|^{\mu-1}|v|)]
\end{aligned}$$

and

$$\begin{aligned}
&|\int_{\mathbb{R}} (W_u(t,u_k) - W_u(t,u), v)dt| \\
&\le c_1[\|a\|_{\frac{2}{2-\gamma}}(\|\omega\|_2^{\gamma-1}\|v\|_2 + \|u\|_2^{\gamma-1}\|v\|_2) \\
&\quad + \|b\|_{\frac{2}{2-\mu}}(\|\omega\|_2^{\mu-1}\|v\|_2 + \|u\|_2^{\mu-1}\|v\|_2)] \\
&\le c[\|\omega\|^{\gamma-1} + \|u\|^{\gamma-1} + \|\omega\|^{\mu-1} + \|u\|^{\mu-1}]\|v\|.
\end{aligned}$$

Note that $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ and (1.2), by the Lebesgue Dominated Convergence Theorem and Theorem A.4 in [15], we have

$$\begin{aligned}
\|D\Psi(u_n) - D\Psi(u)\|_{E^*} &= \sup_{\|v\|=1} |(J(u_n) - J(u))v| \\
&= \sup_{\|v\|=1} |\int_{\mathbb{R}} (W_u(t,u_n(t)) - W_u(t,u(t)), v(t))dt| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Hence $D\Psi(u)$ is weakly continuous. Therefore $\Psi \in C^1(E,\mathbb{R})$, $\Psi'(u) = D\Psi(u) = J(u)$ and (2.4) is verified. Furthermore, Ψ' is compact by the weakly continuity of Ψ' since E is a Hilbert space. Due to the form of Φ, (2.5) is also verified and $\Phi \in C^1(E,\mathbb{R})$.

Lastly, we check that nontrivial critical points of Φ on E are classical homoclinic solutions of (DS). Note that $C_0^\infty \equiv C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ is dense in E. It is well known that $E \subset H^1(\mathbb{R}, \mathbb{R}^N) \subset C(\mathbb{R}, \mathbb{R}^N)$. If $u \in E$ is one critical point of Φ, by (2.5), we have $L(t)u - W_u(t,u)$ is the weak derivative of $\dot{u} + Bu$. Since $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$, $W(\cdot, u) \in C^1(\mathbb{R}^N, \mathbb{R})$, $E \subset C(\mathbb{R}, \mathbb{R}^N)$, we see that $\dot{u} + Bu$ is continuous, which follows that $\dot{u}$ is continuous and $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, *i.e.*, u is a classical solution of (DS). Moreover, it is easy to check that u satisfies $\dot{u} \to 0$ as $|t| \to +\infty$, because $\dot{u}$ is continuous. □

3. Proofs of the main results

In this section we give the proof of our main result.

Proof of Theorem 1.1. Define that

$$
\begin{aligned}
f(u) &= -\Phi(u) \\
&= \int_{\mathbb{R}} W(t, u(t))dt - \frac{1}{2}\int_{\mathbb{R}} [|\dot{u}(t)|^2 + (Bu(t), \dot{u}(t)) + (L(t)u(t), u(t))]dt,
\end{aligned}
$$

for all $u \in E$. By Lemma 2.2, then $f \in C^1(E, \mathbb{R})$ and moreover,

$$
f'(u)v = \Psi'(u)v + \langle u^-, v^- \rangle - \langle u^+, v^+ \rangle. \tag{3.1}
$$

Hence nontrivial critical points of f give rise to homoclinic solutions for (DS). In the following we are looking for nontrivial critical points of f arguing step by step.

Step 1. We show that f is anti-coercive on E^+, *i.e.*, $f(u) \to -\infty$ as $u \in E^+$ and $\|u\| \to \infty$. In fact, by (2.1) and (2.2), for any $u \in E^+$, we have

$$
\begin{aligned}
f(u) &\le c_1(\|a(t)\|_{\frac{2}{2-\gamma}} \|u\|_2^\gamma + \|b(t)\|_{\frac{2}{2-\mu}} \|u\|_2^\mu) - \frac{1}{2}\|u\|^2 \\
&\le c(\|u\|^\gamma + \|u\|^\mu) - \frac{1}{2}\|u\|^2
\end{aligned}
$$

for some constant $c > 0$. Since $\gamma \le \mu < 2$, above inequality implies that $f(u)$ is anti-coercive on E^+.

