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SOME FIXED POINT THEOREMS FOR CONTRACTIVE TYPE CONDITIONS IN COMPLEX VALUED b-METRIC SPACES

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Abstract. In this paper, we prove some fixed point theorems for contractive type conditions in the setting of complex valued b-metric spaces. Our results extend and generalize some previous works from the existing literature.

1. Introduction

Fixed point theory plays a very significant role in the development of nonlinear analysis. In this area, the first important result was proved by Banach in 1922 for contraction mapping in complete metric space, known as the Banach contraction principle [4].

There are many generalizations of the Banach contraction principle specially in metric spaces, for example, b-metric space, cone metric space, rectangular metric space, cone rectangular metric space, rectangular b-metric space, cone b-metric space etc (see, [1, 3, 5, 6, 7, 8, 9, 10]).

In 1989, Bakhtin [3] introduced the concept of b-metric space as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous contraction principle in metric spaces. Czerwik used the concept of b-metric space and generalized the renowned Banach fixed point theorem in b-metric spaces (see, [6, 7]).

In 2011, Azam *et al.* [2] introduced the concept of complex valued metric space and established some fixed point results for mappings satisfying a

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rational inequality. Complex-valued metric space is useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics; as well as in physics, including hydrodynamics, thermodynamics, mechanical engineering and electrical engineering, for some details, see ([13, 14]).

Recently, Rao *et al.* [12] introduced the concept of complex valued b-metric space which is more general than the notion of well known complex valued metric space and proved some common fixed point results.

In this paper, we establish some fixed point theorems for contractive type conditions in the framework of complex valued b-metric spaces.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:
(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$;
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), or (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that
(c_1) $0 \lesssim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$,
(c_2) $z_1 \precsim z_2,\ z_2 \prec z_3 \Rightarrow z_1 \prec z_3$,
(c_3) If $a, b \in \mathbb{R}$ and $a \leq b$ then $az \precsim bz$ for all $z \in \mathbb{C}$.

The following definition is recently introduced by Rao *et al.* [12].

Definition 2.1. ([12]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d\colon X \times X \to \mathbb{C}$ is called a complex valued b-metric (CVbM) if the following conditions are satisfied:
($CVbM1$) $0 \precsim d(x,y)$ and $d(x,y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$;
($CVbM2$) $d(x,y) = d(y,x)$ for all $x, y \in X$;
($CVbM3$) $d(x,y) \precsim s[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.
The pair (X, d) is called a complex valued b-metric space.

Example 2.2. ([12]) Let $X = [0,1]$. Define the mapping $d\colon X \times X \to \mathbb{C}$ by $d(x,y) = |x-y|^2 + i|x-y|^2$ for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 2.3. Let (X, d) be a complex valued b-metric space.

(1) A point $x \in X$ is called an interior point of a subset $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.
(2) A point $x \in X$ is called a limit of A whenever for every $0 \prec r \in \mathbb{C}$ such that $B(x,r) \cap \Big(A - \{x\}\Big) \neq \emptyset$.
(3) The set A is called open whenever each element of A is an interior point of A. A subset B is called closed whenever each limit point of B belongs to B.
(4) A sub-basis for a Hausdorff topology τ on X is a family $\mathcal{F} := \{B(x,r) : x \in X, 0 \prec r\}$.

Definition 2.4. ([12]) Let (X,d) be a complex valued b-metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ is called convergent, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. Also, $\{x_n\}$ converges to x (written as, $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) and x is the limit of $\{x_n\}$.
(ii) $\{x_n\}$ is called a Cauchy sequence in X, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$. If every Cauchy sequence converges in X, then X is called a complete complex valued b-metric space.

Lemma 2.5. ([12]) *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} |d(x_n,x)| = 0$.*

Lemma 2.6. ([12]) *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n\to\infty} |d(x_n, x_{n+m})| = 0$.*

3. Main results

In this section we shall prove some fixed point theorems for contractive type conditions in the framework of complex valued b-metric spaces.

