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ON THE STRONG CONVERGENCE THEOREM OF THE HYBRID VISCOSITY APPROXIMATION METHOD FOR ZEROS OF m-ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper, we give sufficient conditions of the hybrid viscosity approximation method for zeros of m-accretive operators in Banach spaces E, which is introduced by Ceng *et al.* in [4] (Theorem 3.2) when E is a uniformly convex and uniformly smooth Banach space.

1. Introduction

Let E be a real Banach space, $A: \ D(A) \longrightarrow 2^E$ an accretive operator and J_r the resolvent of A. We consider the problem of finding a zero of A, that is, find $p \in D(A)$ such that $0 \in A(p)$.

One popular method of solving equation $0 \in A(x)$ where A is a maximal monotone operator in a Hilbert space H, is the proximal point algorithm. The proximal point algorithm generates, for any starting point $x_0 = x \in E$, a sequence $\{x_n\}$ by the rule

$$x_{n+1} = J_{r_n}(x_n), \text{ for all } n \in \mathbb{N}, \tag{1.1}$$

where $\{r_n\}$ is a sequence of positive real numbers. Some of them dealt with the weak convergence of the sequence $\{x_n\}$ generated by (1.1) and others proved strong convergence theorems by imposing assumptions on A. Note that, this algorithm was first introduced by Martinet [8].

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Further, Rockafellar [10] posed an open question of whether the sequence generated by (1.1) converges strongly or not. In 1991, Güler [6] gave an example showing that Rockafellar's proximal point algorithm does not converge strongly. An example of the authors Bauschke, Matoušková and Reich [1] also showed that the proximal algorithm only converges weakly but not in the norm.

There are many authors who studied and extended the Rockafellar's proximal point algorithm to find zeros of accretive operators. In 2012, Ceng *et al.* introduced a new iterative method to find a zero of an m-accretive operator [4]. They proved the following theorem:

Theorem 1.1. ([4]) *Let E be a uniformly smooth Banach space, let A be an m-accretive operator in E with $C = A^{-1}(0) \neq \emptyset$ and let $f : E \longrightarrow K = \overline{D(A)}$ be a contractive map. Assume that $F : E \longrightarrow E$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Given sequences $\{\lambda_n\}$, $\{\mu_n\}$ in $[0, 1]$, $\{\alpha_n\}$, $\{\beta_n\}$ in $(0, 1]$ and $\{r_n\} \subset [\varepsilon, \infty)$ for some $\varepsilon > 0$, suppose that there hold the following conditions:*

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$*;*

(ii) $\lim_{n\to\infty} \dfrac{\lambda_n \mu_n}{\beta_n} = 0$*;*

(iii) $\{\alpha_n\} \subset [a, b]$*, with* $a, b \in (0, 1)$*;*

(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=0}^{\infty} |\mu_{n+1} - \mu_n| < \infty$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$*.*

Then for any given point $x_0 \in E$, the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)[J_{r_n} y_n - \lambda_n \mu_n F(J_{r_n} y_n)], \ \forall n \geq 0, \end{cases} \tag{1.2}$$

converges strongly to a zero point p of A, which is a unique solution of $VI^*(I - f, C)$, that is

$$\langle (I - f)p, j(p - u) \rangle \leq 0, \ \forall u \in C.$$

The purpose of this paper is to show that if E is a uniformly convex and uniformly smooth, then the conditions i), ii) and iii) in Theorem 1.1 are sufficient conditions to ensure the strong convergence of iterative (1.2).

2. Preliminaries

Let E be a real Banach space with norm $\|.\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f \rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \longrightarrow x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x.

Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \ \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \ \forall x \in E,$$

where $\langle ., .\rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued. In the sequel, we shall denote the single-valued normalized duallity mapping by j.

We always use S_E to denote the unit sphere $S_E = \{x \in E : \ \|x\| = 1\}$ and $Fix(T)$ to denote the set of the fixed point of the mapping $T : \ C \subseteq E \longrightarrow E$, i.e., $Fix(T) = \{x \in C : \ T(x) = x\}$.

