Nonlinear Functional Analysis and Applications
Vol. 22, No. 2 (2017), pp. 335-354
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

ON THE SOLUTION OF STOCHASTIC FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. Using the Picard-Lindelöf successive approximation scheme we prove the existence and uniqueness of solution for the stochastic fractional integrodifferential equations. Further the numerical scheme for approximate solutions of stochastic fractional integrodifferential equations is obtained by using Galerkin method.

1. Introduction

Fractional differential equations have formed a classical research field of differential equations [13, 19, 22, 23]. These equations have important applications in fields such as rheology, viscoelasticity, electro chemistry and electromagnetism. Fractional differential equations and fractional integrodifferential equations have been receiving increasing attention due to their important applications in science and engineering.

In many problems, there are real phenomena depending on the effect of white noise random forces. Stochastic differential equations were first initiated

[0]Received August 4, 2016. Revised October 26, 2016.
[0]2010 Mathematics Subject Classification: 34A08, 60H10.
[0]Keywords: Stochastic fractional differential equations, Picard-Lindelöf successive approximation, Galerkin approximation.

and developed by Itô in 1942 [9]. Stochastic differential equations [5, 7, 8] now find applications in many disciplines including economics and finance, environmetrics, physics, population dynamics, biology and medicine [16, 18, 24]. An important application of stochastic differential equations occurs in the modeling of problems associated with water catchment and the percolation of fluid through porous structures. A natural extension of deterministic ordinary differential equation model is the stochastic differential equation model, where relevant parameters are randomized or modeled as random processes of some suitable form or simply by adding a noise term to the driving equations of the system. Stochastic fractional differential equations are used to model dynamical systems affected by random noises.

These equations have immense physical applications in many fields such as turbulence, heterogeneous flows and materials, viscoelasticity and electromagnetic theory [12]. Complex dynamic processes in sciences and engineering operating under internal structural and external environmental perturbations can be modelled by stochastic fractional differential equations by introducing the concept of dynamics processes operating under a set of linearly independent time-scales.

2. Preliminaries

In this section, we present a few well-known concepts and results in the fields of fractional and stochastic differential equations.

Definition 2.1. (Riemann-Liouville fractional integral) The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ t > 0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. (Riemann-Liouville fractional derivative) The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as

$$\begin{aligned} D_{0+}^{\alpha} f(t) &= \left(\frac{d}{dt}\right)^n I_{0+}^{n-\alpha} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \end{aligned} \tag{2.2}$$

where the function $f(t)$ has absolutely continuous derivatives upto order $(n-1)$.

Definition 2.3. (Caputo fractional derivative) The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as

$$^{C}D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \tag{2.3}$$

where the function $f(t)$ has absolutely continuous derivatives upto order $(n-1)$.

Definition 2.4. ([21], Multi-time scale Integral) For $p \in \mathbb{N}$, $p > 1$, let $\{T_1, T_2, \cdots, T_p\}$ be a set of linearly independent time scales. Let $f : [a, b) \times \mathbb{R}^{p-1} \to \mathbb{R}^n$ be a continuous function defined by $f(t) = f(T_1(t), T_2(t), \cdots, T_p(t))$. The multi-time scale integral of the composite function f over an interval $[t_0, t] \subseteq (a, b)$ is defined as a sum of p integrals with respect to the time-scales $T_1, T_2, \cdots, T_p$. We denote it by If,

$$(If)(t) = \int_{t_0}^t f(s)ds = \sum_{j=1}^{p} (I_j f)(t),$$

where the sense of the integral

$$(I_j f)(t) = \int_{t_0}^t f(s)dT_j(s)$$

depends on the time scale T_j, for each $j = 1, 2, \cdots, p$.