Theorem 3.1. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the condition:*

$$d(Tx,Ty) \precsim \beta \max\left\{d(x,y), \frac{d(x,Tx), d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx), d(y,Ty)}{1+d(Tx,Ty)}\right\} \quad (3.1)$$

for all $x, y \in X$, where $\beta \in [0,1)$ is a constant with $s\beta < 1$. Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (3.1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\precsim \beta \max\Big\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)}{1 + d(x_{n-1}, x_n)}, \\ &\qquad \frac{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)}{1 + d(Tx_{n-1}, Tx_n)}\Big\} \\ &= \beta \max\Big\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n), d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}, \\ &\qquad \frac{d(x_{n-1}, x_n), d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})}\Big\} \\ &\precsim \beta \max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\}. \end{aligned} \tag{3.2}$$

If $\max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\} = d(x_n, x_{n+1})$, then from (3.2), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\precsim \beta\, d(x_n, x_{n+1}) \\ &\prec \frac{1}{s} d(x_n, x_{n+1}) \\ &\prec d(x_n, x_{n+1}), \end{aligned} \tag{3.3}$$

which is a contradiction. Hence

$$\max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\} = d(x_{n-1}, x_n),$$

so from (3.2), we have

$$d(x_n, x_{n+1}) \precsim \beta\, d(x_{n-1}, x_n). \tag{3.4}$$

By induction, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\precsim \beta\, d(x_{n-1}, x_n) \precsim \beta^2\, d(x_{n-2}, x_{n-1}) \precsim \cdots \\ &\precsim \beta^n\, d(x_0, x_1). \end{aligned} \tag{3.5}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\precsim s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\precsim sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \end{aligned}$$

$$\begin{aligned}
&\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\
&\quad + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \\
&\precsim [s\beta^n + s^2\beta^{n+1} + s^3\beta^{n+2} + \cdots + s^m\beta^{n+m-1}] d(x_1, x_0) \\
&= s\beta^n [1 + s\beta + s^2\beta^2 + s^3\beta^3 + \cdots + (s\beta)^{m-1}] d(x_1, x_0) \\
&\precsim \left[\frac{s\beta^n}{1 - s\beta}\right] d(x_1, x_0)
\end{aligned}$$

and so

$$|d(x_n, x_m)| \le \left[\frac{s\beta^n}{1 - s\beta}\right] |d(x_1, x_0)| \to 0 \quad \text{as} \quad m, n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. It follows that $u = Tu$, otherwise $d(u, Tu) = z > 0$ and we would then have

$$\begin{aligned}
z &\precsim sd(u, x_{n+1}) + sd(x_{n+1}, Tu) \\
&= sd(u, x_{n+1}) + sd(Tu, Tx_n) \\
&\precsim sd(u, x_{n+1}) + s\beta \max\Big\{d(u, x_n), \frac{d(u, Tu), d(x_n, Tx_n)}{1 + d(u, x_n)}, \\
&\qquad \frac{d(u, Tu), d(x_n, Tx_n)}{1 + d(Tu, Tx_n)}\Big\} \\
&= sd(u, x_{n+1}) + s\beta \max\Big\{d(u, x_n), \frac{d(u, Tu), d(x_n, x_{n+1})}{1 + d(u, x_n)}, \\
&\qquad \frac{d(u, Tu), d(x_n, x_{n+1})}{1 + d(Tu, x_{n+1})}\Big\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
|z| \le\ & s|d(u, x_{n+1})| + s\beta \max\Big\{|d(u, x_n)|, \frac{|z|, |d(x_n, x_{n+1})|}{1 + |d(u, x_n)|}, \\
&\frac{|z|, |d(x_n, x_{n+1})|}{1 + |d(Tu, x_{n+1})|}\Big\}.
\end{aligned}$$

Letting $n \to \infty$, it follows that

$$|z| \le 0,$$

which is a contradiction and so $|z| = 0$, that is, $u = Tu$.

