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SOME FIXED POINT RESULTS FOR c-DISTANCE IN CONE METRIC SPACES

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Abstract. In this article by using the concept of c-distances in a cone metric space we extend, generalize and improve the corresponding results of Fadail *et al.*[8] and some others, under the continuity condition for maps. In the last section we give an examples in support of our results.

1. Introduction

The Banach contraction principle [4] states that if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping, then T has a unique fixed point. This principle has been generalized by considering contractive mappings on many different metric spaces. In 2007, Huang and Zhang [11] first introduced the concept of cone metric spaces and they established and proved the existence of fixed point theorems which is an extension of the Banach's contraction mapping principle in to cone metric spaces. Cone metric spaces is a generalized version of metric spaces, where each pair of points is assigned to a member of a real Banach space over the cone. Afterward, many authors have generalized and studied fixed point theorems in cone metric spaces (see [1], [2], [3], [12], [16], [20]).

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Recently, Wang and Guo [25] introduced the concept of c-distance in a cone metric spaces (also see [5]) and proved some fixed point theorems in ordered cone metric spaces. This was cone version of w-distance of Kada *et al.*[14]. Then several authors have proved fixed point theorems for c-distance in cone metric spaces (see [8], [9], [10], [22], [23]).

Following are the statements of theorems proved by Fadail *et al.*[8] by using c-distance in cone metric spaces;

Theorem 1.1. *Let (X, d) be a complete cone metric space and q be a c-distance on X. Let $f : X \to X$ be a mapping and suppose that there exists mapping $k : X \to [0, 1)$ such that the following hold:*

(a) $k(fx) \leq k(x)$ *for all* $x \in X$,
(b) $q(fx, fy) \preceq k(x)q(x, y)$ *for all* $x, y \in X$.

Then f has a fixed point x^* in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v, v) = \theta$. The fixed point is unique.

Theorem 1.2. *Let (X, d) be a complete cone metric space and q be a c-distance on X. Let $f : X \to X$ be a continuous mapping and suppose that there exists mapping $k, l, r : X \to [0, 1)$ such that the following hold:*

(a) $k(fx) \leq k(x)$, $l(fx) \leq l(x)$, $r(fx) \leq r(x)$ *for all* $x \in X$,
(b) $(k + l + r)(x) < 1$ *for all* $x \in X$,
(c) $q(fx, fy) \preceq k(x)q(x, y) + l(x)q(x, fx) + r(x)q(y, fy)$ *for all* $x, y \in X$.

Then f has a fixed point x^* in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v, v) = \theta$. The fixed point is unique.

Theorem 1.3. *Let (X, d) be a complete cone metric space and q be a c-distance on X. Let $f : X \to X$ be a mapping and suppose that there exists mapping $k, l, r : X \to [0, 1)$ such that the following hold:*

(a) $k(fx) \leq k(x)$, $l(fx) \leq l(x)$, $r(fx) \leq r(x)$ *for all* $x \in X$,
(b) $(2k + l + r)(x) < 1$ *for all* $x \in X$,
(c) $(1 - r(x))q(fx, fy) \preceq k(x)q(x, fy) + l(x)q(x, fx)$ *for all* $x, y \in X$.

Then f has a fixed point x^* in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v, v) = \theta$. The fixed point is unique.

The purpose of this paper is to extend and generalize some results on c-distance in cone metric spaces.

2. Preliminaries

The following definitions and results will be needed in the sequel. Throughout this paper we assume $\mathbb{R}$ as a set of real numbers and $\mathbb{N}$ as a set of natural numbers.

Definition 2.1. ([11]) Let E be a real Banach space and θ denote to the zero element in E. A cone P is the subset of E such that

(i) P is closed, non empty and $P \neq \{\theta\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$; $x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subseteq E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P.

Definition 2.2. ([11]) The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $||x|| \leq K||y||$. The least positive number satisfying above is called the normal constant of P.

In the following we always suppose E is a Banach Space, P is a cone in E with $intP \neq \phi$ and $\preceq$ is partial ordering with respect to P.