Step 2. We claim that for any finite dimensional subspace $X \subset E$, there exists $\varepsilon = \varepsilon(X) > 0$ such that

$$
m(\{t \in \mathbb{R} : |u(t)|^\mu \ge \varepsilon \ \|u\|^\mu\}) \ge \varepsilon, \quad \forall\, u \in X \setminus \{0\}, \tag{3.2}
$$

where $m(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}$.

If not, for any $n \in \mathbb{N}$, there exists $u_n \in X \setminus \{0\}$ such that

$$
m\bigg(\bigg\{t \in \mathbb{R} : |u_n(t)|^\mu \ge \frac{1}{n} \ \|u_n\|^\mu\bigg\}\bigg) < \frac{1}{n}.
$$

Set $v_n(t) = u_n(t)/\|u_n\| \in X \setminus \{0\}$, then $\|v_n(t)\| = 1$ and

$$m\left(\left\{t \in \mathbb{R} : |v_n(t)|^\mu \geq \frac{1}{n}\right\}\right) < \frac{1}{n}. \tag{3.3}$$

Passing to a subsequence if necessary, we may assume that $v_n \to v_0$ in X since the unit sphere of X is compact. Evidently, $\|v_0\| = 1$. By Lemma 2.1, we have $v_n \to v_0$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ and the equivalence of the norms on X. Hence we have

$$\int_{\mathbb{R}} |v_n(t) - v_0(t)|^\mu dt \leq \|v_n - v_0\|_2^\mu \to 0 \quad \text{as} \quad n \to \infty. \tag{3.4}$$

It is easy to see that there exist $\xi_1, \xi_2 > 0$ such that

$$m(\{t \in \mathbb{R} : |v_0(t)|^\mu \geq \xi_1\}) \geq \xi_2. \tag{3.5}$$

In fact, if not, for any positive integer n, we have

$$m\left(\left\{t \in \mathbb{R} : |v_0(t)|^\mu \geq \frac{1}{n}\right\}\right) = 0.$$

It implies that

$$\int_{\mathbb{R}} |v_0(t)|^{\mu+2} dt \leq \frac{1}{n} \ \|v_0\|_2^2 \leq \frac{\eta_2^2}{n}\|v_0\|^2 = \frac{\eta_2^2}{n} \to 0 \quad \text{as} \ \ n \to \infty.$$

Hence $v_0 = 0$, which is a contradiction proving (3.5).

Let

$$\Omega_0 = \{t \in \mathbb{R} : |v_0(t)|^\mu \geq \xi_1\}, \qquad \Omega_n = \left\{t \in \mathbb{R} : |v_n(t)|^\mu < \frac{1}{n}\right\}$$

and $\Omega_n^c = \mathbb{R} \setminus \Omega_n = \{t \in \mathbb{R} : |v_n(t)|^\mu \geq \frac{1}{n}\}$. By (3.3) and (3.5), we have

$$\begin{aligned} m(\Omega_n \cap \Omega_0) &= m(\Omega_0 \setminus (\Omega_n^c \cap \Omega_0)) \\ &\geq m(\Omega_0) - m(\Omega_n^c \cap \Omega_0) \geq \xi_2 - \frac{1}{n} \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}} |v_n(t) - v_0(t)|^\mu dt &\geq \int_{\Omega_n \cap \Omega_0} |v_n(t) - v_0(t)|^\mu dt \\ &\geq \frac{1}{2^\mu} \int_{\Omega_n \cap \Omega_0} |v_0(t)|^\mu dt - \int_{\Omega_n \cap \Omega_0} |v_n(t)|^\mu dt \\ &\geq \left(\frac{1}{2^\mu}\xi_1 - \frac{1}{n}\right) \ m(\Omega_n \cap \Omega_0) \\ &\geq \left(\frac{1}{2^\mu}\xi_1 - \frac{1}{n}\right) \ \left(\xi_2 - \frac{1}{n}\right) \\ &\to \frac{1}{2^\mu}\xi_1 \ \xi_2 > 0 \end{aligned}$$

as $n \to \infty$. This is in contradiction to (3.4). Therefore (3.2) holds.