To prove the uniqueness of fixed point of T, assume that u^* is another fixed point of T, that is, $Tu^* = u^*$ such that $u \neq u^*$. Then

$$\begin{aligned} d(u,u^*) &= d(Tu,Tu^*) \\ &\precsim \beta \max\Big\{d(u,u^*), \frac{d(u,Tu),d(u^*,Tu^*)}{1+d(u,u^*)}, \frac{d(u,Tu),d(u^*,Tu^*)}{1+d(Tu,Tu^*)}\Big\} \\ &= \beta \max\Big\{d(u,u^*), \frac{d(u,u),d(u^*,u^*)}{1+d(u,u^*)}, \frac{d(u,u),d(u^*,u^*)}{1+d(u,u^*)}\Big\} \\ &= \beta \max\Big\{d(u,u^*),0,0\Big\} \\ &\precsim \beta\, d(u,u^*), \end{aligned}$$

so that $|d(u,u^*)| \leq \beta\,|d(u,u^*)| < |d(u,u^*)|$, since $0 < \beta < 1$, which is a contradiction and hence $d(u,u^*) = 0$. Thus $u = u^*$. This shows the uniqueness of fixed point of T. This completes the proof. □

If $\max\Big\{d(x,y), \frac{d(x,Tx),d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx),d(y,Ty)}{1+d(Tx,Ty)}\Big\} = d(x,y)$, then from Theorem 3.1, we have the following result as corollary.

Corollary 3.2. *Let (X,d) be a complete complex valued b-metric space (CVbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$d(Tx,Ty) \precsim \beta\, d(x,y)$$

for all $x,y \in X$, where $\beta \in [0,1)$ is a constant with $s\beta < 1$. Then T has a unique fixed point in X.

Remark 3.3. Corollary 3.2 extends well known Banach contraction principle from complete metric space to that setting of complete complex valued b-metric space considered in this paper.

Corollary 3.4. *Let (X,d) be a complete complex valued b-metric space (CVbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies (for fixed n):*

$$d(T^n x,T^n y) \precsim \beta \max\Big\{d(x,y), \frac{d(x,T^n x),d(y,T^n y)}{1+d(x,y)}, \frac{d(x,T^n x),d(y,T^n y)}{1+d(T^n x,T^n y)}\Big\}$$

for all $x,y \in X$, where $\beta \in [0,1)$ is a constant with $s\beta < 1$. Then T has a unique fixed point in X.

Proof. By Theorem 3.1, there exists $q \in X$ such that $T^n q = q$. Then

$$\begin{aligned} d(Tq,q) &= d(TT^nq,T^nq) = d(T^nTq,T^nq) \\ &\precsim \beta \max\Big\{d(Tq,q), \frac{d(Tq,T^nTq),d(q,T^nq)}{1+d(Tq,q)}, \frac{d(Tq,T^nTq),d(q,T^nq)}{1+d(T^nTq,T^nq)}\Big\} \\ &= \beta \max\Big\{d(Tq,q), \frac{d(Tq,TT^nq),d(q,T^nq)}{1+d(Tq,q)}, \frac{d(Tq,TT^nq),d(q,T^nq)}{1+d(TT^nq,T^nq)}\Big\} \\ &= \beta \max\Big\{d(Tq,q), \frac{d(Tq,Tq),d(q,q)}{1+d(Tq,q)}, \frac{d(Tq,Tq),d(q,q)}{1+d(Tq,q)}\Big\} \\ &= \beta \max\Big\{d(Tq,q),0,0\Big\} \\ &\precsim \beta\, d(Tq,q) \end{aligned}$$

so that $|d(Tq,q)| \leq \beta\,|d(Tq,q)| < |d(Tq,q)|$, since $0 < \beta < 1$, which is a contradiction and hence $d(Tq,q) = 0$. Thus $Tq = q$. This shows that T has a unique fixed point in X. This completes the proof. □

