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ON MONOTONIC SOLUTIONS OF SOME NONLINEAR FRACTIONAL INTEGRAL EQUATIONS

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Abstract. It is well known that nonlinear integral and differential equations create an important branch of nonlinear analysis. A lot of nonlinear problems arising from areas of the real world are generally represented with integral and differential equations. Especially, integral and differential equations of fractional order play a very important role in modelling of some problems in physics, mechanics and other fields in natural sciences. For instance, these equations are used in describing of some problems in theory of neutron transport, the theory of radioactive transfer, the kinetic theory of gases [18], the traffic theory and so on.

In this study, we examine the solvability of the following nonlinear integral equation of fractional order in $C[0,a]$ which is the space of real valued and continuous functions defined on the interval $[0,a]$

$$x(t) = f\left(t, x(t)\right) + \frac{\left(Tx\right)(t)}{\Gamma\left(\alpha\right)} \int_0^t \frac{u(t,s,\left(Gx\right)(s))}{\left(t-s\right)^{1-\alpha}} ds.$$

We present some sufficient conditions for existence of nondecreasing solutions of the above equation. Then using a Darbo type fixed point theorem associated with the measure of noncompactness we prove that this equation has at least one nondecreasing solution in $C[0,a]$. Finally we give some examples to show that our result is applicable.

1. Introduction

Operator equations create a very important part of mathematical modelling of nonlinear problems arising from the most areas of natural sciences such as the theory of radioactive transfer, engineering, mechanics, physics, and so on, [1, 18]. Especially, integral and differential equations of fractional order

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play a very important role in describing these problems. For example, some problems in physics, mechanics and other fields can be described with the help of integral and differential equations of fractional order. Some of these problems are theory of neutron transport, the theory of radioactive transfer, the kinetic theory of gases [18], the traffic theory and so on.

On the other hand the measure of noncompactness used in the paper allows us not only to obtain the existence of solutions of the mentioned integral equation but also to characterize those solutions in terms of monotonicity. In this paper we will use a fixed point theorem of Darbo type associated with measures of noncompactness as the main tool.

Banaś et al. dealt with the following equations,

$$\begin{aligned} x(t) &= h(t) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(s,x(s))}{(t-s)^{1-\alpha}} ds, \ t \in [0,1], \\ x(t) &= h(t) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}} ds, \ t \in [0,\infty), \\ x(t) &= f(t,x(t)) \left(p(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,(Gx)(s))}{(t-s)^{1-\alpha}} ds \right), \ t \in [0,1], \end{aligned}$$

in [3, 4, 5], respectively. Moreover Darwish et al. considered the following equations,

$$\begin{aligned} x(t) &= f(t) + \frac{x(t)}{\Gamma(\alpha)} \int_0^t \frac{u(t,x(t))}{(t-s)^{1-\alpha}} ds, \ t \in [0,T], \\ x(t) &= f(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}} ds, \ t \in [0,\infty), \\ x(t) &= a(t) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{g(k(t,s))}{(t-s)^{1-\alpha}} |x(s)| \, ds, \ t \in [0,1], \\ x(t) &= a(t) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s),x(\lambda x))}{(t-s)^{1-\alpha}} ds, \ t \in [0,\infty), \\ x(t) &= a(t) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)u(t,s,x(s),x(\lambda x))}{(t-s)^{1-\alpha}} ds, \ t \in [0,1], \\ x(t) &= g(t,x(t)) + \frac{(Tx)(t)}{\Gamma(\alpha)} \int_0^t \frac{h(u(t,s))}{(t-s)^{1-\alpha}} (Hx)(s) ds, \ t \in [0,1], \\ x(t) &= g(t,x(t)) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,(Hx)(s))}{(t-s)^{1-\alpha}} ds, \ t \in [0,1], \end{aligned}$$

in [11]-[17], respectively.

Also, in 2010 Blachandran et al. [2] and Olszowy [20], for $t \in [0,\infty)$, discussed the following equations,

$$\begin{aligned} x(t) &= g(t,x(\alpha(t))) + \frac{f(t,x(\beta(t)))}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(\gamma(s)))}{(t-s)^{1-\alpha}} ds, \\ x_i(t) &= g_i\left(t,x_1(t),x_2(t),\cdots\right) + \int_0^t \frac{u_i(t,x_1(\tau),x_2(\tau),\cdots)d\tau}{(t-\tau)^{\alpha}},\ i=1,2,\cdots \end{aligned}$$

respectively.

