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GENERALIZED VECTOR B-VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, the concept of generalized vector B-variational inequality problems are introduced and are studied their existence theorems. The concept of pre b-η-invex function and generalized T-(η, θ, ρ)-B-invex are introduced and are used to prove the existence theorems of generalized vector B-variational inequality problems.

1. Introduction

In 1964, the variational inequality problem (VIP) was introduced by Stampacchia [13]. To develop the VIP in vector spaces, Gianessi ([10], 1980) has introduced the vector variational inequality problems and has studied its existence theorems in finite dimensional vector spaces $\mathbb{R}^n$. The theory of variational inequalities is applied to study various types of problems arises in physical sciences, engineering branches, economics, optimization etc. The variational inequality problems studied by Stampacchia [13] is defined as follows.

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Let K be nonempty closed and convex subset of a reflexive real Banach space X with dual X^* and $T : K \to X^*$ a nonlinear map. Let the value of $f \in X^*$ at $x \in X$ be denoted by $\langle f, x\rangle$. The *Variational Inequality Problem* **(*VIP*)** is to:

find $x_0 \in K$ such that for all $x \in K$,

$$\langle T(x_0), x - x_0\rangle \geq 0. \qquad (VIP)$$

Earlier convexity property of a function has played an important role to study various types of results in optimization theory. Many significant results have been derived under convexity assumption. In 1981, Hanson [11] used the concept of generalized convex function as invex function in the place of convex and concave as follows: the differentiable function f from $\mathbb{R}^n$ to $\mathbb{R}$ is invex if there exists a vector valued function $\eta(x, u) \in \mathbb{R}^n$ such that

$$f(x) - f(u) \geq \nabla' f(u)(\eta(x, u))$$

for all $x, u \in \mathbb{R}^n$, where $\nabla' f(u)$ stands for the transpose of the gradient of f at $u \in \mathbb{R}^n$. But later Craven [6] coined the name as invex (invariant convex) function if any function f is satisfying the above equation and studied many duality theorems for functional programs using the functions. In 1988, Weir and Mond [14] have introduced the concept of η-invex set, pre-invex functions and have studied the multiple objective optimization problems. Inspired by the work of Hanson [11], the researchers have defined generalized variational inequalities replacing $x - y$ by $\eta(x, y)$ and have shown various useful existence theorems.

1.1. **Definition of the Problems.** For simplicity, we recall the following terminologies given by Behera and Das [3]. Let (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \varnothing$. Let $y \in Y$. Then

(a) $y \notin -intP$ if and only if $y \geq_P 0$;
(b) $y \in intP$ if and only if $y >_P 0$;
(c) $y \notin intP$ if and only if $y \leq_P 0$;
(d) $y \in -intP$ if and only if $y <_P 0$;
(e) $y - z \notin -intP$ if and only if $y - z \geq_P 0$ $(i.e.,\ y \geq_P z)$;
(f) $y - z \notin intP$ if and only if $y - z \leq_P 0$ $(i.e.,\ y \leq_P z)$;
(g) $y - z \notin (-intP \bigcup intP)$ if and only if $y - z =_P 0$, $(i.e.,\ y =_P z)$.

Throughout this paper, the domain and functions are defined as follows. Let X be a topological vector space, K be any subset of X and (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \varnothing$. Let $L(X, Y)$ be the set of continuous linear functionals from X to Y and $\eta : K \times K \to X$ and $\theta : K \times K \to Y$ be the vector valued functions. Let $T : K \to L(X,Y)$ be any operator. Let $\rho \in \mathbb{R}$ be any real number.

We consider the following generalized vector variational inequalities.

(i) The *generalized primal vector variational inequality problems* is to: find $u \in K$ such that for all $x \in K$,

$$\langle T(u), \eta(x,u)\rangle + \rho\theta(x,u) \notin -intP. \qquad (GPVVIP)$$

(ii) The *generalized dual vector variational inequality problems* is to: find $u \in K$ such that for all $x \in K$,

$$\langle T(x), \eta(u,x)\rangle + \rho\theta(u,x) \notin -intP. \qquad (GDVVIP)$$