Example 2.5. Let $J = [0, T]$. For $p = 3$, consider the linearly independent set consisting of time scales $T_1(t) := t$ signifying the ideal and controlled environmental condition, $T_2(t) := W(t)$, where W is the standard Wiener process and $T_3(t) := t^{\alpha}$, $0 < \alpha < 1$ indicates the time varying delay or lagged process. In this case, $f(t) = f(T_1(t), T_2(t), T_3(t))$ and

$$(If)(t) = (I_1 f)(t) + (I_2 f)(t) + (I_3 f)(t),$$

where the integrals

$$(I_1 f)(t) = \int_{t_0}^t f(s)ds, \quad (I_2 f)(t) = \int_{t_0}^t f(s)dW(s),$$
$$(I_3 f)(t) = \int_{t_0}^t \frac{(t-s)^{\alpha-1} f(s)}{\Gamma(\alpha)} ds$$

are Cauchy-Riemann/Lebesgue, Itô-Doob and Riemann-Liouville type respectively.

Under the set of time scales in Example 2.5, we have the following stochastic fractional differential equation

$$dx(t) = b(t, x(t))dt + \sigma_1(t, x(t))dW(t) \\ + \sigma_2(t, x(t))(dt)^\alpha, \quad t \in J = [0, T], \tag{2.4}$$

with the initial condition $x(0) = x_0$, where $\alpha \in (1/2,\ 1)$, $b,\ \sigma_2 \in C(J \times \mathbb{R}^n,\ \mathbb{R}^n)$, $\sigma_1 \in C(J \times \mathbb{R}^n,\ \mathbb{R}^{nm})$ and $W = \{W(t),\ t \geq 0\}$ is an m-dimensional Brownian motion on a complete probability space $\Omega \equiv (\Omega,\ \mathcal{F},\ P)$.

3. Existence Theorem

The problems of existence and uniqueness of solutions of initial value problem provide the basis for the model validation and further undertaking a study of the corresponding dynamic processes. Øksendal [20], Kloeden and Platen [14] and Mao [17] have established the existence and uniqueness results for stochastic differential equations. The results are obtained by using the successive approximation scheme. The same technique was used by Arnold [2] and Evans [6] to obtain the existence and uniqueness of solution for the stochastic differential equations. The existence and uniqueness of stochastic fractional differential equation was studied by Pedjeu and Ladde [21]. Kamrani [12] discussed the numerical solution of stochastic fractional differential equation. Here the classical Picard-Lindelöf method of successive approximation scheme [25, 26] is used to obtain the existence and uniqueness of solution for the given stochastic fractional integrodifferential equation.

Consider the stochastic fractional integrodifferential equation of the form

$$dx(t) = b\left(t, x(t), \int_0^t f_1(t, s, x(s))ds\right)dt + \sigma_1\left(t, x(t), \int_0^t f_2(t, s, x(s))ds\right)dW(t) \\ + \sigma_2\left(t, x(t), \int_0^t f_3(t, s, x(s))ds\right)(dt)^\alpha, \quad t \in J = [0, T],$$

with the initial condition $x(0) = x_0$, where $\alpha \in (1/2,\ 1),\ b,\ \sigma_2 \in C(J \times \mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^n)$, $\sigma_1 \in C(J \times \mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^{nm})$, $f_1,\ f_2,\ f_3 \in C(J \times J \times \mathbb{R}^n,\ \mathbb{R}^n)$ and $W = \{W(t),\ t \geq 0\}$ is an m-dimensional Brownian motion on a complete probability space $\Omega \equiv (\Omega,\ \mathcal{F},\ P)$. We can rewrite the above equation in its equivalent integral form as follows [10, 11]:

$$x(t) = x_0 + \int_0^t b\left(s, x(s), \int_0^s f_1(s, \tau, x(\tau))d\tau\right)ds \\ + \int_0^t \sigma_1\left(s, x(s), \int_0^s f_2(s, \tau, x(\tau))d\tau\right)dW(s) \\ + \alpha\int_0^t (t-s)^{\alpha-1}\sigma_2\left(s, x(s), \int_0^s f_3(s, \tau, x(\tau))d\tau\right)ds.$$