On the other hand, the authors considered the following equation in [21] and [8],

$$x(t) = g(t,x(\beta(t))) + f(t,x(\alpha(t))) \int_0^{\varphi(t)} u(t,s,x(\gamma(s)))ds,\ t\in[0,a].$$

Then Özdemir and Çakan dealt with the following equations,

$$\begin{aligned} x(t) &= g(t,x(\alpha(t))) + f\left(t, \int_0^{\varphi(t)} u(t,s,x(\gamma(s)))ds, x(\beta(t))\right), \\ x(t) &= g(t,x(\beta_1(t)),\cdots,x(\beta_s(t))) + f(t,x(\xi_1(t)),\cdots,x(\xi_m(t))) \\ &\quad \times \int_0^{\varphi(t)} u(t,\tau,x(\gamma_1(\tau)),\cdots,x(\gamma_n(\tau)))d\tau \end{aligned} \tag{1.1}$$

for t$\in [0,a]$ in [9, 22], respectively and so on [10, 23].

In this paper, we will consider the equation

$$x(t) = f\left(t,x(t)\right) + \frac{(Tx)(t)}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,(Gx)(s))}{(t-s)^{1-\alpha}} ds \tag{1.2}$$

for $t \in [0,a]$ and $0 < \alpha \leq 1$. We present some definitions and preliminary results about the concept of measure of noncompactness and fractional integral equation in the next section. In the last section, we give our main results concerning with the existence of nondecreasing and continuous solutions of integral equation (1.2) by applying a Darbo type fixed point theorem associated with the measures of noncompactness defined by Banaś et al. [6] and [7] as well as some examples to show that this result is applicable.

2. Definitions and auxiliary facts

Definition 2.1. ([19]) Let $x \in C[a,b]$ and $a < t < b$, then

$$I_{a^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(s)}{(t-s)^{1-\alpha}} ds,\ \alpha\in(-\infty,\infty)$$

is called the Riemann-Liouville fractional integral of order α, where symbol of Γ denote the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

Let $(E, \|.\|)$ be an infinite dimensional Banach space with zero element θ. We write $B(x, r)$ to denote the closed ball centered at x with radius r and especially, we write B_r in case of $x = \theta$. We write $\overline{X}$ and $\operatorname{Conv} X$ to denote the closure and convex closure of X, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty bounded subsets of E and $\mathfrak{N}_E$ indicate its subfamily of all relatively compact sets. Finally, the standard algebraic operations on sets are denoted by λX and $X + Y$, respectively.

We use the following definition of the measure of noncompactness, given in [6].

Definition 2.2. A mapping $\mu : \mathfrak{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(1) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(3) $\mu(X) = \mu(\overline{X}) = \mu(\operatorname{Conv} X)$.
(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
(5) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \cdots)$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $\cap_{n=1}^{\infty} X_n$ is nonempty.

Theorem 2.3. ([6]) *Let C be a nonempty, closed, bounded and convex subset of the Banach space E and $F : C \to C$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(FX) \leq k\mu(X) \tag{2.1}$$

for any nonempty subset X of C, where μ is a measure of noncompactness in E. Then F has a fixed point in set C.

It is known that the family of all real valued and continuous functions defined on interval $[a, b]$ is a Banach space with the standard norm

$$\|x\| = \max\{|x(t)| : t \in [a, b]\}.$$

Let X a fixed subset of $\mathfrak{M}_{C[a,b]}$. For $\varepsilon > 0$ and $x \in X$, we denote by $\omega(x, \varepsilon)$ the modulus of continuity of function x defined by

$$\omega(x, \varepsilon) = \sup\{|x(t_1) - x(t_2)| : t_1, t_2 \in [a, b] \text{ and } |t_1 - t_2| \leq \varepsilon\}.$$

Furthermore, let $\omega(X,\varepsilon)$ and $\omega_0(X)$ are defined by

$$\omega(X,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon\to 0} \omega(X,\varepsilon). \tag{2.2}$$

Then, function ω_0 is a measure of noncompactness in space $C[a,b]$ ([6]).

For $x \in X$ let us consider the following quantities

$$\begin{aligned} d(x) &= \sup\{|x(s)-x(t)| - [x(s)-x(t)] : t,s \in [a,b] \text{ and } t \le s\},\\ i(x) &= \sup\{|x(s)-x(t)| - [x(t)-x(s)] : t,s \in [a,b] \text{ and } t \le s\}. \end{aligned}$$

The quantity $d(x)$ represents the degree of decrease of the function x while $i(x)$ represents the degree of increase. Moreover, $d(x)=0$ if and only if x is nondecreasing on $[a,b]$ and similarly $i(x)=0$ if and only if x is nonincreasing on $[a,b]$. Further, let us put

$$\begin{aligned} d(X) &= \sup\{d(x) : x \in X\},\\ i(X) &= \sup\{i(x) : x \in X\}. \end{aligned}$$

Finally, let us denote

$$\begin{aligned} \mu_d(X) &= \omega_0(X) + d(X), \qquad (2.3)\\ \mu_i(X) &= \omega_0(X) + i(X). \end{aligned}$$

The authors have shown in [7] that above functions $\mu_d(X)$ and $\mu_i(X)$ are measures of noncompactness in the space $C[a,b]$.