A Banach space E is said to be strictly convex if

$$x, y \in S_E \text{ with } x \neq y, \text{ implies that } \|(1-t)x + ty\| < 1 \text{ for all } t \in (0,1).$$

A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0, 2]$ the inequalities $\|x\| \leq 1, \ \|y\| \leq 1, \ \|x - y\| \geq \varepsilon$ imply there exists a $\delta = \delta(\varepsilon) \geq 0$ such that

$$\frac{\|x+y\|}{2} \leq 1 - \delta.$$

A Banach E is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_E. In this case, the norm of E is said to be Gâteaux differentiable. It is said to be uniformly Fréchet differentiable (and E is called uniformly smooth) if this limit is attained uniformly for all x and y in S_E.

For an operator $A : \ E \longrightarrow 2^E$, we define its domain, range and graph as follows:

$$D(A) = \{x \in E : \ Ax \neq \emptyset\},$$
$$R(A) = \cup\{Az : \ z \in D(A)\}$$

and

$$G(A) = \{(x, y) \in E \times E : \ x \in D(A), \ y \in Ax\},$$

respectively. The inverse A^{-1} of A is defined by

$$x \in A^{-1}y, \text{ if and only if } y \in Ax.$$

The operator A is said to be accretive if, for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that $\langle u - v, j(x-y)\rangle \geq 0$ for all $u \in Ax$ and $v \in Ay$. We denote by I the identity operator on E. An accretive operator A is said to be maximal accretive if there is no proper accretive extension of A and m-accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$. If A is m-accretive, then it is maximal accretive, but the converse is not true in general. If A is accretive, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda : \ R(I + \lambda A) \longrightarrow D(A)$ by

$$J_\lambda = (I + \lambda A)^{-1},$$

it is called the resolvent of A. An accretive A defined on a Banach space E is said to satisfy the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. We know that for an accretive operator A which satisfies the range condition, $A^{-1}(0) = F(J_\lambda)$ for all $\lambda > 0$. It's easy to see that if A is m-accretive operator, then A satisfies the range condition.

Recall that a mapping $F: E \longrightarrow E$ is said to be δ-strongly accretive if for each $x, y \in E$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle F(x) - F(y), j(x - y)\rangle \geq \delta \|x - y\|^2$$

for some $\delta \in (0, 1)$. A mapping $F: E \longrightarrow E$ is said to be λ-strictly pseudocontractive [2] if for each $x, y \in E$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle F(x) - F(y), j(x - y)\rangle \leq \|x - y\|^2 - \lambda \|x - y - (F(x) - F(y))\|$$

for some $\lambda \in (0, 1)$ and F is said to be pseudocontractive if for each $x, y \in E$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle F(x) - F(y), j(x - y)\rangle \leq \|x - y\|^2.$$

So, if F is a nonexpansive mapping, that is $\|F(x) - F(y)\| \leq \|x - y\|$ for all $x, y \in E$, then F is a pseudocontractive mapping.

The following lemmas will be needed in the sequel for the proof of the main results in this paper.

Lemma 2.1. ([3]) *Let E be a real smooth Banach space and $F: E \longrightarrow E$ be a mapping. If F is δ-strongly accretive and λ- strictly pseudocontractive with $\delta + \lambda > 1$, then for any fixed number $\tau \in (0, 1]$, $I - \tau F$ is contractive with constant* $1 - \tau\left(1 - \sqrt{\dfrac{1-\delta}{\lambda}}\right)$.

Lemma 2.2. ([9]) *Let E be a Banach space. For every $x, y \in E$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,$$

for all $j(x + y) \in J(x + y)$.

Lemma 2.3. ([5]) *Let $A: D(A) \subset E \longrightarrow 2^E$ be an accretive operator. Let $r, t > 0$. If E is uniformly convex, then there exists a continuous, strictly increasing and convex function $\varphi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\varphi(0) = 0$ so that*

$$\|J_r^A x - J_r^A y\|^2 \leq \|x - y\|^2 - \varphi(\|(I - J_r^A)x - (I - J_r^A)y\|),$$

for all $x, y \in R(I + rA)$ with $\max\{\|x\|, \|y\|\} \leq t$.