Theorem 3.1. (Existence and Uniqueness) *Assume that* $(t, x) \in J \times \mathbb{R}^n$, $\alpha \in (1/2,\ 1)$, $b,\ \sigma_2 \in C(J \times \mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^n), \sigma_1 \in C(J \times \mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^{nm}),\ f_1,\ f_2,\ f_3 \in C(J \times J \times \mathbb{R}^n,\ \mathbb{R}^n)$ *and* $W = \{W(t),\ t \geq 0\}$ *is an* m*-dimensional Brownian motion on a complete probability space* $\Omega \equiv (\Omega,\ \mathcal{F},\ P)$*. Suppose the following inequalities hold:*

(i) *Linear growth condition : For* $y_j = \int_0^t f_j(t, s, x(s))ds,\ j = 1,\ 2,\ 3,$

$$
\begin{aligned}
&|b(t, x, y_1)|^2 + |\sigma_1(t, x, y_2)|^2 + |\sigma_2(t, x, y_3)|^2 \\
&\leq K_1^2(1 + |x|^2 + |y_1|^2 + |y_2|^2 + |y_3|^2), \qquad (3.1) \\
&|y_j|^2 \leq K_2^2(1 + |x|^2), \quad j = 1,\ 2,\ 3, \qquad (3.2)
\end{aligned}
$$

for some constants $K_1, K_2 > 0$.

(ii) *The Lipschitz condition : For* $y_i = \int_0^t f_i(t, s, x(s))ds,\ i = 1,\ 2,\ 3$ *and* $\tilde{y_i} = \int_0^t f_i(t, s, \tilde{x}(s))ds,\ \ i = 1,\ 2,\ 3,$

$$
\begin{aligned}
&|b(t, x, y_1) - b(t, \tilde{x}, \tilde{y_1})|^2 + |\sigma_1(t, x, y_2) - \sigma_1(t, \tilde{x}, \tilde{y_2})|^2 \\
&+ |\sigma_2(t, x, y_3) - \sigma_2(t, \tilde{x}, \tilde{y_3})|^2 \\
&\leq L_1^2(|x_1 - x_2|^2 + |y_1 - \tilde{y_1}|^2 + |y_2 - \tilde{y_2}|^2 + |y_3 - \tilde{y_3}|^2), \qquad (3.3) \\
&|y_i - \tilde{y_i}|^2 \leq L_2^2(|x_1 - x_2|^2),\ i = 1,\ 2,\ 3, \qquad (3.4)
\end{aligned}
$$

for some constants $L_1,\ L_2 > 0$*. Let* x_0 *be a random variable defined on* $(\Omega, \mathcal{F}, \mathcal{P})$ *and independent of the* σ*-algebra* $\mathcal{F}_s^t \subset \mathcal{F}$ *generated by* $\{W(s), t \geq s \geq 0\}$ *and such that* $E|x_0|^2 < \infty$.

Then the initial value problem

$$
\left.
\begin{aligned}
dx(t) &= b\left(t, x(t), \int_0^t f_1(t, s, x(s))ds\right) dt \\
&\quad + \sigma_1\left(t, x(t), \int_0^t f_2(t, s, x(s))ds\right) dW(t) \\
&\quad + \sigma_2\left(t, x(t), \int_0^t f_3(t, s, x(s))ds\right) (dt)^\alpha, \\
x(0) &= x_0,
\end{aligned}
\right\} \qquad (3.5)
$$

has a unique solution which is t*-continuous with the property that* $x(t, \omega)$ *is adapted to the filtration* $\mathcal{F}_t^{x_0}$ *generated by* x_0 *and* $\{W(s)(.), s \leq t\}$ *and*

$$
\sup_{0 \leq t \leq T} E[|x(t)|^2] < \infty. \qquad (3.6)
$$

Proof. ***Existence***: First we imitate the classical method of successive approximations to establish the existence of solution of the initial value problem (3.5).