3. Main Results

Throughout this section we denote by I the interval $[0,a]$. We study functional integral equation (1.2) under the following conditions:

($\mathbf{a_1}$) $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and there exist nonnegative constant k such that

$$|f(t,x) - f(t,y)| \le k\,|x-y|$$

for all $t \in I$ and $x,y \in \mathbb{R}_+$. Also f is nondecreasing according to second variable.

($\mathbf{a_2}$) The function $u : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing according to first variable. Moreover there exists a functions $h : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing on $\mathbb{R}_+$ such that the inequality

$$u(t,s,x) \le h(x)$$

holds for all $t,s \in I$ and $x \in \mathbb{R}_+$.

($\mathbf{a_3}$) The operators $T : C(I) \to C(I)$ and $G : C(I) \to C(I)$ are continuous and Tx, Gx are nonnegative functions for all $x \in C(I)$. Also there exist functions $b_1, b_2 : \mathbb{R}_+ \to \mathbb{R}_+$ which are nondecreasing on $\mathbb{R}_+$ such that the inequalities

$$|(Tx)(t)| \leq b_1(\|x\|)$$

and

$$|(Gx)(t)| \leq b_2(\|x\|)$$

hold for all $x \in C(I)$ and $t \in I$.

($\mathbf{a_4}$) There exists a positive solution r_0 of the inequality

$$kr + \frac{b_1(r)h\left(b_2(r)\right)a^\alpha}{\Gamma(\alpha+1)} + M \leq r, \tag{3.1}$$

where M is the positive constant such that $|f(t,0)| \leq M$ for all $t \in I$.

($\mathbf{a}_5$) T satisfies Darbo condition on B_{r_0} with respect to measure of noncompactness μ_d given by (2.3) with nonnegative constant l. Also

$$k + \frac{lh\left(b_2(r_0)\right)a^\alpha}{\Gamma(\alpha+1)} < 1.$$

Theorem 3.1. *Under assumptions* ($\mathbf{a_1}$)-($\mathbf{a_4}$), *Eq.*(1.2) *has at least one solution* $x = x(t)$ *which belongs to* $B_{r_0} \subset C(I)$.

Proof. Note that we will use Theorem 2.3 as our main tool. We define operators A, U and F as

$$(Ax)(t) = (Fx)(t) + \frac{(Tx)(t)}{\Gamma(\alpha)}(Ux)(t)$$

and

$$(Ux)(t) = \int_0^t \frac{u(t,s,(Gx)(s))}{(t-s)^{1-\alpha}}ds, \ (Fx)(t) = f(t,x(t))$$

for $x \in C(I)$. By using the conditions of Theorem 3.1 we infer that Fx and Ux are continuous on I. For any $x \in B_{r_0}$, we have

$$\begin{aligned}
&|(Ax)(t)| \\
&= \left|(Fx)(t) + \frac{(Tx)(t)}{\Gamma(\alpha)}(Ux)(t)\right| \\
&\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| + \frac{1}{\Gamma(\alpha)}|(Tx)(t)(Ux)(t)| \\
&\leq k|x(t)| + \frac{1}{\Gamma(\alpha)}|(Tx)(t)(Ux)(t)| + M \\
&\leq k|x(t)| + \frac{|(Tx)(t)|}{\Gamma(\alpha)}\int_0^t \frac{|u(t,s,(Gx)(s))|}{(t-s)^{1-\alpha}}ds + M
\end{aligned}$$

$$
\begin{aligned}
&\leq k\left|x\left(t\right)\right|+\frac{\left|\left(Tx\right)\left(t\right)\right|}{\Gamma\left(\alpha\right)}\int_0^t\frac{h\left(\left|\left(Gx\right)\left(s\right)\right|\right)}{\left(t-s\right)^{1-\alpha}}ds+M\\
&\leq k\left\|x\right\|+\frac{b_1(\|x\|)}{\Gamma\left(\alpha\right)}\int_0^t\frac{h\left(b_2(\|x\|)\right)}{\left(t-s\right)^{1-\alpha}}ds+M\\
&\leq kr_0+\frac{b_1(r_0)h\left(b_2(r_0)\right)a^\alpha}{\Gamma\left(\alpha+1\right)}+M\\
&\leq r_0.
\end{aligned}
\tag{3.2}
$$