(iii) The *generalized primal vector B-variational inequality problems* is to: find $u \in K$ such that for all $x \in K$, $\bar{b}(u,x) > 0$ and

$$\bar{b}(u,x)\,[\langle T(u), \eta(x,u)\rangle + \rho\theta(x,u)] \notin -intP. \qquad (GPVVIP_b)$$

(iv) The *generalized dual vector B-variational inequality problems* is to: find $u \in K$ such that for all $x \in K$, $\bar{b}(x,u) < 0$ and

$$\bar{b}(x,u)\,[\langle T(x), \eta(u,x)\rangle + \rho\theta(u,x)] \notin intP. \qquad (GDVVIP_b)$$

Remark 1.1. (1) Let $F : K \to Y$. If $\rho = 1$, $b(x,u) = 1$ for all $x,u \in K$ and $\theta(x,u) = F(x) - F(u)$, then $(GPVVIP)$ coincides the F-GVIP studied by Behera and Das [3].
(2) Let $\xi : K \to L(X,Y)$. If $\rho = 1$, $b(x,u) = 1$ for all $x,u \in K$ and $\theta(x,u) = \langle \xi(u), \eta(x,u)\rangle$, then $(GPVVIP)$ coincides the generalized nonlinear variational inequality problems studied by Das and Kodamasingh [9].

2. η-semiinvex set, $(\bar{b},\eta)$-semipreinvex function and $(\bar{b},\eta)$-monotone function

In fact, preinvexity is a generalization of invexity for nondifferentiable function. In 1992, Yang and Chen [15] presented a wide class of generalized convex set and functions, called γ-semiconnected set and semipreinvex functions as follows.

Definition 2.1. ([15]) Let K be a nonempty subset of $\mathbb{R}^n$.

(1) K is to satisfy *γ-semiconnected property*, if for all $x,y \in K$ and $t \in [0,1]$, there exists a vector path function

$$\gamma : K \times K \times [0,1] \to \mathbb{R}^n$$

such that

$$y + t\gamma(x,y,t) \in K.$$

(2) Assume that the set K satisfies γ-*semiconnected* property. A real valued function $F : K \to \mathbb{R}$ is said to be *semipreinvex* with respect to γ if for all $x, y \in K$ and $t \in [0,1]$,

$$F(y + t\gamma(x,y,t)) \leq (1-t)F(y) + tF(x)$$

holds and

$$\lim_{t \downarrow 0} t\ \gamma(x,y,t) = 0.$$

Let

$$B = \left\{ b : X \times X \times [0,1] \to \mathbb{R} \mid \bar{b}(x,u) = \lim_{\lambda \to 0^+} b(x,u,\lambda),\ \left|\bar{b}(x,u)\right| < \infty \right\} \neq \varnothing.$$

For our need, we define γ-*semiinvex* set and $(b;\gamma)$-*semipreinvex* as follows.

Definition 2.2. A set $K \subset X$ is said to be

(a) γ-*semiaffine* set in X if for all $x, u \in K$, there exists a vector valued map $\gamma : K \times K \times \mathbb{R} \to X$ satisfying the condition $\lim_{t \to 0} t\ \gamma(x,u,t) = 0$ such that $u + t\ \gamma(x,u,t) \in K$ for all $t \in \mathbb{R}$.

(b) γ-*semiinvex* set in X if for all $x, u \in K$, there exists a vector valued map $\gamma : K \times K \times [0,1] \to X$ satisfying the condition $\lim_{t \to 0} t\ \gamma(x,u,t) = 0$ such that $u + t\ \gamma(x,u,t) \in K$ for all $t \in [0,1]$.

(c) η-limiting γ-*semiaffine* (or η-*semiaffine*) set in X if for all $x, u \in K$, there exists a vector valued map $\gamma : K \times K \times \mathbb{R} \to X$ satisfying the condition

$$\lim_{t \to 0} \gamma(x,u,t) = \eta(x,u) \quad \text{and} \quad \lim_{t \to 0} t\ \gamma(x,u,t) = 0$$

such that $u + t\ \gamma(x,u,t) \in K$ for all $t \in \mathbb{R}$.

(d) η-limiting γ-*semiinvex* (or η-*semiinvex*) set in X if for all $x, u \in K$, there exists a vector valued map $\gamma : K \times K \times [0,1] \to X$ satisfying the condition

$$\lim_{t \to 0} \gamma(x,u,t) = \eta(x,u) \quad \text{and} \quad \lim_{t \to 0} t\ \gamma(x,u,t) = 0$$

such that $u + t\ \gamma(x,u,t) \in K$ for all $t \in [0,1]$.