Definition 2.3. ([11]) Let X be a non empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(i) If $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.4. Let $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}^2$ and suppose that $d : X \times X \to E$ is defined by $d(x, y) = d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|, \alpha \max\{|x_1 - y_1|, |x_2 - y_2|\})$ where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space. It is easy to see that d is a cone metric, and hence (X, d) becomes a cone metric space over (E, P). Also, we have P is a solid and normal cone where the normal constant $K = 1$.

Definition 2.5. ([11]) Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x ($i.e.$ $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$).

(2) for all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) If every Cauchy sequence in X is convergent in X then (X, d) is called a complete cone metric space.

Lemma 2.6. ([13])

(1) *If E be a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

(2) *If $c \in intP$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.*

Next, we give the notion of c-distance on a cone metric space (X, d) of Wang and Guo in [25], which is a generalization of w-distance of Kada et al. [14] and some properties.

Definition 2.7. ([25]) Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:

(q_1) $\theta \preceq q(x, y)$ for all $x, y \in X$,

(q_2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$,

(q_3) for each $x \in X$ and $n \geq 1$ if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,

(q_4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.8. ([25]) Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}, X = [0, \infty)$ and define a mapping $d : X \times X \to E$ is defined by $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance on X.

Example 2.9. ([25]) Let (X, d) be a cone metric space and P be a normal cone. Define a mapping $q : X \times X \to E$ by $q(x, y) = d(u, y)$ for all $x, y \in X$, where u is a fixed point in X. Then q is c-distance.

Lemma 2.10. ([25]) *Let (X, d) be a cone metric space and q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be a sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then the following hold:*

(1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.*

(2) *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$ then $\{y_n\}$ converges to z.*

(3) *If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

(4) *If* $q(y, x_n) \preceq u_n$, *then* $\{x_n\}$ *is a Cauchy sequence in* X.

Remark 2.11. ([25])

(1) If $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$.
(2) If $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Now we are ready to state and prove our main results.

3. Main Results

Theorem 3.1. *Let* (X, d) *be a complete cone metric space and* q *be a* c*-distance on* X. *Let* $f : X \to X$ *be a continuous mapping and suppose that there exists mapping* $k, l : X \to [0, 1)$ *such that the following conditions hold:*

(*a*) $k(fx) \leq k(x)$, $l(fx) \leq l(x)$ *for all* $x \in X$;
(*b*) $(k + 2l)(x) < 1$ *for all* $x \in X$;
(*c*) $q(fx, fy) \preceq k(x)q(x, y) + l(x)[q(fx, y) + q(fy, x)]$ *for all* $x, y \in X$.

Then the map f *has a fixed point* x^* *in* X *and for any* $x \in X$, *iterative sequence* $\{f^n x\}$ *converges to the fixed point. If* $v = fv$, *then* $q(v, v) = \theta$. *The fixed point is unique.*

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, \cdots, x_{n+1} = fx_n = f^{n+1}x_0$. Then we have

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq k(x_{n-1})q(x_{n-1}, x_n) + l(x_{n-1})[q(fx_{n-1}, x_n) + q(fx_n, x_{n-1})] \\ &= k(fx_{n-2})q(x_{n-1}, x_n) + l(fx_{n-2})[q(x_{n+1}, x_{n-1})] \\ &\preceq k(x_{n-2})q(x_{n-1}, x_n) + l(x_{n-2})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})], \end{aligned}$$

continuing in this manner we can get,

$$q(x_n, x_{n+1}) \preceq k(x_0)q(x_{n-1}, x_n) + l(x_0)q(x_{n-1}, x_n) + l(x_0)q(x_n, x_{n+1})$$

and hence

$$\begin{aligned} q(x_n, x_{n+1}) &\preceq \frac{k(x_0) + l(x_0)}{1 - l(x_0)} q(x_{n-1}, x_n) \\ &= hq(x_{n-1}, x_n) \preceq h^2 q(x_{n-2}, x_{n-1}) \preceq \cdots \preceq h^n q(x_0, x_1), \end{aligned}$$

where $h = \frac{k(x_0)+l(x_0)}{1-l(x_0)} < 1$. Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq (h^n(x_0) + h^{n+1}(x_0) + \cdots + h^{m-1}(x_0))q(x_0, x_1) \\ &\preceq \frac{h^n(x_0)}{1 - h(x_0)} q(x_0, x_1). \end{aligned}$$