So, $|(Ax)(t)|\leq r_0$ for all $t\in I$ and this implies that $Ax\in B_{r_0}$. Now, we prove that operator $A:B_{r_0}\to B_{r_0}$ is continuous. Let y be any fixed element of B_{r_0}. Since T and G are continuous on B_{r_0}, for any $\varepsilon>0$ and y, there exist $0<\delta_1(\varepsilon,y)<\varepsilon$ and $0<\delta_2(\varepsilon,y)<\varepsilon$ such that $\|Tx-Ty\|\leq\varepsilon$ for $\|x-y\|\leq\delta_1(\varepsilon,y)$ and $\|Gx-Gy\|\leq\varepsilon$ for $\|x-y\|\leq\delta_2(\varepsilon,y)$. If we take $\delta(\varepsilon,y)=\min\left\{\delta_1(\varepsilon,y),\ \delta_2(\varepsilon,y)\right\}$ then, by using conditions of Theorem 3.1, we obtain the following inequalities for $\|x-y\|\leq\delta(\varepsilon,y)$.

$$
\begin{aligned}
&|(Ax)(t)-(Ay)(t)|\\
&=\left|(Fx)(t)+\frac{\left(Tx\right)\left(t\right)}{\Gamma\left(\alpha\right)}\left(Ux\right)\left(t\right)-\left((Fy)(t)+\frac{\left(Ty\right)\left(t\right)}{\Gamma\left(\alpha\right)}\left(Uy\right)\left(t\right)\right)\right|\\
&\leq k\left|x(t)-y(t)\right|+\left|\frac{\left(Tx\right)\left(t\right)}{\Gamma\left(\alpha\right)}\left(Ux\right)\left(t\right)-\frac{\left(Ty\right)\left(t\right)}{\Gamma\left(\alpha\right)}\left(Uy\right)\left(t\right)\right|\\
&\leq k\left|x(t)-y(t)\right|+\left|\frac{\left(Tx\right)\left(t\right)}{\Gamma\left(\alpha\right)}\left(Ux\right)\left(t\right)-\frac{\left(Tx\right)\left(t\right)}{\Gamma\left(\alpha\right)}\left(Uy\right)\left(t\right)\right|\\
&\quad+\left|\frac{\left(Tx\right)\left(t\right)}{\Gamma\left(\alpha\right)}\left(Uy\right)\left(t\right)-\frac{\left(Ty\right)\left(t\right)}{\Gamma\left(\alpha\right)}\left(Uy\right)\left(t\right)\right|\\
&\leq k\left|x(t)-y(t)\right|+\frac{\left|\left(Tx\right)\left(t\right)\right|}{\Gamma\left(\alpha\right)}\int_0^t\frac{\left|u(t,s,\left(Gx\right)\left(s\right))-u(t,s,\left(Gy\right)\left(s\right))\right|}{\left(t-s\right)^{1-\alpha}}ds\\
&\quad+\frac{\left|\left(Tx\right)\left(t\right)-\left(Ty\right)\left(t\right)\right|}{\Gamma\left(\alpha\right)}\int_0^t\frac{\left|u(t,s,\left(Gy\right)\left(s\right))\right|}{\left(t-s\right)^{1-\alpha}}ds\\
&\leq k\left\|x-y\right\|+\frac{b_1(\|x\|)}{\Gamma\left(\alpha\right)}\int_0^t\frac{\left|u(t,s,\left(Gx\right)\left(s\right))-u(t,s,\left(Gy\right)\left(s\right))\right|}{\left(t-s\right)^{1-\alpha}}ds\\
&\quad+\frac{\|Tx-Ty\|}{\Gamma\left(\alpha\right)}\int_0^t\frac{h\left(\left|\left(Gy\right)\left(s\right)\right|\right)}{\left(t-s\right)^{1-\alpha}}ds\\
&\leq k\varepsilon+\frac{b_1(\|x\|)\omega_{u_3}(I,\varepsilon)a^\alpha}{\Gamma\left(\alpha+1\right)}+\frac{\varepsilon h\left(b_2(\|x\|)\right)a^\alpha}{\Gamma\left(\alpha+1\right)}\\
&\leq k\varepsilon+\frac{b_1(r_0)\omega_{u_3}(I,\varepsilon)a^\alpha+\varepsilon h\left(b_2(r_0)\right)a^\alpha}{\Gamma\left(\alpha+1\right)},
\end{aligned}
\tag{3.3}
$$

where

$$\omega_{u_3}(I,\varepsilon)=\sup\left\{|u(t,s,x)-u(t,s,y)|:t,s\in I,\ x,\ y\in R \text{ and } |x-y|\leq\varepsilon\right\}$$

such that $R=\left[-\left\|GB_{r_0}\right\|,\left\|GB_{r_0}\right\|\right]$. On the other hand, due to the fact that the function $u=u(t,s,x)$ is uniformly continuous on $I\times I\times R$, we infer that $\omega_{u_3}(I,\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence, above estimate (3.3) proves that operator F is continuous on B_{r_0}. Moreover, we show that operator F satisfies (2.1) on B_{r_0} with respect to measure of noncompactness μ_d. To do this, fix arbitrary $\varepsilon>0$. Let us consider $x\in X$ and $t_1,t_2\in I$ with $t_2\leq t_1$ and $|t_1-t_2|\leq\varepsilon$, for any nonempty subset X of B_{r_0};