Let $x^{(0)}(t) = x_0$ and $x^{(k)}(t) = x^{(k)}(t,\omega)$ inductively as follows:

$$x^{(k+1)}(t) = x_0 + \int_0^t b\left(s, x^{(k)}(s), \int_0^s f_1(s,\tau,x^{(k)}(\tau))d\tau\right) ds + \int_0^t \sigma_1\left(s, x^{(k)}(s), \int_0^s f_2(s,\tau,x^{(k)}(\tau))d\tau\right) dW(s) + \alpha\int_0^t (t-s)^{\alpha-1}\sigma_2\left(s, x^{(k)}(s), \int_0^s f_3(s,\tau,x^{(k)}(\tau))d\tau\right) ds, \tag{3.7}$$

for $k = 0,\ 1,\ 2, \cdots$. If, for fixed $n \geq 0$, the approximation $x^{(k)}(t)$ is $\mathcal{F}_t$-measurable and continuous on J, then it follows, from (3.1)-(3.4), that the integrals in (3.7) are meaningful and that the resulting process $x^{(k+1)}(t)$ is $\mathcal{F}_t$-measurable and continuous on J. As $x^{(0)}(t)$ is obviously $\mathcal{F}_t$-measurable and continuous on J, it follows by induction that so too is each $x^{(k)}(t)$ for $k = 1,\ 2,\ \cdots$. Since x_0 is $\mathcal{F}_t$-measurable with $E(|x_0|^2) < \infty$, it is clear that

$$\sup_{0\le t\le T} E(|x^{(0)}(t)|^2) < \infty.$$

Applying the algebraic inequality $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$, the Cauchy-Schwarz inequality, the Itô isometry and the linear growth condition (3.1), (3.2) and (3.7), we obtain

$$E(|x^{(k+1)}(t)|^2) \le 4E[|x_0|^2] + 4tE\left[\int_0^t \left|b(s, x^{(k)}(s), \int_0^s f_1(s,\tau,x^{(k)}(\tau))d\tau)\right|^2 ds\right] + 4E\left[\int_0^t \left|\sigma_1(s, x^{(k)}(s), \int_0^s f_2(s,\tau,x^{(k)}(\tau))d\tau)\right|^2 ds\right] + 4\alpha^2\frac{t^{2\alpha-1}}{2\alpha-1}E\left[\int_0^t \left|\sigma_2(s, x^{(k)}(s), \int_0^s f_3(s,\tau,x^{(k)}(\tau))d\tau)\right|^2 ds\right].$$

Therefore

$$E(|x^{(k+1)}(t)|^2) \le 4E[|x_0|^2] + 4K_1^2(1+K_2^2)\left(1+t+\alpha^2\frac{t^{2\alpha-1}}{2\alpha-1}\right)\int_0^t E\left[|x^{(k)}(s))|^2\right]ds,$$

for $k = 0,\ 1,\ 2, \cdots$. By induction, we have $\sup_{0\le t\le T} E(|x^{(k)}(t)|^2) \le C_0 < \infty$, for $k = 1,\ 2,\ 3, \cdots$. Applying the Schwarz inequality and Itô isometry, we obtain

$$E[|x^{(k+1)}(t) - x^{(k)}(t)|^2] \le 3t\int_0^t E\left[\left|b\left(s, x^{(k)}(s), \int_0^s f_1(s,\tau,x^{(k)}(\tau))d\tau\right)\right.\right.$$

$$
\begin{aligned}
&- b\left(s, x^{(k-1)}(s), \int_0^s f_1(s,\tau,x^{(k-1)}(\tau))d\tau\right)\Bigg|^2\Bigg] ds \\
&+ 3\int_0^t E\left[\left|\sigma_1\left(s, x^{(k)}(s), \int_0^s f_2(s,\tau,x^{(k)}(\tau))d\tau\right)\right.\right. \\
&- \sigma_1\left(s, x^{(k-1)}(s), \int_0^s f_2(s,\tau,x^{(k-1)}(\tau))d\tau\right)\Bigg|^2\Bigg] ds \\
&+ 3\frac{\alpha^2 t^{2\alpha-1}}{2\alpha-1}\int_0^t E\left[\left|\sigma_2\left(s, x^{(k)}(s), \int_0^s f_3(s,\tau,x^{(k)}(\tau))d\tau\right)\right.\right. \\
&- \sigma_2\left(s, x^{(k-1)}(s), \int_0^s f_3(s,\tau,x^{(k-1)}(\tau))d\tau\right)\Bigg|^2\Bigg] ds.
\end{aligned}
$$