Definition 2.3. A map $F : K \to Y$ is said to be

(a) $(b;\gamma)$-*semipreaffine* on K if there exists a vector valued map $\gamma : K \times K \times \mathbb{R} \to X$ satisfying the condition $\lim_{t \to 0} t\,\gamma(x,u,t) = 0$ and a scalar valued map $b \in B$ with $\lim_{t \to 0} b(x,u,t) = \bar{b}(x,u)$ for all $x, u \in K$ such that for all $t \in \mathbb{R}$, we have

$$t\, b(x,u,t)F(u) + (1-t)\, b(u,x,t)F(x) - F(u + t\gamma(x,u,t)) =_P 0;$$

(b) $(b;\gamma)$-*semipreinvex* on K if there exists a vector valued map $\gamma : K \times K \times [0,1] \to X$ satisfying the condition $\lim_{t \to 0} t\,\gamma(x,u,t) = 0$ and a scalar valued map $b \in B$ with $\lim_{t \to 0} b(x,u,t) = \bar{b}(x,u)$ for all $x, u \in K$ such that for all $t \in [0,1]$, we have

$$t\, b(x,u,t)F(u) + (1-t)\, b(u,x,t)F(x) - F(u + t\gamma(x,u,t)) \notin -intP;$$

(c) $(\bar{b};\eta)$-limiting $(b;\gamma)$-*semipreaffine* (or $(\bar{b};\eta)$-*semipreaffine*) on K if there exists a vector valued map $\gamma : K \times K \times \mathbb{R} \to X$ satisfying the condition $\lim_{t \to 0} \gamma(x,u,t) = \eta(x,u)$ and $\lim_{t \to 0} t\,\gamma(x,u,t) = 0$ and a scalar valued map $b \in B$ with $\lim_{t \to 0} b(x,u,t) = \bar{b}(x,u)$ for all $x, u \in K$ such that for all $t \in \mathbb{R}$, we have

$$t\, b(x,u,t)F(u) + (1-t)\, b(u,x,t)F(x) - F(u + t\gamma(x,u,t)) =_P 0;$$

(d) $(\bar{b};\eta)$-limiting $(b;\gamma)$-*semipreinvex* (or $(\bar{b};\eta)$-*semipreinvex*) on K if there exists a vector valued map $\gamma : K \times K \times [0,1] \to X$ satisfying the condition $\lim_{t \to 0} \gamma(x,u,t) = \eta(x,u)$ and $\lim_{t \to 0} t\,\gamma(x,u,t) = 0$ and a scalar valued map $b \in B$ with $\lim_{t \to 0} b(x,u,t) = \bar{b}(x,u)$ for all $x, u \in K$ such that for all $t \in [0,1]$, we have

$$t\, b(x,u,t)F(u) + (1-t)\, b(u,x,t)F(x) - F(u + t\gamma(x,u,t)) \notin -intP.$$

Let there exists a map $\gamma : K \times K \times [0,1] \to X$ satisfying

$$\lim_{t \to 0} \gamma(x,u,t) = \eta(x,u)$$

with $\lim_{t \to 0} t\,\gamma(x,u,t) = 0$, and a scalar valued map $b \in B$ with

$$\lim_{t \to 0} b(x,u,t) = \bar{b}(x,u)$$

for all $x, u \in K$. The concept of $\bar{b}$-η-monotonicity of T is defined as follows.

Definition 2.4. The mapping T is $(\bar{b};\eta)$-*monotone* associated with (ρ,θ) on K if for all $x, u \in K$, we have

$$\bar{b}(u,x)\,[\langle T(u), \eta(x,u)\rangle + \rho\theta(x,u)] + \bar{b}(x,u)\,[\langle T(x), \eta(u,x)\rangle + \rho\theta(u,x)] \notin intP.$$