$$\begin{aligned}
&|(Ax)(t_1)-(Ax)(t_2)|\\
&=\left|(Fx)(t_1)+\frac{(Tx)(t_1)}{\Gamma(\alpha)}(Ux)(t_1)-\left((Fx)(t_2)+\frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_2)\right)\right|\\
&\leq|(Fx)(t_1)-(Fx)(t_2)|+\left|\frac{(Tx)(t_1)}{\Gamma(\alpha)}(Ux)(t_1)-\frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_2)\right|\\
&\leq\omega(Fx,\varepsilon)+\left|\frac{(Tx)(t_1)}{\Gamma(\alpha)}(Ux)(t_1)-\frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_2)\right|\\
&\leq\omega(Fx,\varepsilon)+\left|\frac{(Tx)(t_1)}{\Gamma(\alpha)}(Ux)(t_1)-\frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_1)\right|\\
&\quad+\left|\frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_1)-\frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_2)\right|\\
&\leq\omega(Fx,\varepsilon)+\frac{|(Tx)(t_1)-(Tx)(t_2)|}{\Gamma(\alpha)}\int_0^{t_1}\frac{|u(t_1,s,(Gx)(s))|}{(t_1-s)^{1-\alpha}}ds\\
&\quad+\frac{|(Tx)(t_2)|}{\Gamma(\alpha)}|(Ux)(t_1)-(Ux)(t_2)|\\
&\leq\omega(Fx,\varepsilon)+\frac{|(Tx)(t_1)-(Tx)(t_2)|}{\Gamma(\alpha)}\int_0^{t_1}\frac{|u(t_1,s,(Gx)(s))|}{(t_1-s)^{1-\alpha}}ds\\
&\quad+\frac{|(Tx)(t_2)|}{\Gamma(\alpha)}\int_0^{t_2}\left|\frac{u(t_1,s,(Gx)(s))}{(t_1-s)^{1-\alpha}}-\frac{u(t_2,s,(Gx)(s))}{(t_2-s)^{1-\alpha}}\right|ds\\
&\quad+\frac{|(Tx)(t_2)|}{\Gamma(\alpha)}\int_{t_2}^{t_1}\frac{|u(t_1,s,(Gx)(s))|}{(t_1-s)^{1-\alpha}}ds\\
&\leq\omega(Fx,\varepsilon)+\frac{|(Tx)(t_1)-(Tx)(t_2)|}{\Gamma(\alpha)}\int_0^{t_1}\frac{|u(t_1,s,(Gx)(s))|}{(t_1-s)^{1-\alpha}}ds\\
&\quad+\frac{|(Tx)(t_2)|}{\Gamma(\alpha)}\int_0^{t_2}\left|\frac{u(t_1,s,(Gx)(s))}{(t_1-s)^{1-\alpha}}-\frac{u(t_1,s,(Gx)(s))}{(t_2-s)^{1-\alpha}}\right|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left|(Tx)(t_2)\right|}{\Gamma(\alpha)}\int_0^{t_2}\left|\frac{u(t_1,s,(Gx)(s))}{(t_2-s)^{1-\alpha}}-\frac{u(t_2,s,(Gx)(s))}{(t_2-s)^{1-\alpha}}\right|ds\\
&+\frac{\left|(Tx)(t_2)\right|}{\Gamma(\alpha)}\int_{t_2}^{t_1}\frac{\left|u(t_1,s,(Gx)(s))\right|}{(t_1-s)^{1-\alpha}}ds\\
\leq{}&\omega(Fx,\varepsilon)+\frac{\omega(Tx,\varepsilon)}{\Gamma(\alpha)}\int_0^{t_1}\frac{h(\left|(Gx)(s)\right|)}{(t_1-s)^{1-\alpha}}ds\\
&+\frac{b_1(\|x\|)}{\Gamma(\alpha)}\int_0^{t_2}\left|u(t_1,s,(Gx)(s))\right|\left(\frac{1}{(t_2-s)^{1-\alpha}}-\frac{1}{(t_1-s)^{1-\alpha}}\right)ds\\
&+\frac{b_1(\|x\|)}{\Gamma(\alpha)}\int_0^{t_2}\frac{\left|u(t_1,s,(Gx)(s))-u(t_2,s,(Gx)(s))\right|}{(t_2-s)^{1-\alpha}}ds\\
&+\frac{b_1(\|x\|)}{\Gamma(\alpha)}\int_{t_2}^{t_1}\frac{h(\left|(Gx)(s)\right|)}{(t_1-s)^{1-\alpha}}ds\\
\leq{}&\omega(Fx,\varepsilon)+\frac{\omega(Tx,\varepsilon)\,h(b_2(r_0))\,a^\alpha}{\Gamma(\alpha+1)}\\
&+\frac{b_1(r_0)h(b_2(r_0))\left[(t_1-t_2)^\alpha-(t_1^\alpha-t_2^\alpha)\right]}{\Gamma(\alpha+1)}\\
&+\frac{b_1(r_0)\omega_{u_1}(I,\varepsilon)\,a^\alpha}{\Gamma(\alpha+1)}+\frac{b_1(r_0)h(b_2(r_0))(t_1-t_2)^\alpha}{\Gamma(\alpha+1)}\\
\leq{}&\omega(Fx,\varepsilon)+\frac{\omega(Tx,\varepsilon)\,h(b_2(r_0))\,a^\alpha}{\Gamma(\alpha+1)}\\
&+\frac{b_1(r_0)}{\Gamma(\alpha+1)}\left\{2h(b_2(r_0))\,\varepsilon^\alpha+\omega_{u_1}(I,\varepsilon)\,a^\alpha\right\}
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
&\omega_{u_1}(I,\varepsilon)\\
&=\sup\left\{\left|u(t_1,s,x)-u(t_2,s,x)\right|: t_1,t_2,s\in I,\ x\in R \text{ and } \left|t_1-t_2\right|\leq\varepsilon\right\}.
\end{aligned}
$$