Lemma 2.4. ([7]) *Let $\{s_n\}$ be a real sequence that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_k}\}$ so that*

$$s_{n_k} \leq s_{n_k+1}, \ \forall k \geq 0.$$

For every $n > n_0$ define an integer sequence $\{\tau(n)\}$ as

$$\tau(n) = \max\{n_0 \leq k \leq n : \ s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ for all $n > n_0$,

$$\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}.$$

3. Main Results

We have the following theorem:

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, let A be an m-accretive operator in E with $C = A^{-1}(0) \neq \emptyset$ and let $f : E \longrightarrow K = \overline{D(A)}$ be a contractive map with the contraction coefficient $\beta \in [0, 1)$. Assume that $F : E \longrightarrow E$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Given sequences $\{\lambda_n\}$, $\{\mu_n\}$ in $[0, 1]$, $\{\alpha_n\}$, $\{\beta_n\}$ in $(0, 1]$ and $\{r_n\} \subset [\varepsilon, \infty)$ for some $\varepsilon > 0$, suppose that there hold the following conditions:*

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(ii) $\lim_{n\to\infty} \dfrac{\lambda_n \mu_n}{\beta_n} = 0$;

(iii) $\{\alpha_n\} \subset [a, b]$, *with* $a, b \in (0, 1)$;

Then for any given point $x_0 \in E$, the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)[J_{r_n} y_n - \lambda_n \mu_n F(J_{r_n} y_n)], \ \forall n \geq 0, \end{cases} \tag{3.1}$$

converges strongly to a zero point p of A, which is a unique solution of $VI^*(I - f, C)$, that is

$$\langle (I - f)p, j(p - u)\rangle \leq 0, \ \forall u \in C.$$

Proof. The first, we show that $\{x_n\}$ is bounded. Indeed, taking a fixed $u \in C$, we have

$$\begin{aligned} \|y_n - u\| &= \|\alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n - u\| \\ &= \|\alpha_n (x_n - u) + (1 - \alpha_n)(J_{r_n} x_n - u)\| \\ &\leq \alpha_n \|x_n - u\| + (1 - \alpha_n)\|J_{r_n} x_n - J_{r_n} u\| \\ &\leq \|x_n - u\|. \end{aligned}$$

So, from Lemma 2.1, we have

$$\begin{aligned}
&\|x_{n+1} - u\| \\
&= \|\beta_n f(x_n) + (1-\beta_n)[J_{r_n}y_n - \lambda_n\mu_n F(J_{r_n}y_n)] - u\| \\
&\leq \beta_n\|f(x_n) - u\| + (1-\beta_n)\|\lambda_n(I-\mu_n F)J_{r_n}y_n + (1-\lambda_n)J_{r_n}y_n - u\| \\
&\leq \beta_n\beta\|x_n - u\| + \beta_n\|f(u) - u\| \\
&\quad + (1-\beta_n)[\lambda_n\|(I-\mu_n F)J_{r_n}y_n - u\| + (1-\lambda_n)\|J_{r_n}y_n - u\|] \\
&\leq \beta_n\beta\|x_n - u\| + \beta_n\|f(u) - u\| \\
&\quad + (1-\beta_n)[\lambda_n\|(I-\mu_n F)J_{r_n}y_n - (I-\mu_n F)u\| \\
&\quad + \lambda_n\mu_n\|F(u)\| + (1-\lambda_n)\|J_{r_n}y_n - u\|] \\
&\leq \beta_n\beta\|x_n - u\| + \beta_n\|f(u) - u\| \\
&\quad + (1-\beta_n)\Bigg[\lambda_n\Bigg(1 - \mu_n\Bigg(1 - \sqrt{\frac{1-\delta}{\lambda}}\Bigg)\Bigg)\|J_{r_n}y_n - u\| \\
&\quad + \lambda_n\mu_n\|F(u)\| + (1-\lambda_n)\|J_{r_n}y_n - u\|\Bigg] \\
&= \beta_n\beta\|x_n - u\| + \beta_n\|f(u) - u\| \\
&\quad + (1-\beta_n)\Bigg[\Bigg(1 - \lambda_n\mu_n\Bigg(1 - \sqrt{\frac{1-\delta}{\lambda}}\Bigg)\Bigg)\|J_{r_n}y_n - u\| + \lambda_n\mu_n\|F(u)\|\Bigg] \\
&\leq \beta_n\beta\|x_n - u\| + \beta_n\|f(u) - u\| \\
&\quad + (1-\beta_n)\max\Bigg\{\|J_{r_n}y_n - u\|, \Bigg(1 - \sqrt{\frac{1-\delta}{\lambda}}\Bigg)^{-1}\|F(u)\|\Bigg\} \\
&\leq \beta_n\beta\|x_n - u\| + \beta_n\|f(u) - u\| \\
&\quad + (1-\beta_n)\max\Bigg\{\|x_n - u\|, \Bigg(1 - \sqrt{\frac{1-\delta}{\lambda}}\Bigg)^{-1}\|F(u)\|\Bigg\} \\
&\leq \beta_n\|f(u) - u\| \\
&\quad + (1-(1-\beta)\beta_n)\max\Bigg\{\|x_n - u\|, \Bigg(1 - \sqrt{\frac{1-\delta}{\lambda}}\Bigg)^{-1}\|F(u)\|\Bigg\} \\
&\leq \max\Bigg\{\|x_n - u\|, \frac{\|f(u) - u\|}{1-\beta}, \Bigg(1 - \sqrt{\frac{1-\delta}{\lambda}}\Bigg)^{-1}\|F(u)\|\Bigg\}.
\end{aligned}$$