Definition 2.5. The mapping T is pseudo $\bar{b}$-η-*monotone* associated with (ρ,θ) on K if for all $x,u\in K$, we have

$$\bar{b}(u,x)\left[\langle T(u),\eta(x,u)\rangle+\rho\theta(x,u)\right]\notin -intP,$$

implying

$$\bar{b}(x,u)\left[\langle T(x),\eta(u,x)\rangle+\rho\theta(u,x)\right]\notin intP.$$

Definition 2.6. The mapping T is quasi $\bar{b}$-η-*monotone* associated with (ρ,θ) on K if for all $x,u\in K$, we have

$$\bar{b}(x,u)\left[\langle T(x),\eta(u,x)\rangle+\rho\theta(u,x)\right]\notin intP,$$

implying

$$\bar{b}(u,x)\left[\langle T(u),\eta(x,u)\rangle+\rho\theta(x,u)\right]\notin -intP.$$

3. Main Results

The concept of η-*hemicontinuous* at any point $x_0\in K\subset X$ is defined as follows.

Definition 3.1. The mapping $T:K\to L(X,Y)$ is η-*hemicontinuous* at x_0 if for any sequence $\{x_n\}$ converging to x_0 along a line, the sequence $\{T(x_n)\}$ weakly converges to $T(x_0)$, *i.e.*, the map $\lambda\mapsto T(y+\lambda v)$ of $[0,1]$ into Y is continuous for $y\in K$, $v\in M$, where

$$M=\{z:z=\eta(x,y)\in X, x\in K\}$$

when Y is endowed with its weak topology.

Theorem 3.2. *Let K be a η-semiinvex set in X. The problems $(GPVVIP)$ and $(GDVVIP)$ are equivalent under the following conditions:*

(a) $\langle T(x),\eta(x,x)\rangle+\rho\,\theta(x,x)=_P 0$ *for all* $x\in K$,
(b) T *is* $(\bar{b};\eta)$*-monotone associated with* (ρ,θ) *on* K,
(c) $\bar{b}$ *is antisymmetric on* K *where* $\bar{b}$ *satisfies* $\bar{b}(x,u)<0$,
(d) *the mapping* $\langle T(u),\eta(\text{-},u)\rangle:K\to Y$ *is* η*-semipreinvex on* K,
(e) *the mapping* $\theta(\text{-},u):K\to Y$ *is* η*-semipreinvex on* K,
(f) *for each* $u\in K$, *the mapping*

$$u\mapsto\langle T(u),\eta(x,u)\rangle+\rho\,\theta(x,u)$$

is η*-hemicontinuous for all* $x\in K$.

Proof. Let $u\in K$ solves $(GPVVIP)$, *i.e.*,

$$\langle T(u),\eta(x,u)\rangle+\rho\theta(x,u)\notin -intP$$

for all $x \in K$. Since T is $(\bar{b};\eta)$-monotone associated with (ρ,θ) on K, *i.e.*, there exists a map $\gamma : K \times K \times [0,1] \to X$ satisfying $\lim_{t\to 0} \gamma(x,u,t) = \eta(x,u)$ with $\lim_{t\to 0} t\,\gamma(x,u,t) = 0$ and a scalar valued map $b \in B$ with $\lim_{t\to 0} b(x,u,t) = \bar{b}(x,u)$ such that for all $x \in K$,

$$\bar{b}(u,x)\,[\langle T(u),\eta(x,u)\rangle + \rho\theta(x,u)] + \bar{b}(x,u)\,[\langle T(x),\eta(u,x)\rangle + \rho\theta(u,x)] \notin intP.$$

Since $\bar{b}$ is antisymmetric on K with $\bar{b}(x,u) < 0$, we get $\bar{b}(u,x) > 0$. Thus

$$\bar{b}(u,x)\,[\langle T(u),\eta(x,u)\rangle + \rho\theta(x,u)] \notin -intP$$

for all $x \in K$, implying

$$\bar{b}(x,u)\,[\langle T(x),\eta(u,x)\rangle + \rho\theta(u,x)] \notin intP$$

for all $x \in K$. Since $\bar{b}(x,u) < 0$, we get

$$\langle T(x),\eta(u,x)\rangle + \rho\theta(u,x) \notin -intP$$

for all $x \in K$. Hence u solves $(GDVVIP)$. Conversely, let $u \in K$ solves $(GDVVIP)$, *i.e.*,