Thus, by using above estimate (3.4), we get

$$
\begin{aligned}
\omega(AX,\varepsilon)\leq\omega(FX,\varepsilon)&+\frac{\omega(TX,\varepsilon)\,h(b_2(r_0))\,a^\alpha}{\Gamma(\alpha+1)}\\
&+\frac{b_1(r_0)}{\Gamma(\alpha+1)}\left\{2h(b_2(r_0))\,\varepsilon^\alpha+\omega_{u_1}(I,\varepsilon)\,a^\alpha\right\}.
\end{aligned}
$$

We obtain that $\omega_{u_1}(I,\varepsilon)\to 0$ as $\varepsilon\to 0$ because of the fact that function u is uniformly continuous on sets $I\times I\times R$. So, we conclude

$$
\omega_0(AX)\leq\omega_0(FX)+\frac{h(b_2(r_0))\,a^\alpha}{\Gamma(\alpha+1)}\omega_0(TX). \tag{3.5}
$$

On the other hand, we show that F satisfies inequality (2.1) according to measure of noncompactness μ_d with constant k on B_{r_0}. To do this, let us consider $x \in X$ and $t_1, t_2 \in I$ with $t_2 \leq t_1$ for any nonempty subset X of B_{r_0}. Then we have

$$\begin{aligned}
&|(Fx)(t_1) - (Fx)(t_2)| - [(Fx)(t_1) - (Fx)(t_2)] \\
&= \frac{|(Fx)(t_1) - (Fx)(t_2)|}{|x(t_1) - x(t_2)|} |x(t_1) - x(t_2)| - \frac{[(Fx)(t_1) - (Fx)(t_2)]}{[x(t_1) - x(t_2)]} [x(t_1) - x(t_2)] \\
&= (|x(t_1) - x(t_2)| - [x(t_1) - x(t_2)]) \frac{|(Fx)(t_1) - (Fx)(t_2)|}{|x(t_1) - x(t_2)|} \\
&\leq d(x) \frac{k|x(t_1) - x(t_2)|}{|x(t_1) - x(t_2)|} \\
&= kd(x).
\end{aligned}$$

Taking supremum with respect to x over the set X, we obtain

$$d(FX) \leq kd(X).$$

Also we get

$$\begin{aligned}
&(Ux)(t_1) - (Ux)(t_2) \\
&= \int_0^{t_1} \frac{u(t_1, s, (Gx)(s))}{(t_1 - s)^{1-\alpha}} ds - \int_0^{t_2} \frac{u(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds \\
&= \int_0^{t_2} \frac{u(t_1, s, (Gx)(s))}{(t_1 - s)^{1-\alpha}} ds - \int_0^{t_2} \frac{u(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds + \int_{t_2}^{t_1} \frac{u(t_1, s, (Gx)(s))}{(t_1 - s)^{1-\alpha}} ds \\
&\quad + \int_0^{t_2} \frac{u(t_1, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_2} \frac{u(t_1, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds \\
&\geq m \left\{ \int_0^{t_2} \frac{1}{(t_1 - s)^{1-\alpha}} ds - \int_0^{t_2} \frac{1}{(t_2 - s)^{1-\alpha}} ds + \int_{t_2}^{t_1} \frac{1}{(t_1 - s)^{1-\alpha}} ds \right\} \\
&\quad + \int_0^{t_2} \frac{u(t_1, s, (Gx)(s)) - u(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds \\
&= m \frac{t_1^\alpha - t_2^\alpha}{\alpha} + \int_0^{t_2} \frac{u(t_1, s, (Gx)(s)) - u(t_2, s, (Gx)(s))}{(t_2 - s)^{1-\alpha}} ds,
\end{aligned}$$