By induction, we get

$$\|x_n - u\| \leq \max\Bigg\{\|x_0 - u\|, \frac{\|f(u) - u\|}{1-\beta}, \Bigg(1 - \sqrt{\frac{1-\delta}{\lambda}}\Bigg)^{-1}\|F(u)\|\Bigg\},\ \forall n \geq 0.$$

Thus, $\{x_n\}$ is bounded.

Now, let p is a unique solution of $\mathrm{VI}^*(I-f,C)$, that is

$$\langle (I-f)p, j(p-u)\rangle \le 0,\ \forall u\in C,$$

from Lemma 2.2, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\beta_n)^2\|J_{r_n}y_n-p\|^2 + 2\beta_n\langle f(x_n)-p, j(x_{n+1}-p)\rangle\\
&\quad - 2(1-\beta_n)\lambda_n\mu_n\langle F(J_{r_n}y_n), j(x_{n+1}-p)\rangle\\
&= (1-\beta_n)^2\|y_n-p\|^2 + 2\beta_n\langle f(x_n)-f(p), j(x_{n+1}-p)\rangle\\
&\quad + 2\beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\quad - 2(1-\beta_n)\lambda_n\mu_n\langle F(J_{r_n}y_n), j(x_{n+1}-p)\rangle\\
&\le (1-\beta_n)^2\|y_n-p\|^2 + \beta_n\beta(\|x_n-p\|^2+\|x_{n+1}-p\|^2)\\
&\quad + 2\beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\quad + 2(1-\beta_n)\lambda_n\mu_n\|F(J_{r_n}y_n)\|.\|x_{n+1}-p\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \frac{1}{1-\beta_n\beta}\big((1-\beta_n)^2\|y_n-p\|^2+\beta_n\beta\|x_n-p\|^2\big)\\
&\quad + \frac{2\beta_n}{1-\beta_n\beta}\langle f(p)-p, j(x_{n+1}-p)\rangle \qquad (3.2)\\
&\quad + \frac{2\lambda_n\mu_n}{1-\beta_n\beta}\|F(J_{r_n}y_n)\|.\|x_{n+1}-p\|.
\end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned}
\|y_n-p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)J_{r_n}x_n - p\|^2\\
&\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|J_{r_n}x_n-p\|^2\\
&\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|x_n-p\|^2 \qquad (3.3)\\
&\quad - (1-\alpha_n)\varphi(\|x_n-J_{r_n}x_n\|)\\
&\le \|x_n-p\|^2 - (1-b)\varphi(\|x_n-J_{r_n}x_n\|).
\end{aligned}$$

From (3.2) and (3.3), we get

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le \frac{1-(2-\beta)\beta_n}{1-\beta_n\beta}\|x_n-p\|^2 + \frac{2\beta_n}{1-\beta_n\beta}\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\quad + \frac{M}{1-\beta}\beta_n^2 + \frac{M}{1-\beta}\lambda_n\mu_n - (1-\beta_n)^2(1-b)\varphi(\|x_n-J_{r_n}x_n\|),
\end{aligned}$$

where $M > \max\{\|x_n-p\|^2, 2\|F(J_{r_n}y_n)\|.\|x_{n+1}-p\|\}$. We have the following inequality

$$s_{n+1} \le (1-b_n)s_n + b_nc_n - \sigma_n, \qquad (3.4)$$

where $s_n = \|x_n - p\|^2$, $b_n = \dfrac{2(1-\beta)\beta_n}{1-\beta_n\beta}$,

$$c_n = \frac{M(1-\beta\beta_n)}{2(1-\beta)^2}\left(\beta_n + \frac{\lambda_n\mu_n}{\beta_n}\right) + \frac{1}{1-\beta}\langle f(p)-p, j(x_{n+1}-p)\rangle,$$

and

$$\sigma_n = (1-\beta_n)^2(1-b)\varphi(\|x_n - J_{r_n}x_n\|).$$

We will show that $s_n \to 0$ by considering two possible cases.