Thus, Lemma 2.10 (3) shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since f is continuous, then $x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x^*)$. Therefore x^* is a fixed point of f. Suppose that $v = fv$, then

$$\begin{aligned} q(v,v) &= q(fv,fv) \\ &\preceq k(x_0)q(v,v) + l(x_0)[q(fv,v) + q(fv,v)] \\ &= (k+2l)(x_0)q(v,v,). \end{aligned}$$

Since $(k+2l)(x_0) < 1$, Lemma 2.6 (1) shows that $q(v,v) = \theta$. Finally, suppose there is another fixed point y^* of f, then we have,

$$\begin{aligned} q(x^*,y^*) &= q(fx^*,fy^*) \\ &\preceq k(x^*)q(x^*,y^*) + l(x^*)[q(fx^*,y^*) + q(fy^*,x^*)] \\ &= k(x^*)q(x^*,y^*) + l(x^*)[q(x^*,y^*) + q(y^*,x^*)] \\ &= (k+2l)(x^*)q(x^*,y^*). \end{aligned}$$

Since $(k+2l)(x^*) < 1$, Lemma 2.6 (1) shows that $q(x^*,y^*) = \theta$ and also we have $q(x^*,x^*) = \theta$, hence by Lemma 2.10 (1), $x^* = y^*$. Therefore the fixed point is unique. □

Corollary 3.2. *Let (X,d) be a complete cone metric space and q be a c-distance on X. Let $f : X \to X$ be a continuous mapping and suppose that there exists mapping $k,l : X \to [0,1)$ such that the following conditions hold:*

(a) *$k(fx) \leq k(x)$, $l(fx) \leq l(x)$ for all $x \in X$;*
(b) *$(k+2l)(x) < 1$ for all $x \in X$;*
(c) *$q(fx,fy) \preceq k(x)q(x,y) + l(x)[q(x,fx) + q(y,fy)]$ for all $x,y \in X$.*

Then the map f has a fixed point x^* in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v,v) = \theta$. The fixed point is unique.

Theorem 3.3. *Let (X,d) be a complete cone metric space and q be a c-distance on X. Let $f : X \to X$ be a continuous mapping and suppose that there exists mapping $k,l,r : X \to [0,1)$ such that the following conditions hold:*

(a) *$k(fx) \leq k(x)$, $l(fx) \leq l(x)$, $r(fx) \leq r(x)$ for all $x \in X$;*
(b) *$(k+2l+2r)(x) < 1$ for all $x \in X$;*
(c) *$q(fx,fy) \preceq k(x)q(x,y) + l(x)[q(x,fy) + q(y,fx)] + r(x)[q(x,fx) + q(y,fy)]$ for all $x,y \in X$.*

Then the map f has a fixed point x^* in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v,v) = \theta$. The fixed point is unique.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, \cdots, x_{n+1} = fx_n = f^{n+1}x_0$. Then we have,

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq k(x_{n-1})q(x_{n-1}, x_n) + l(x_{n-1})[q(x_{n-1}, fx_n) + q(x_n, fx_{n-1})] \\ &\quad + r(x_{n-1})[q(x_{n-1}, fx_{n-1}) + q(x_n, fx_n)] \\ &= k(fx_{n-2})q(x_{n-1}, x_n) + l(fx_{n-2})[q(x_{n-1}, x_{n+1}) + q(x_n, x_n)] \\ &\quad + r(fx_{n-2})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \\ &\preceq k(x_{n-2})q(x_{n-1}, x_n) + l(x_{n-2})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \\ &\quad + r(x_{n-2})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})], \end{aligned}$$