Theorem 3.5. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies the condition:*

$$d(Tx,Ty) \precsim \alpha_1\, d(x,y) + \alpha_2\, d(x,Tx) + \alpha_3\, d(y,Ty) \tag{3.6}$$

for all $x, y \in X$, where α_1, α_2, $\alpha_3 \in [0,1)$ are constants with $s\alpha_1 + s\alpha_2 + \alpha_3 < 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be any arbitrary point. We define the iterative sequence $\{x_n\}$ such that $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (3.6), we have

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\precsim \alpha_1\, d(x_{n-1},x_n) + \alpha_2\, d(x_{n-1},Tx_{n-1}) + \alpha_3\, d(x_n,Tx_n) \\ &= \alpha_1\, d(x_{n-1},x_n) + \alpha_2\, d(x_{n-1},x_n) + \alpha_3\, d(x_n,x_{n+1}) \\ &= (\alpha_1+\alpha_2)\, d(x_{n-1},x_n) + \alpha_3\, d(x_n,x_{n+1}). \end{aligned} \tag{3.7}$$

This implies that

$$\begin{aligned} d(x_n,x_{n+1}) &\precsim \Big(\frac{\alpha_1+\alpha_2}{1-\alpha_3}\Big) d(x_{n-1},x_n) \\ &= \theta\, d(x_{n-1},x_n), \end{aligned} \tag{3.8}$$

where $\theta = \Big(\frac{\alpha_1+\alpha_2}{1-\alpha_3}\Big)$. As $s\alpha_1+s\alpha_2+\alpha_3 < 1$, this implies that $\theta = \Big(\frac{\alpha_1+\alpha_2}{1-\alpha_3}\Big) < \frac{1}{s}$, that is, $0 < s\theta < 1$. By induction, we have

$$d(x_{n+1},x_{n+2}) \precsim \theta\, d(x_n,x_{n+1}) \precsim \cdots \precsim \theta^{n+1} d(x_0,x_1). \tag{3.9}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned}
d(x_n, x_m) &\precsim s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\
&= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\
&\precsim sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\
&= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\
&\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\
&\quad + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \\
&\precsim [s\theta^n + s^2\theta^{n+1} + s^3\theta^{n+2} + \cdots + s^m\theta^{n+m-1}] d(x_1, x_0) \\
&= s\theta^n [1 + s\theta + s^2\theta^2 + s^3\theta^3 + \cdots + (s\theta)^{m-1}] d(x_1, x_0) \\
&\precsim \Big[\frac{s\theta^n}{1 - s\theta}\Big] d(x_1, x_0).
\end{aligned}$$

and so

$$|d(x_n, x_m)| \leq \Big[\frac{s\theta^n}{1 - s\theta}\Big] |d(x_1, x_0)| \to 0 \quad \text{as} \quad m, n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. It follows that $v = Tv$, otherwise $d(v, Tv) = z > 0$ and we would then have

$$\begin{aligned}
z &\precsim sd(v, x_{n+1}) + sd(x_{n+1}, Tv) \\
&= sd(v, x_{n+1}) + sd(Tv, Tx_n) \\
&\precsim sd(v, x_{n+1}) + \alpha_1\, d(v, x_n) + \alpha_2\, d(v, Tv) + \alpha_3\, d(x_n, Tx_n) \\
&= sd(v, x_{n+1}) + \alpha_1\, d(v, x_n) + \alpha_2\, d(v, Tv) + \alpha_3\, d(x_n, x_{n+1}).
\end{aligned}$$

This implies that

$$|z| \leq s|d(v, x_{n+1})| + \alpha_1\, |d(v, x_n)| + \alpha_2\, |z| + \alpha_3\, |d(x_n, x_{n+1})|.$$

Letting $n \to \infty$, it follows that

$$\begin{aligned}
|z| &\leq \alpha_2\, |z| \\
&\leq (s\alpha_1 + s\alpha_2 + \alpha_3)|z| \\
&< |z|,
\end{aligned}$$

which is a contradiction and so $|z| = 0$, that is, $v = Tv$.