Case 1. $\{s_n\}$ is eventually decreasing, *i.e.*, there exists $N_0 \geq 0$ such that $\{s_n\}$ is decreasing for $n \geq N_0$ and thus $\{s_n\}$ must be convergent. It then follows from (3.4) that

$$0 \leq \sigma_n \leq (s_n - s_{n+1}) + b_n(c_n - s_n) \to 0,$$

which implies that

$$\|x_n - J_{r_n}x_n\| \to 0. \tag{3.5}$$

Now we show that

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n - p)\rangle \leq 0. \tag{3.6}$$

Indeed, puting $x_{t,n} = tf(x_{t,n}) + (1-t)[J_{r_n}x_{t,n} - \theta_t F(J_{r_n}x_{t,n})]$. Then by Theorem 3.1 in [4], $x_{t,n}$ converges strongly to a unique solution of $\mathrm{VI}^*(I-f, C)$, as $t \to 0$. We have

$$x_{t,n} - x_n = t(f(x_{t,n}) - x_n) + (1-t)(J_{r_n}x_{t,n} - x_n) - (1-t)\theta_t F(J_{r_n}x_{t,n}).$$

So, by Lemma 2.2, we have

$$\begin{aligned}
&\|x_{t,n} - x_n\|^2 \\
&\leq (1-t)^2\|J_{r_n}x_{t,n} - x_n\|^2 + 2t\langle f(x_{t,n}) - x_n, j(x_{t,n} - x_n)\rangle \\
&\quad - 2(1-t)\theta_t\langle F(J_{r_n}x_{t,n}), j(x_{t,n} - x_n)\rangle \\
&\leq (1-t)^2(\|J_{r_n}x_{t,n} - J_{r_n}x_n\|^2 + \|J_{r_n}x_n - x_n\|^2) \\
&\quad + 2t\langle f(x_{t,n}) - x_{t,n}, j(x_{t,n} - x_n)\rangle + 2t\|x_{t,n} - x_n\|^2 \\
&\quad + 2\theta_t\|F(J_{r_n}x_{t,n})\|.\|x_{t,n} - x_n\| \\
&\leq (1+t^2)\|x_{t,n} - x_n\|^2 \\
&\quad + \|J_{r_n}x_n - x_n\|(2\|x_{t,n} - x_n\| + \|J_{r_n}x_n - x_n\|) \\
&\quad + 2t\langle f(x_{t,n}) - x_{t,n}, j(x_{t,n} - x_n)\rangle + 2\theta_t\|F(J_{r_n}x_{t,n})\|.\|x_{t,n} - x_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\langle f(x_{t,n}) - x_{t,n}, j(x_{t,n} - x_n)\rangle \\
&\le \frac{t}{2}\|x_{t,n} - x_n\|^2 + \frac{1}{2t}\|J_{r_n}x_n - x_n\|(2\|x_{t,n} - x_n\| + \|J_{r_n}x_n - x_n\|) \\
&\quad + \frac{\theta_t}{t}\|F(J_{r_n}x_{t,n})\|.\|x_{t,n} - x_n\|.
\end{aligned} \tag{3.7}$$

From (3.5) and in (3.7) letting $n \to \infty$, we obtain that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(x_{t,n}) - x_{t,n}, j(x_{t,n} - x_n)\rangle \\
&\le \frac{t}{2}\limsup_{n\to\infty}\|x_{t,n} - x_n\|^2 + \limsup_{n\to\infty}\frac{\theta_t}{t}\|F(J_{r_n}x_{t,n})\|.\|x_{t,n} - x_n\| \\
&\le \left(\frac{t}{2} + \frac{\theta_t}{t}\right)M_1,
\end{aligned} \tag{3.8}$$