continuing in this manner we can get,

$$\begin{aligned} q(x_n, x_{n+1}) \preceq\ & k(x_0)q(x_{n-1}, x_n) + l(x_0)[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \\ & + r(x_0)[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \end{aligned}$$

and hence

$$\begin{aligned} q(x_n, x_{n+1}) &\preceq \frac{k(x_0) + l(x_0) + r(x_0)}{1 - l(x_0) - r(x_0)} q(x_{n-1}, x_n) \\ &= hq(x_{n-1}, x_n) \\ &\preceq h^2 q(x_{n-2}, x_{n-1}) \preceq \cdots \preceq h^n q(x_0, x_1), \end{aligned}$$

where $h = \frac{k(x_0)+l(x_0)+r(x_0)}{1-l(x_0)-r(x_0)} < 1$. Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq (h^n(x_0) + h^{n+1}(x_0) + \cdots + h^{m-1}(x_0))q(x_0, x_1) \\ &\preceq \frac{h^n(x_0)}{1 - h(x_0)} q(x_0, x_1). \end{aligned}$$

Thus, Lemma 2.10 (3) shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since f is continuous, then $x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x^*)$. Therefore x^* is a fixed point of f. Suppose that $v = fv$, then

$$\begin{aligned} q(v, v) &= q(fv, fv) \\ &\preceq k(x_0)q(v, v) + l(x_0)[q(v, fv) + q(v, fv)] \\ &\quad + r(x_0)[q(v, fv) + q(v, fv)] \\ &= (k + 2l + 2r)(x_0)q(v, v). \end{aligned}$$

Since $(k+2l+2r)(x_0) < 1$, Lemma 2.6 (1) shows that $q(v,v) = \theta$. Finally, suppose there is another fixed point y^* of f, then we have,

$$\begin{aligned}
q(x^*, y^*) &= q(fx^*, fy^*) \\
&\preceq k(x^*)q(x^*, y^*) + l(x^*)[q(x^*, fy^*) + q(y^*, fx^*)] \\
&\quad + r(x^*)[q(x^*, fx^*) + q(y^*, fy^*)] \\
&= k(x^*)q(x^*, y^*) + l(x^*)[q(x^*, y^*) + q(y^*, x^*)] \\
&\quad + r(x^*)[q(x^*, x^*) + q(y^*, y^*)] \\
&= (k+2l)(x^*)q(x^*, y^*) \\
&\preceq (k+2l+2r)(x^*)q(x^*, y^*).
\end{aligned}$$

Since $(k+2l+2r)(x^*) < 1$, Lemma 2.6 (1) shows that $q(x^*, y^*) = \theta$ and also we have $q(x^*, x^*) = \theta$, hence by Lemma 2.10 (1), $x^* = y^*$. Therefore the fixed point is unique. □

Corollary 3.4. *Let (X, d) be a complete cone metric space and q be a c-distance on X. Let $f : X \to X$ be a continuous mapping and suppose that there exists mapping $k, l, r : X \to [0, 1)$ such that the following conditions hold:*

- *(a) $k(fx) \le k(x)$, $l(fx) \le l(x)$, $r(fx) \le r(x)$ for all $x \in X$;*
- *(b) $(k+2l+2r)(x) < 1$ for all $x \in X$;*
- *(c) $q(fx, fy) \preceq k(x)q(x,y) + l(x)[q(x,fx) + q(x,fy)] + r(x)[q(y,fx) + q(y,fy)]$ for all $x, y \in X$.*

Then the map f has a fixed point x^* in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v,v) = \theta$. The fixed point is unique.