where

$$m = \min \{u(t, s, x) : t, s \in I \text{ and } x \in R\}.$$

Taking into account that the function $t \to u(t, s, (Gx)(s))$ is nondecreasing on I, we conclude

$$(Ux)(t_1) - (Ux)(t_2) \geq 0$$

and so

$$d(UX) = 0. \tag{3.6}$$

Now, let us consider the degree of decrease of the function Ax on I for any $x \in X \subset B_{r_0}$. To do this let us choose any elements $t_1, t_2 \in I$ with $t_2 \leq t_1$. Then using (3.2) and (3.6) we can write

$$\begin{aligned}
&|(Ax)(t_1) - (Ax)(t_2)| - [(Ax)(t_1) - (Ax)(t_2)] \\
&= \left|(Fx)(t_1) + \frac{(Tx)(t_1)}{\Gamma(\alpha)}(Ux)(t_1) - \left((Fx)(t_2) + \frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_2)\right)\right| \\
&\quad - \left[(Fx)(t_1) + \frac{(Tx)(t_1)}{\Gamma(\alpha)}(Ux)(t_1) - \left((Fx)(t_2) + \frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_2)\right)\right] \\
&\leq |(Fx)(t_1) - (Fx)(t_2)| - [(Fx)(t_1) - (Fx)(t_2)] \\
&\quad + \left|\frac{(Tx)(t_1)}{\Gamma(\alpha)}(Ux)(t_1) - \frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_1)\right| \\
&\quad + \left|\frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_1) - \frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_2)\right| \\
&\quad - \left[\frac{(Tx)(t_1)}{\Gamma(\alpha)}(Ux)(t_1) - \frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_1)\right] \\
&\quad - \left[\frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_1) - \frac{(Tx)(t_2)}{\Gamma(\alpha)}(Ux)(t_2)\right] \\
&= |(Fx)(t_1) - (Fx)(t_2)| - [(Fx)(t_1) - (Fx)(t_2)] \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(|(Tx)(t_1) - (Tx)(t_2)| - [(Tx)(t_1) - (Tx)(t_2)]\right)(Ux)(t_1) \\
&\quad + \left(|(Ux)(t_1) - (Ux)(t_2)| - [(Ux)(t_1) - (Ux)(t_2)]\right)\frac{(Tx)(t_2)}{\Gamma(\alpha)} \\
&\leq d(Fx) + \frac{(Ux)(t_1)}{\Gamma(\alpha)} d(Tx) + d(Ux)\frac{(Tx)(t_2)}{\Gamma(\alpha)} \\
&\leq d(Fx) + \frac{h(b_2(r_0))\, a^{\alpha}}{\Gamma(\alpha+1)} d(Tx) + \frac{b_1(r_0)}{\Gamma(\alpha)} d(Ux) \\
&= d(Fx) + \frac{h(b_2(r_0))\, a^{\alpha}}{\Gamma(\alpha+1)} d(Tx). \qquad (3.7)
\end{aligned}$$

Taking supremum with respect to x over the set X, we obtain from (3.7)

$$d(AX) \leq d(FX) + \frac{h(b_2(r_0))\, a^{\alpha}}{\Gamma(\alpha+1)} d(TX). \qquad (3.8)$$

If we consider (3.5), (3.8) and condition $(\mathbf{a}_5)$, we obtain

$$\mu_d(AX) \leq \mu_d(FX) + \frac{h(b_2(r_0))\, a^{\alpha}}{\Gamma(\alpha+1)} \mu_d(TX)$$

$$\leq \left(k + \frac{lh\left(b_2(r_0)\right)a^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)\mu_d\left(X\right)$$

So operator A is a contraction on ball B_{r_0} with respect to measure of noncompactness μ_d. Thus, nonlinear functional integral equation (1.2) has at least one nonnegative solution in $B_{r_0} \subset C\left(I\right)$. Finally, we prove that these solutions are nondecreasing on I. Let $D = \{x \in B_{r_0} : Ax = x\}$ then $A\left(D\right) = D$. Since D is nonempty subset of B_{r_0}, we write

$$\mu_d\left(AD\right) = \mu_d\left(D\right) \leq \left(k + \frac{lh\left(b_2(r_0)\right)a^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)\mu_d\left(D\right). \tag{3.9}$$

Taking (3.9) into account, we can write $\mu_d(D) = 0$. This means that for every $x \in D$, $d(x) = 0$ and so x is nondecreasing on I. This completes the proof. □

4. Examples

Example 4.1. Consider the following nonlinear functional integral equation (given in [16]) in $C\left[0,1\right]$:

$$x(t) = \frac{tx\left(t\right)}{1+t^2} + \frac{\int_0^t \left|x\left(s\right)\right| ds}{2\Gamma\left(\frac{1}{2}\right)} \int_0^t \frac{\ln\left(1+\sqrt{t+s}\right)}{\sqrt{t-s}} \left(\int_0^s \tau x^2\left(\tau\right)d\tau\right) ds. \tag{4.1}$$