where $M_1 = \max\{\sup_n\{\|x_{t,n} - x_n\|^2\}, \sup_n\{\|F(J_{r_n}x_{t,n})\|.\|x_{t,n} - x_n\|\}\} < \infty$ for all $t \in (0, a]$ (see [4], Theorem 3.1). Taking the $\limsup$ as $t \to 0$ in (3.8) and by the duality map j is norm-to-norm uniformly continuous on bounded sets, we obtain (3.6). We have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \|x_{n+1} - J_{r_n}y_n\| + \|J_{r_n}y_n - J_{r_n}x_n\| + \|x_n - J_{r_n}x_n\| \\
&\le K\beta_n + \|y_n - x_n\| + \|x_n - J_{r_n}x_n\| \\
&= K\beta_n + (1 - \alpha_n)\|x_n - J_{r_n}x_n\| + \|x_n - J_{r_n}x_n\| \\
&\le K\beta_n + (2 - a)\|x_n - J_{r_n}x_n\| \to 0,
\end{aligned} \tag{3.9}$$

where $K = \sup_n\{\|f(x_n)\| + \|J_{r_n}y_n\|\} + F(J_{r_n}y_n)\|\}$. Thus,

$$\|x_{n+1} - x_n\| \to 0. \tag{3.10}$$

By (3.6) and the duality map j is uniformly norm-to-norm continuous on bounded set, we get

$$\limsup_{n\to\infty}\langle f(p) - p, j(x_{n+1} - p)\rangle \le 0. \tag{3.11}$$

Now, from (3.4), we have

$$s_n \le \frac{s_n - s_{n+1}}{b_n} + c_n.$$

Note that $\liminf_{n\to\infty}\dfrac{s_n - s_{n+1}}{b_n} = 0$, because $\sum_{n=0}^{\infty} b_n = \infty$. Thus, from the last inequality, we obtain that

$$\lim_{n\to\infty} s_n \le \liminf_{n\to\infty}\left(\frac{s_n - s_{n+1}}{b_n} + c_n\right) \le \liminf_{n\to\infty}\frac{s_n - s_{n+1}}{b_n} + \limsup_{n\to\infty} c_n \le 0.$$

So, $\{s_n\}$ converges to 0.

Case 2. $\{s_n\}$ is not eventually decreasing. Hence, there exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k} \leq s_{n_k+1}$ for all $k \geq 0$. By Lemma 2.4, we can define a subsequence $\{s_{\tau(n)}\}$ such that

$$\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}, \ \forall n \geq n_0. \tag{3.12}$$

From (3.4), we have

$$0 \leq \sigma_{\tau(n)} \leq b_{\tau(n)}(c_{\tau(n)} - s_{\tau(n)}) \to 0, \tag{3.13}$$

thus $\sigma_{\tau(n)} \to 0$. By similar argument to Case 1, we get

$$\limsup_{n\to\infty} \langle f(p) - p, j(x_{\tau(n)+1} - p)\rangle \leq 0,$$

or

$$\limsup_{n\to\infty} c_{\tau(n)} \leq 0.$$

By (3.13), $\limsup_{n\to\infty} s_{\tau(n)} \leq 0$, thus

$$\lim_{n\to\infty} s_{\tau(n)} = 0.$$

In a similar to (3.9), we have

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0.$$

Thus, by $\{x_n\}$ is bounded, we get

$$\begin{aligned} |s_{\tau(n)+1} - s_{\tau(n)}| &= |\|x_{\tau(n)+1} - p\|^2 - \|x_{\tau(n)+1} - p\|^2| \\ &\leq \|x_{\tau(n)+1} - x_{\tau(n)}\|(\|x_{\tau(n)+1} - p\| + \|x_{\tau(n)+1} - p\|) \\ &\to 0. \end{aligned}$$

Hence, $|s_{\tau(n)+1} - s_{\tau(n)}| \to 0$. From (3.12), for all $n \geq n_0$, we have

$$0 \leq s_n \leq s_{\tau(n)+1} = s_{\tau(n)} + (s_{\tau(n)+1} - s_{\tau(n)}) \to 0,$$

which implies that $s_n \to 0$. Consequently, we obtain $s_n \to 0$ in both cases, that is $x_n \to p$. This completes the proof. □

Now, we have the following corollary:

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, let A be an m-accretive operator in E with $C = A^{-1}(0) \neq \emptyset$ and let $f : E \longrightarrow K = \overline{D(A)}$ be a contractive map with the contraction coefficient $\beta \in [0, 1)$. Given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $(0, 1]$ and $\{r_n\} \subset [\varepsilon, \infty)$ for some $\varepsilon > 0$, suppose that there hold the following conditions:*

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $\{\alpha_n\} \subset [a, b]$, *with* $a, b \in (0, 1)$.