Theorem 3.5. *Let (X, d) be a complete cone metric space and q be a c-distance on X. Let $f : X \to X$ be a continuous mapping and suppose that there exists mapping $k, r, l, t : X \to [0, 1)$ such that the following conditions hold:*

- *(a) $k(fx) \le k(x)$, $r(fx) \le r(x)$, $l(fx) \le l(x)$, $t(fx) \le t(x)$ for all $x \in X$;*
- *(b) $(k+r+l+2t)(x) < 1$ for all $x \in X$;*
- *(c) $q(fx, fy) \preceq k(x)q(x,y) + r(x)q(fx,x) + l(x)q(fy,y) + t(x)[q(fx,y) + q(fy,x)]$ for all $x, y \in X$.*

Then the map f has a fixed point x^* in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v,v) = \theta$. The fixed point is unique.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, \cdots, x_{n+1} = fx_n = f^{n+1}x_0$. Then we have,

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq k(x_{n-1})q(x_{n-1}, x_n) + r(x_{n-1})q(fx_{n-1}, x_{n-1}) \\ &\quad + l(x_{n-1})q(fx_n, x_n) + t(x_{n-1})[q(fx_{n-1}, x_n) + q(fx_n, x_{n-1})] \\ &= k(fx_{n-2})q(x_{n-1}, x_n) + r(fx_{n-2})q(x_n, x_{n-1}) \\ &\quad + l(fx_{n-2})q(x_{n+1}, x_n) + t(fx_{n-2})[q(x_n, x_n) + q(x_{n+1}, x_{n-1})] \\ &\preceq k(x_{n-2})q(x_{n-1}, x_n) + r(x_{n-2})q(x_{n-1}, x_n) \\ &\quad + l(x_{n-2})q(x_n, x_{n+1}) + t(x_{n-2})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})], \end{aligned}$$

continuing in this manner we can get,

$$\begin{aligned} q(x_n, x_{n+1}) \preceq\ & k(x_0)q(x_{n-1}, x_n) + r(x_0)q(x_{n-1}, x_n) + l(x_0)q(x_n, x_{n+1}) \\ & + t(x_0)[q(x_{n-1}, x_n) + q(x_n, x_{n+1})], \end{aligned}$$

and hence,

$$\begin{aligned} q(x_n, x_{n+1}) &\preceq \frac{k(x_0) + r(x_0) + t(x_0)}{1 - l(x_0) - t(x_0)} q(x_{n-1}, x_n) \\ &= hq(x_{n-1}, x_n) \\ &\preceq h^2 q(x_{n-2}, x_{n-1}) \preceq \cdots \preceq h^n q(x_0, x_1), \end{aligned}$$

where $h = \frac{k(x_0)+r(x_0)+t(x_0)}{1-l(x_0)-t(x_0)} < 1$. Let $m > n \geq 1$. Then it follows that,

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq (h^n(x_0) + h^{n+1}(x_0) + \cdots + h^{m-1}(x_0))q(x_0, x_1) \\ &\preceq \frac{h^n(x_0)}{1 - h(x_0)} q(x_0, x_1). \end{aligned}$$

Thus, Lemma 2.10 (3) shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since f is continuous, then $x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x^*)$. Therefore x^* is a fixed point of f. Suppose that $v = fv$, then,

$$\begin{aligned} q(v, v) &= q(fv, fv) \\ &\preceq k(x_0)q(v, v) + r(x_0)q(fv, v) + l(x_0)q(fv, v) \\ &\quad + t(x_0)[q(fv, v) + q(fv, v)] \\ &= (k + r + l + 2t)(x_0)q(v, v). \end{aligned}$$

Since $(k + r + l + 2t)(x_0) < 1$, Lemma 2.6 (1) shows that $q(v, v) = \theta$. Finally, suppose there is another fixed point y^* of f, then we have,

$$\begin{aligned} q(x^*,y^*) &= q(fx^*,fy^*) \\ &\preceq k(x^*)q(x^*,y^*)+r(x^*)q(fx^*,x^*)+l(x^*)q(fy^*,y^*) \\ &\quad + t(x^*)[q(fx^*,y^*)+q(fy^*,x^*)] \\ &= k(x^*)q(x^*,y^*)+r(x^*)q(x^*,x^*)+l(x^*)q(y^*,y^*) \\ &\quad + t(x^*)[q(x^*,y^*)+q(y^*,x^*)] \\ &= (k+2t)(x^*)q(x^*,y^*) \\ &\preceq (k+r+l+2t)(x^*)q(x^*,y^*). \end{aligned}$$