To prove the uniqueness of fixed point of T, assume that v_1 is another fixed point of T, that is, $Tv_1 = v_1$ such that $v \neq v_1$. Then

$$\begin{aligned} d(v, v_1) &= d(Tv, Tv_1) \\ &\precsim \alpha_1\, d(v, v_1) + \alpha_2\, d(v, Tv) + \alpha_3\, d(v_1, Tv_1) \\ &= \alpha_1\, d(v, v_1) + \alpha_2\, d(v, v) + \alpha_3\, d(v_1, v_1) \\ &= \alpha_1\, d(v, v_1) \end{aligned}$$

so that $|d(v, v_1)| \leq \alpha_1 |d(v, v_1)| < |d(v, v_1)|$, since $0 < \alpha_1 < 1$, which is a contradiction and hence $d(v, v_1) = 0$. Thus $v = v_1$. This shows that T has a unique fixed point in X. This completes the proof. □

Corollary 3.6. *Let (X, d) be a complete complex valued b-metric space (CVbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T \colon X \to X$ satisfies (for fixed n):*

$$d(T^n x, T^n y) \precsim \alpha_1\, d(x, y) + \alpha_2\, d(x, T^n x) + \alpha_3\, d(y, T^n y)$$

for all $x, y \in X$, where α_1, α_2, $\alpha_3 \in [0, 1)$ are constants with $s\alpha_1 + s\alpha_2 + \alpha_3 < 1$. Then T has a unique fixed point in X.

Proof. By Theorem 3.5, there exists $w \in X$ such that $T^n w = w$. The rest of the proof follows from Corollary 3.4. This completes the proof. □

If we put $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 = \lambda$ in Theorem 3.5, then we have the following result as corollary.

Corollary 3.7. *Let (X, d) be a complete complex valued b-metric space (CVbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T \colon X \to X$ satisfies:*

$$d(Tx, Ty) \precsim \lambda\, [d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where $\lambda \in [0, 1)$ is a constant with $\lambda < \frac{1}{s+1}$. Then T has a unique fixed point in X.

Remark 3.8. Corollary 3.7 extends Kannan contraction [11] from complete metric space to that setting of complete complex valued b-metric space considered in this paper.

Finally, we give examples in support of Theorem 3.1 and 3.5.

Example 3.9. Let $X = \{0, \frac{1}{2}, 2\}$ and partial order $'\precsim'$ is defined as $x \precsim y$ iff $x \geq y$. Let the complex valued b-metric be given as

$$d(x, y) = |x - y|^2 (1 + i) \text{ for } x, y \in X.$$

Let $s = 2$ and $T\colon X \to X$ be defined as follows:

$$T(0) = 0, \quad T(\frac{1}{2}) = 0, \quad T(2) = \frac{1}{2}.$$

Case I. Take $x = \frac{1}{2}$, $y = 0$, $T(0) = 0$ and $T(\frac{1}{2}) = 0$ in Theorem 3.1, then we have

$$d(Tx, Ty) = 0 \leq \beta \max\Big\{\frac{1+i}{4}, 0, 0\Big\} = \beta.\frac{1+i}{4}.$$

This implies that $\beta \geq 0$. If we take $0 < \beta < \frac{1}{2}$, we get $0 < s\beta < 1$, then all the conditions of Theorem 3.1 are satisfied and of course 0 is the unique fixed point of T.