For the ε given in (3.2), let

$$\Omega(u) = \{t \in \mathbb{R} : |u(t)|^\mu \ge \varepsilon \|u\|^\mu\}, \quad \forall\, u \in X \setminus \{0\}.$$

Thus, for $u \in X \setminus \{0\}$, we have

$$\begin{aligned} \Psi(u) &= \int_{\mathbb{R}} W(t, u(t))dt \\ &\ge \int_{\mathbb{R}} c_1 |u(t)|^\mu dt \ge \int_{\Omega(u)} c_1 |u(t)|^\mu dt \\ &\ge \varepsilon\ c_1 \|u\|^\mu\ m(\Omega(u)) \ge c_1\ \varepsilon^2\ \|u\|^\mu \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} f(u) &= \Psi(u) + \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\|u^+\|^2 \\ &\ge c_1\ \varepsilon^2\ \|u\|^\mu + \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\|u^+\|^2. \end{aligned} \tag{3.7}$$

Step 3. We claim that if $\{u_n\} \subset E$ is a bounded sequence with $f'(u_n) \to 0$, then $\{u_n\} \subset E$ has a convergent subsequence.

By Lemma 2.2, passing to a subsequence if necessary, we can assume that there exists a point $u \in E$ such that $\{u_n\}$ converges weakly to u in E and $\|\Psi'(u_n) - \Psi'(u)\|_{E^*} \to 0$ as $n \to \infty$. That is

$$u_n^- \to u^-, \quad u_n^0 \to u^0, \quad u_n^+ \rightharpoonup u^+ \quad \text{as} \quad n \to \infty$$

for $u_n = u_n^- + u_n^0 + u_n^+$ and $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$, since $E^- \oplus E^0$ is finite dimensional. Hence we have

$$\begin{aligned} &\|u_n^+ - u^+\|^2 \\ &= (f'(u) - f'(u_n))(u_n^+ - u^+) + (\Psi'(u_n) - \Psi'(u))(u_n^+ - u^+) \\ &\le f'(u)(u_n^+ - u^+) + \|f'(u_n)\|_{E^*}\|u_n^+ - u^+\| \\ &\quad + \|\Psi'(u_n) - \Psi'(u)\|_{E^*}\|u_n^+ - u^+\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This means that $u_n^+ \to u^+$ in E and proving the claim.

Step 4. f satisfies (PS). Let $\{u_n\} \subset E$ such that $\{f(u_n)\}$ is bounded and $f'(u_n) \to 0$, we need to prove that $\{u_n\}$ possesses a convergent subsequence. By Step 3, we know that it is sufficient to verify that $\{u_n\}$ is bounded in E. Arguing indirectly, assume as a contradiction that $\|u_n\| \to \infty$. Let $u_n =$

$u_n^- + u_n^0 + u_n^+ \in E = E^- \oplus E^0 \oplus E^+$, by $f'(u_n) \to 0$ and (1.2), we have

$$\begin{aligned}\|u_n^-\| \geq f'(u_n)u_n^- &= \int_{\mathbb{R}} (W_u(t, u_n(t)), u_n^-(t))dt + \langle u_n^-, u_n^- \rangle \\ &\geq \|u_n^-\|^2 - c_1\|u_n\|^{\gamma-1}\|u_n^-\| - c_2\|u_n\|^{\mu-1}\|u_n^-\|.\end{aligned}$$

Hence $\|u_n^-\| \leq 1 + c\|u_n\|^{\mu-1}$ which implies that $\|u_n^-\|^2 \leq 1 + c\|u_n\|^{2\mu-2}$ for n large since $\|u_n\| \to \infty$ and $\gamma \leq \mu < 2$. Note that $2\mu - 2 < \mu < 2$, we have

$$\lim_{n\to\infty} \frac{\|u_n^-\|^2}{\|u_n\|^\mu} = \lim_{n\to\infty} \frac{\|u_n^-\|^2}{\|u_n\|^2} = 0.$$

Similarly, we have

$$\begin{aligned}\|u_n^+\| \quad &\geq -f'(u_n)u_n^+ = \langle u_n^+, u_n^+ \rangle - \textstyle\int_{\mathbb{R}} (W_u(t, u_n(t)), u_n^+(t))dt \\ &\geq \|u_n^+\|^2 - c_1\|u_n\|^{\gamma-1}\|u_n^+\| - c_2\|u_n\|^{\mu-1}\|u_n^+\|\end{aligned}$$

which implies that $\frac{\|u_n^+\|^2}{\|u_n\|^\mu} \to 0$ and $\frac{\|u_n^+\|^2}{\|u_n\|^2} \to 0$, hence

$$\lim_{n\to\infty} \frac{\|u_n^0\|^2}{\|u_n\|^2} = 1. \tag{3.8}$$

Therefore, $\|u_n^0\| \to \infty$. Note that $|f(u_n)| \leq C$ and

$$\frac{f(u_n)}{\|u_n\|^\mu} = \int_{\mathbb{R}} \frac{W(t, u_n(t))}{\|u_n\|^\mu} dt + \frac{\|u_n^-\|^2 - \|u_n^+\|^2}{2\|u_n\|^\mu},$$

which with above formulas imply that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{W(t, u_n(t))}{\|u_n\|^\mu} dt = 0.$$