Since $(k+r+l+2t)(x^*) < 1$, Lemma 2.6 (1) shows that $q(x^*,y^*) = \theta$ and also we have $q(x^*,x^*) = \theta$, hence by Lemma 2.10 (1), $x^* = y^*$. Therefore, the fixed point is unique. □

Example 3.6. Let $E = \mathbb{R}$ and $P = \{x \in E, x \geq 0\}$. Let $X = [0,1]$ and define a mapping $d : X \times X \to E$ by $d(x,y) = |x-y|$ for all $x, y \in X$. Then (X,d) is a complete cone metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = 2d(x,y)$ for all $x, y \in X$. Then q is c-distance. In fact, (q_1)-(q_3) are immediate. Let $c \in E$ with $0 \ll c$ and put $e = \frac{c}{2}$. If $q(z,x) \ll e$ and $q(z,y) \ll e$, then we have

$$\begin{aligned} d(x,y) \leq 2d(x,y) &= 2|x-y| \\ &\leq 2|x-z| + 2|z-y| \\ &= q(z,x) + q(z,y) \ll e + e = c. \end{aligned}$$

This shows that (q_4) holds. Therefore q is the c-distance. Let $f : X \to X$ defined by $f(x) = \frac{x^2}{16}$ for all $x \in X$. Take mappings $k, r, l, t : X \to [0,1)$ by

$$k(x) = \frac{x+1}{16}, \quad r(x) = \frac{2x+3}{16}, \quad l(x) = \frac{3x+2}{16}, \quad t(x) = \frac{x}{16}$$

for all $x \in X$. Observe that

(i) $k(fx) = \big(\frac{x^2}{16}+1\big)/16 = \frac{1}{16}\big(\frac{x^2}{16}+1\big) \leq \frac{1}{16}(x+1) = k(x)$ for all $x \in X$,
(ii) $t(fx) = \big(\frac{x^2}{16}\big)/16 = \frac{1}{16}\big(\frac{x^2}{16}\big) \leq \frac{1}{16}(x) = t(x)$ for all $x \in X$,
(iii) $r(fx) = \big(2\big(\frac{x^2}{16}\big)+3\big)/16 = \frac{1}{16}\big(\frac{2x^2}{16}+3\big) \leq \frac{1}{16}(2x+3) = r(x)$ for all $x \in X$,
(iv) $l(fx) = \big(3\big(\frac{x^2}{16}\big)+2\big)/16 = \frac{1}{16}\big(\frac{3x^2}{16}+2\big) \leq \frac{1}{16}(3x+2) = l(x)$ for all $x \in X$,
(v) $(k+r+l+2t)(x) = \big(\frac{x+1}{16}\big) + \big(\frac{2x+3}{16}\big) + \big(\frac{3x+2}{16}\big) + 2\big(\frac{x}{16}\big) = \big(\frac{8x+6}{16}\big) < 1$ for all $x \in X$,

(vi) for all $x, y \in X$, we have

$$\begin{aligned} q(fx, fy) &= 2\left|\frac{x^2}{16} - \frac{y^2}{16}\right| \\ &\preceq \frac{2|x+y||x-y|}{16} = \left(\frac{x+y}{16}\right)2|x-y| \\ &\preceq \left(\frac{x+1}{16}\right)2|x-y| = k(x)q(x,y) \\ &\preceq k(x)q(x,y) + r(x)q(fx,x) + l(x)q(fy,y) \\ &\quad + t(x)[q(fx,y) + q(fy,x)]. \end{aligned}$$

Therefore, all the conditions of Theorem 3.5 are satisfied. Hence f has a unique fixed point $x = 0$ with $q(0,0) = 0$.

4. Conclusion

In this attempt, we prove some fixed point results in cone metric spaces. These results generalizes and improves the recent results of Fadail *et al.* [8] in the sense that in our results, we employing c-distance and in contractive conditions, replacing the constants with functions, which extend the further scope of our results. In the last section of the paper, an example is given to support the presented results.

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