$$\langle T(x),\eta(u,x)\rangle + \rho\theta(u,x) \notin -intP$$

for all $x \in K$. Since

$$\langle T(x),\eta(x,x)\rangle + \rho\,\theta(x,x) =_P 0$$

for all $x \in K$. Since K is η-semiinvex, $x_t = u + t\gamma(x,u,t) \in K$ for all $x,u \in K$ and $t \in [0,1]$. Replacing x by x_t in the above equation, we get

$$\begin{aligned} 0 \;&=_P\; \langle T(x_t),\eta(x_t,x_t)\rangle + \rho\,\theta(x_t,x_t) \\ &\leq_P\; t\langle T(x_t),\eta(u,x_t)\rangle + (1-t)\langle T(x_t),\eta(x,x_t)\rangle \\ &\quad + \rho\,[t\theta(u,x_t) + (1-t)\theta(x,x_t)] \end{aligned}$$

for all $x \in K$. Since T and θ are η-hemicontinuous, taking limit as $t \to 0$, we get

$$0 \leq_P \langle T(u),\eta(x,u)\rangle + \rho\,\theta(x,u)$$

for all $x \in K$, implying

$$\langle T(u),\eta(x,u)\rangle + \rho\,\theta(x,u) \notin -intP$$

for all $x \in K$. This showing u solves $(GPVVIP)$. This completes the proof of the theorem. □

Theorem 3.3. *Let K be a η-semiinvex set in X. The problems $(GPVVIP_b)$ and $(GDVVIP_b)$ are equivalent under the following conditions:*

(a) $\bar{b}(x,x)\,[\langle T(x),\eta(x,x)\rangle + \rho\,\theta(x,x)] =_P 0$ *for all* $x \in K$*,*

(b) *T is $(\bar{b};\eta)$-monotone associated with (ρ,θ) on K,*
(c) *the mapping $\bar{b}(\text{-},u)\langle T(u),\eta(\text{-},u)\rangle : K \to Y$ is $(\bar{b};\eta)$-semipreinvex on K,*
(d) *the mapping $\bar{b}(\text{-},u)\theta(\text{-},u) : K \to Y$ is $(\bar{b};\eta)$-semipreinvex on K,*
(e) *for each $u \in K$, the mapping*

$$u \mapsto \bar{b}(x,u)\,[\langle T(u),\eta(x,u)\rangle + \rho\,\theta(x,u)]$$

is η-hemicontinuous for all $x \in K$.

Proof. Let $u \in K$ solves $(GPVVIP_b)$, *i.e.*,

$$\bar{b}(u,x)\,[\langle T(u),\eta(x,u)\rangle + \rho\theta(x,u)] \notin -intP$$

for all $x \in K$. Since T is $(\bar{b};\eta)$-monotone associated with (ρ,θ) on K, *i.e.*, there exists a map $\gamma : K \times K \times [0,1] \to X$ satisfying $\lim_{t\to 0}\gamma(x,u,t) = \eta(x,u)$ with $\lim_{t\to 0} t\,\gamma(x,u,t) = 0$ and a scalar valued map $b \in B$ with $\lim_{t\to 0} b(x,u,t) = \bar{b}(x,u)$ such that for all $x \in K$,

$$\bar{b}(u,x)\,[\langle T(u),\eta(x,u)\rangle + \rho\theta(x,u)] + \bar{b}(x,u)\,[\langle T(x),\eta(u,x)\rangle + \rho\theta(u,x)] \notin intP.$$

Since

$$\bar{b}(u,x)\,[\langle T(u),\eta(x,u)\rangle + \rho\theta(x,u)] \notin -intP$$

for all $x \in K$, we get

$$\bar{b}(x,u)\,[\langle T(x),\eta(u,x)\rangle + \rho\theta(u,x)] \notin intP$$

for all $x \in K$. Hence u solves $(GDVVIP_b)$. Conversely, let $u \in K$ solves $(GDVVIP_b)$, *i.e.*,

$$\bar{b}(x,u)\,[\langle T(x),\eta(u,x)\rangle + \rho\theta(u,x)] \notin intP$$

for all $x \in K$. Since

$$\bar{b}(x,x)\,[\langle T(x),\eta(x,x)\rangle + \rho\,\theta(x,x)] =_P 0$$

for all $x \in K$. Since K is η-semiinvex, $x_t = u + t\gamma(x,u,t) \in K$ for all $x,u \in K$ and $t \in [0,1]$. Replacing x by x_t in the above equation, we get