Then for any given point $x_0 \in E$, the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) J_{r_n} y_n, \ \forall n \geq 0, \end{cases} \tag{3.14}$$

converges strongly to a zero point p of A, which is a unique solution of $VI^*(I-f, C)$, that is

$$\langle (I-f)p, j(p-u) \rangle \leq 0, \ \forall u \in C.$$

Proof. Applying Theorem 3.1 with $\lambda_n = 0$ or $\mu_n = 0$ for all n, we get the Corollary 3.2. ☐

4. Numerical test

Example 4.1. Consider the problem of finding an element

$$x^* \in S = \ \operatorname{argmin}_{x \in \mathbb{R}^3} \Theta(x),$$

where Θ is defined by

$$\Theta(x) = \langle Qx, x \rangle + \langle B, x \rangle + C, \ \forall x \in \mathbb{R}^3,$$

with

$$Q = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} -4 & -4 & 4 \end{pmatrix} \text{ and } C \text{ is any constant.}$$

Since $\bigtriangledown^2 \Theta = 2Q$ is positive semi-definite matrix, Θ is a convex function. Hence, $\bigtriangledown \Theta$ is a maximal monotone operator in $\mathbb{R}^3$ and the above problem is equivalent to the following problem:

Finding an element $x^* \in S = (\bigtriangledown \Theta)^{-1}(0)$.

It is easy to show that

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 + x_2 - x_3 = 2\}.$$

We can see that $\lambda_n = \mu_n = \beta_n = \dfrac{1}{\sqrt{n}}$, $\alpha_n = \dfrac{1}{4} + \dfrac{1}{2\sqrt{n}}$ and $r_n = n$ for all $n \geq 1$ satisfy all conditions in Theorem 3.1, but the conditions

$$\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \quad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$$
$$\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \quad \sum_{n=0}^{\infty} |\mu_{n+1} - \mu_n| < \infty$$

are not satisfied.

Remark 4.2. The sequence $\{x_n\}$ generated by (3.1) converges strongly to a zero point p of $\bigtriangledown\Theta$, which is a unique solution of the following inequality

$$\langle p - v, p - u\rangle \leq 0, \ \forall u \in C.$$

By the property of metric projection, we get that $p = P_S v = (-1, 1, -2)$, where P_C is the metric projection from $\mathbb{R}^3$ onto S.

Apply the iterative (3.1), with $f(x) = v = (-2, 0, -1)$ and $F(x) = \frac{3}{4}x$ $\left(\delta = \frac{3}{4} \text{ and } \lambda = \frac{1}{2}\right)$ for all $x \in \mathbb{R}^3$, and $x^0 = (1, 0, -1)$, we have the following figure of numerical results:

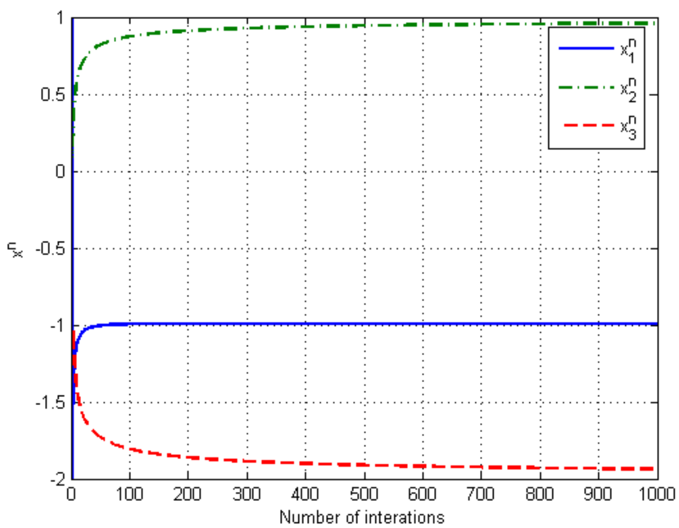

FIGURE 1. $x^{1000} = (-0.994144, \ 0.960298, \ -1.937520)$

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