Put

$$f(t,x) = \frac{tx\left(t\right)}{1+t^2}, \qquad \left(Tx\right)\left(t\right) = \frac{1}{2}\int_0^t \left|x\left(s\right)\right| ds,$$

$$u(t,s,x) = \ln\left(1+\sqrt{t+s}\right)x, \qquad \left(Gx\right)\left(s\right) = \int_0^s \tau x^2\left(\tau\right)d\tau,$$

$$h\left(x\right) = \ln\left(1+\sqrt{2}\right)x$$

and

$$a = 1,\ \alpha = \frac{1}{2},\ k = \frac{1}{2},\ M = \frac{1}{10} \quad \text{and} \quad l = 0.$$

Since

$$\left|u(t,s,x)\right| \leq \ln\left(1+\sqrt{t+s}\right)\|x\|$$

for all $t, s \in [0,1]$ and $x \in \mathbb{R}_+$. Also,

$$\begin{aligned} |(Tx)(t)| &= \frac{1}{2}\int_0^t |x(s)|\,ds \le \frac{\|x\|}{2}, \\ |(Gx)(s)| &= \int_0^s \tau x^2(\tau)\,d\tau \le \frac{\|x\|^2}{2} \end{aligned}$$

satisfy for all $x \in C[0,1]$. So b_1 and b_2 can be choosen as $b_1(x) = x/2$ and $b_2(x) = x^2/2$.

It is easy to verify that any number $0.204236 \le r_0 \le 1.30491$ satisfies the inequality

$$\frac{r}{2} + \frac{\ln\left(1+\sqrt{2}\right) r^3}{2\Gamma\left(\frac{3}{2}\right)} + \frac{1}{10} \le r,$$

which is equivalent to (3.1) for Eq.(4.1). On the other hand it is easy to verify that the other assumptions of Theorem 3.1 hold. Therefore, Theorem 3.1 guarantees that Eq.(4.1) has at least one nonnegative and nondecreasing solution $x = x(t) \in B_{r_0} \subset C[0,1]$ for any fixed r_0 such that $0.204236 \le r_0 \le 1.30491$.

Example 4.2. Consider the following functional integral equation in $C\left[0, \frac{\pi}{2}\right]$

$$\begin{aligned} x(t) = &\frac{x(t)}{10}\sin\left(\frac{1}{1+t}\right) \\ &+ \frac{e^{x(t)}}{16\Gamma\left(\frac{3}{2}\right)}\int_0^t \frac{\left(\ln\left(1+\sqrt{t+s}\right) + \int_0^s |\sin x(\tau)|\,d\tau\right)}{\sqrt{t-s}}ds. \end{aligned} \tag{4.2}$$

For this equation, we have

$$f(t,x) = \frac{x(t)}{10}\sin\left(\frac{1}{1+t}\right),\ (Tx)(t) = \frac{e^{x(t)}}{16},$$

$$u(t,s,x) = \ln\left(1+\sqrt{t+s}\right) + x,\ \ (Gx)(s) = \int_0^s |\sin x(\tau)|\,d\tau,$$

and

$$b_1(x) = \frac{e^x}{16},\ b_2(x) = \frac{\pi x}{2},\ h(x) = \ln\left(1+\sqrt{\pi}\right) + x.$$

Also,

$$a = \frac{\pi}{2},\ \alpha = \frac{1}{2} \quad \text{and} \quad k = M = \frac{1}{10}.$$

Inequality (3.1) is equivalent to

$$\frac{r}{10}+\frac{e^{r}\left(\ln\left(1+\sqrt{\pi}\right)+\frac{\pi r}{2}\right)\sqrt{\frac{\pi}{2}}}{16\Gamma\left(\frac{3}{2}\right)}+\frac{1}{10}\leq r.$$

It is easy to verify that $r_0 = 1/2$ satisfies the above inequality and T satisfies Darbo condition on B_{r_0} with respect to measure of noncompactness μ_d with $l = e^{\frac{3}{2}}/16$. Moreover

$$\frac{1}{10}+\frac{e^{\frac{3}{2}}\left(\ln\left(1+\sqrt{\pi}\right)+\frac{\pi}{4}\right)\sqrt{\frac{\pi}{2}}}{16\Gamma\left(\frac{3}{2}\right)}<1.$$

On the other hand it is clear that the other assumptions of Theorem 3.1 hold. Therefore, Theorem 3.1 guarantees that Eq.(4.2) has at least one nonnegative and nondecreasing solution $x = x(t) \in B_{\frac{1}{2}} \subset C\left[0, \frac{\pi}{2}\right]$.

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