$$\begin{aligned} 0 \;&=_P\; \bar{b}(x_t,x_t)\,[\langle T(x_t),\eta(x_t,x_t)\rangle + \rho\,\theta(x_t,x_t)] \\ &\le_P\; tb(x,u,t)\langle T(x_t),\eta(u,x_t)\rangle + (1-t)b(u,x,t)\langle T(x_t),\eta(x,x_t)\rangle \\ &\quad + \rho\,[tb(x,u,t)\theta(u,x_t) + (1-t)b(u,x,t)\theta(x,x_t)] \end{aligned}$$

for all $x \in K$. Since T and θ are η-hemicontinuous, taking limit as $t \to 0$, we get

$$0 \le_P \bar{b}(u,x)\langle T(u),\eta(x,u)\rangle + \rho\,\bar{b}(u,x)\theta(x,u)$$

for all $x \in K$, implying

$$\bar{b}(u,x)\,[\langle T(u),\eta(x,u)\rangle + \rho\,\theta(x,u)] \notin -intP$$

for all $x \in K$. This showing u solves $(GPVVIP_b)$. This completes the proof of the theorem. □

4. T-(η, θ, ρ)-B-invex function and its associated generalized variational inequalities

Kaul and Kaur [12] called these functions η-convex and defined η-pseudoconvex and η-quasiconvex functions. As an extension, the concept of ρ-(η, θ)-invexity was introduced by Zalmai [16] which is generalization of invexity.

The class of convex functions have also been further extended to the class of B-invex functions by Bector *et al.* [1, 2]. A class of pseudo B-invex and quasi B-invex functions are studied by Bector *et al.* [2], which are generalization of pseudoinvex and quasiinvex functions respectively. Bector *et al.* [2] have introduced the sufficient optimality conditions and duality results for a nonlinear programming problem using B-invex functions. Behera, Nahak and Nanda [5] introduced the concept of generalized (ρ, θ)-η-B-bexity and generalized (ρ, θ)-η-B-preivexity to study the optimality of the problems.

Behera and Das [3] have defined the T-η-invex functions to study the variational inequality problems arises in ordered topological vector spaces. Later the extensions of T-η-invex functions are studied by Behera and Das [4], Das and Sahu [8], Das and Behera [7].

In this section, we define the concept of T-(η, θ, ρ)-B-semiinvex function and study the existence theorems of generalized variational inequalities associated with it. Let $F : K \to Y$ be any map. Let there exists a map $\gamma : K \times K \times [0, 1] \to X$ satisfying $\lim_{t \to 0} \gamma(x, u, t) = \eta(x, u)$ with $\lim_{t \to 0} t\, \gamma(x, u, t) = 0$ and a scalar valued map $b \in B$ with $\lim_{t \to 0} b(x, u, t) = \bar{b}(x, u)$ for all $x, u \in K$. The concept of *T-η-θ-invex* function *relaxed with* (ρ, b) of F is defined as follows.

Definition 4.1. Let $\theta : K \times K \to Y \backslash (-int\, P)$ be any map and $\rho \in \mathbb{R}$. The mapping $F : K \to Y$ is said to be *T-η-θ-invex relaxed with* $(\rho, \bar{b})$ (in short; T-(η, θ, ρ)-B-invex) at $u \in K$ if for all $x \in K$, we have

$$\bar{b}(x, u)\,[F(x) - F(u)] - \langle T(u), \eta(x, u) \rangle - \rho\theta(x, u) \notin -intP.$$

(1) F is *weak T-η-θ-invex relaxed with* $(\rho, \bar{b})$ if $\rho < 0$,
(2) F is *strong T-η-θ-invex relaxed with* $(\rho, \bar{b})$ if $\rho > 0$.