Case II. Take $x = 2$, $y = \frac{1}{2}$, $T(2) = \frac{1}{2}$ and $T(\frac{1}{2}) = 0$ in Theorem 3.1, then we have

$$d(Tx, Ty) = \frac{1+i}{4} \leq \beta \max\Big\{\frac{9(1+i)}{4}, \frac{81+117i}{500}, \frac{36+45i}{82}\Big\} = \beta.\frac{9(1+i)}{4}.$$

This implies that $\beta \geq \frac{1}{9}$. If we take $\frac{1}{9} \leq \beta < \frac{1}{2}$, we get $0 < s\beta < 1$, then all the conditions of Theorem 3.1 are satisfied and of course 0 is the unique fixed point of T.

Case III. Take $x = 2$, $y = 0$, $T(2) = \frac{1}{2}$ and $T(0) = 0$ in Theorem 3.1, then we have

$$d(Tx, Ty) = \frac{1+i}{4} \leq \beta \max\Big\{4(1+i), 0, 0\Big\} = \beta.4(1+i).$$

This implies that $\beta \geq \frac{1}{16}$. If we take $\frac{1}{16} \leq \beta < \frac{1}{2}$, we get $0 < s\beta < 1$, then all the conditions of Theorem 3.1 are satisfied and of course 0 is the unique fixed point of T.

Example 3.10. Let $X = \{0, \frac{1}{2}, 2\}$ and partial order $'\precsim'$ is defined as $x \precsim y$ iff $x \geq y$. Let the complex valued b-metric be given as

$$d(x, y) = |x - y|^2(1+i) \;\text{ for }\; x, y \in X.$$

Let $s = 2$ and $T\colon X \to X$ be defined as follows:

$$T(0) = 0, \quad T(\frac{1}{2}) = 0, \quad T(2) = \frac{1}{2}.$$

Case I. Take $x = \frac{1}{2}$, $y = 0$, $T(0) = 0$ and $T(\frac{1}{2}) = 0$ in Theorem 3.5, then we have

$$d(Tx, Ty) = 0 \leq \alpha_1.\frac{1+i}{4} + \alpha_2.\frac{1+i}{4} + \alpha_3.0.$$

This implies that $\alpha_1 + \alpha_2 \geq 0$. If we take $\alpha_1 = \alpha_2 = \frac{1}{7}$, and $0 \leq \alpha_3 \leq \frac{2}{7}$, then all the conditions of Theorem 3.5 are satisfied and of course 0 is the unique fixed point of T.

Case II. Take $x = 2$, $y = \frac{1}{2}$, $T(2) = \frac{1}{2}$ and $T(\frac{1}{2}) = 0$ in Theorem 3.5, then we have

$$d(Tx, Ty) = \frac{1+i}{4} \leq \alpha_1.\frac{9(1+i)}{4} + \alpha_2.\frac{9(1+i)}{4} + \alpha_3.\frac{1+i}{4}.$$

This implies that $9\alpha_1 + 9\alpha_2 + \alpha_3 \geq 1$. If we take $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{18}$, then all the conditions of Theorem 3.5 are satisfied and of course 0 is the unique fixed point of T.

Case III. Take $x = 2$, $y = 0$, $T(2) = \frac{1}{2}$ and $T(0) = 0$ in Theorem 3.5, then we have

$$d(Tx, Ty) = \frac{1+i}{4} \leq \alpha_1.4(1+i) + \alpha_2.\frac{9(1+i)}{4} + \alpha_3.0.$$

This implies that $16\alpha_1 + 9\alpha_2 \geq 1$. If we take $\alpha_1 = \alpha_2 = \frac{1}{24}$, and $0 \leq \alpha_3 \leq \frac{19}{24}$, then all the conditions of Theorem 3.5 are satisfied and of course 0 is the unique fixed point of T.

4. Conclusion

In this paper, we prove some fixed point theorems for contractive type conditions in the setting of complex-valued b-metric spaces and give some examples in support of our results. Our results extend and generalize several results from the current existing literature.

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