Set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and $\|v_n\|_p \leq \eta_p\|v_n\| = \eta_p$ for each $p \in [1, \infty]$. Since E is a Hilbert space and $\|v_n\| = 1$, passing to a subsequence, we can set $v_n \rightharpoonup v$(weakly) in E, by Lemma 2.1, $v_n \to v$ (strongly) in $L^p(\mathbb{R}, \mathbb{R}^N)$ for $p \in [1, \infty]$ and $v_n(t) \to v(t)$ a.e. $t \in \mathbb{R}$. Set $\Omega = \{t \in \mathbb{R} : v(t) \neq 0\}$. If $\text{meas}(\Omega) > 0$, then $u_n(t) \to \infty$ for a.e. $t \in \Omega$. Then by (1.1) we have that

$$\int_{\mathbb{R}} \frac{W(t, u_n(t))}{\|u_n\|^\mu} dt \geq c_1\|v_n\|_\mu^\mu.$$

Consequently, we have

$$\begin{aligned}0 &= \lim_{n\to\infty} \int_{\mathbb{R}} \frac{W(t, u_n(t))}{\|u_n\|^\mu} dt \\ &\geq \lim_{n\to\infty} (c_1\|v_n\|_\mu^\mu) = c_1\|v\|_\mu^\mu = c_1 \int_\Omega |v|^\mu dt > 0,\end{aligned}$$

a contradiction. Hence meas$(\Omega) = 0$ and $v(t) = 0$ a.e. $t \in \mathbb{R}$, which implies that

$$\lim_{n\to\infty} \frac{\|u_n\|_2^2}{\|u_n\|^2} = \|v(t)\|_2^2 = 0. \tag{3.9}$$

Now note that $E = E^- \oplus E^0 \oplus E^+$ is an orthogonal decomposition in $L^2(\mathbb{R}, \mathbb{R}^N)$, we have

$$\|u_n^0\|_2^2 = (u_n^0, u_n)_2 \le \|u_n^0\|_2 \|u_n\|_2,$$

which implies that $\|u_n^0\|^2 \le c\|u_n\|_2^2$, since E^0 is finite dimensional. Both side divided by $\|u_n\|^2$ and passing to limit, (3.8) and (3.9) imply that $1 \le 0$, this is a contradiction. The desired conclusion now follows.

Step 5. If $E^- \oplus E^0 = \{0\}$, by Steps 1 and 2, it shows that f has a maximum (> 0) which yields a homoclinic solution for (DS).

If $E^- \oplus E^0 \ne \{0\}$, take $e = e_{\overline{n}+1} \in E^+$ and set $X = E^- \oplus E^0 \oplus \mathbb{R}\ e \subset E$, by (3.2)

$$\begin{aligned} f(u) &\ge c_1\varepsilon^2\ \|u\|^\mu + \frac{1}{2}\|u^-\|^2 - \frac{1}{2}s^2 \\ &\ge c_1\varepsilon^2\ s^\mu + \frac{1}{2}\|u^-\|^2 - \frac{1}{2}s^2 \end{aligned}$$

for all $u = u^- + u^0 + se \in X$ with $s > 0$. Therefore, there exist $s_0 > 0$ small and $\sigma > 0$ such that

$$f(u) \ge \sigma, \quad \forall\, u \in S_e \equiv E^- \oplus E^0 \oplus s_0\ e.$$

In addition, by Step 1, one can choose $r > s_0$ large such that

$$f(u) \le 0, \quad \forall\, u \in E^+ \ \text{ as } \ \|u\| \ge r.$$

Let $Q = B_r \cap E^+$. Then S_e and ∂Q link. By the linking theorem, f has a critical point u such that $f(u) \ge \sigma$ which is a nontrivial homoclinic solution for (DS). The proof is complete. □

Remark 3.1. Moreover, suppose that $W(t, u)$ is even with respect to $u \in \mathbb{R}^N$, then f is even. From above Steps, f satisfies the assumptions of Lemma 2.4 in [6]. Therefore f possesses infinitely many (pairs) critical points which are homoclinic solutions for (DS). Hence, we obtain infinitely many homoclinic solutions for (DS) by using linking Theorem, do not use the variant fountain theorem as usual as used in [4], genus argument used in [5] and [22].

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