Definition 4.2. Let $\theta : K \times K \to Y \backslash (-int\, P)$ be any map and $\rho \in \mathbb{R}$. The mapping $F : K \to Y$ is said to be *T-η-θ-invex relaxed with* $(\rho, \bar{b})$ (in short; T-(η, θ, ρ)-$\bar{b}$-invex) on K if for all $x, u \in K$, we have

$$\bar{b}(x, u)\,[F(x) - F(u)] - \langle T(u), \eta(x, u) \rangle - \rho\theta(x, u) \notin -intP.$$

Remark 4.3. If $\bar{b} = 1$ and $\rho \geq 0$, then Definition 4.2 coincides with the definition of T-η-invexity of F introduced and studied by [3].

Remark 4.4. If $\rho \geq 0$, then Definition 4.2 coincides with the definition of T-η-invexity of F of order $\lambda > 0$ where $\lambda = 1/\bar{b}$ introduced and studied by [8].

Proposition 4.5. *Let $b \in B$ with $\bar{b} > 0$. Let F be T-(η, θ, ρ)-$\bar{b}$-invex on K, then T is $(\bar{b}; \eta)$-monotone associated with (ρ, θ) on K.*

Proof. F is T-(η, θ, ρ)-$\bar{b}$-invex on K, *i.e.*, there exists a map $\gamma : K \times K \times [0, 1] \to X$ satisfying $\lim_{t\to 0} \gamma(x, u, t) = \eta(x, u)$ with $\lim_{t\to 0} t\, \gamma(x, u, t) = 0$ and a scalar valued map $b \in B$ with $\lim_{t\to 0} b(x, u, t) = \bar{b}(x, u)$ for all $x, u \in K$, and

$$\bar{b}(x, u)\, [F(x) - F(u)] - \langle T(u), \eta(x, u)\rangle - \rho\theta(x, u) \notin -intP,$$

i.e.,

$$F(x) - F(u) - \frac{\langle T(u), \eta(x, u)\rangle + \rho\theta(x, u)}{\bar{b}(x, u)} \notin -intP.$$

Interchanging x and u in the above equation, we get

$$F(u) - F(x) - \frac{\langle T(x), \eta(u, x)\rangle + \rho\theta(u, x)}{\bar{b}(u, x)} \notin -intP.$$

Adding the above two equations, we obtain

$$\frac{\langle T(u), \eta(x, u)\rangle + \rho\theta(x, u)}{\bar{b}(x, u)} + \frac{\langle T(x), \eta(u, x)\rangle + \rho\theta(u, x)}{\bar{b}(u, x)} \notin intP$$

for all $x, u \in K$. Since $\bar{b}(x, u)\bar{b}(u, x) > 0$, we have

$$\bar{b}(u, x)\, [\langle T(u), \eta(x, u)\rangle + \rho\theta(x, u)] + \bar{b}(x, u)\, [\langle T(x), \eta(u, x)\rangle + \rho\theta(u, x)] \notin intP$$

for all $x, u \in K$. Hence T is $(\bar{b}; \eta)$-monotone associated with (ρ, θ) on K. This completes the proof. □

Theorem 4.6. *Let K be a η-semiinvex set in X. The problems $(GPVVIP_b)$ and $(GDVVIP_b)$ are equivalent under the following conditions:*

(a) $\bar{b}(x, x)\, [\langle T(x), \eta(x, x)\rangle + \rho\, \theta(x, x)] =_P 0$ *for all* $x \in K$,
(b) *F is T-(η, θ, ρ)-$\bar{b}$-invex on K,*
(c) *the mapping $\bar{b}(\text{-}, u)\langle T(u), \eta(\text{-}, u)\rangle : K \to Y$ is $(\bar{b}; \eta)$-semipreinvex on K,*
(d) *the mapping $\bar{b}(\text{-}, u)\theta(\text{-}, u) : K \to Y$ is $(\bar{b}; \eta)$-semipreinvex on K,*

(e) *for each $u \in K$, the mapping*

$$u \mapsto \bar{b}(x,u)\,[\langle T(u), \eta(x,u)\rangle + \rho\,\theta(x,u)]$$

is η-hemicontinuous for all $x \in K$.

Proof. By Proposition 4.5, F is T-(η,θ,ρ)-$\bar{b}$-invex on K, implying T is $(\bar{b};\eta)$-monotone associated with (ρ,θ) on K which is a condition in Theorem 3.3. Since all the conditions of Theorem 3.3 are satisfied, the problems $(GPVVIP_b)$ and $(GDVVIP_b)$ are equivalent. This completes the proof of the theorem. □

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