By using the Lipschitz continuity assumptions (3.3), (3.4) and $\alpha > 1/2$, we have

$$
\begin{aligned}
&E[|x^{(k+1)}(t) - x^{(k)}(t)|^2] \\
&\leq 3L_1^2(1+L_2^2)\left[1 + t + \frac{\alpha^2 t^{2\alpha-1}}{2\alpha-1}\right]\int_0^t E\left[|x^{(k)}(s) - x^{(k-1)}(s)|^2\right] ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&E[|x^{(k+1)}(t) - x^{(k)}(t)|^2] \\
&\leq 3C_1^2\left[1 + T + \frac{\alpha^2 T^{2\alpha-1}}{2\alpha-1}\right]\int_0^t E\left[|x^{(k)}(s) - x^{(k-1)}(s)|^2\right] ds,
\end{aligned}
\tag{3.8}
$$

where $C_1^2 = L_1^2(1+L_2^2)$. From (3.7), by applying again the Schwarz inequality, the Itô isometry together with the growth conditions (3.1) and (3.2) for $k = 1$, we have

$$
\begin{aligned}
E[|x^{(1)}(t) - x^{(0)}(t)|^2] &\leq 3t\int_0^t E\left[\left|b(s, x_0, \int_0^s f_1(s,\tau,x_0)d\tau)\right|^2\right] ds \\
&+ 3\int_0^t E\left[\left|\sigma_1(s, x_0, \int_0^s f_2(s,\tau,x_0)d\tau)\right|^2\right] ds \\
&+ 3\frac{\alpha^2 T^{2\alpha-1}}{2\alpha-1}\int_0^t E\left[\left|\sigma_2(s, x_0, \int_0^s f_1(s,\tau,x_0)d\tau)\right|^2\right] ds \\
&\leq 3C_1^2\left(1 + T + \frac{\alpha^2 T^{2\alpha-1}}{2\alpha-1}\right)3K_1^2(1+K_2^2)\int_0^t E(1+|x_0|^2)ds \\
&\leq 3^2 C_2^2 C_1^2\left(1 + T + \frac{\alpha^2 T^{2\alpha-1}}{2\alpha-1}\right)(t)(1+E|x_0|^2),
\end{aligned}
\tag{3.9}
$$

with $C_2^2 = K_1^2(1+K_2^2)$. Now, for $k=1$, replacing $E[|x^{(1)}(t)-x^{(0)}(t)|^2]$ in the inequality (3.8) with the value on the right hand side of inequality (3.9) and integrating, we obtain

$$\begin{aligned}
&E[|x^{(2)}(t)-x^{(1)}(t)|^2] \\
&\leq 3C_1^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t E[|x^{(1)}(s)-x^{(0)}(s)|^2]ds \\
&\leq C_2^2(1+E|x_0|^2)\left[3^2C_1^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)\right]^2\int_0^t sds \\
&\leq C_2^2(1+E|x_0|^2)\left[3^2C_1^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)\right]^2\times\frac{t^2}{2!}. \qquad (3.10)
\end{aligned}$$

For $k=2$, proceeding as before, we have

$$E[|x^{(3)}(t)-x^{(2)}(t)|^2] \leq C_2^2(1+E|x_0|^2)\left[3^2C_1^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)\right]^3\times\frac{t^3}{3!}. \qquad (3.11)$$

Thus, by the principle of mathematical induction, we have

$$E[|x^{(k+1)}(t)-x^{(k)}(t)|^2] \leq \frac{BM^{k+1}t^{(k+1)}}{(k+1)!},\ k=0,1,2,\cdots,\ 0\leq t\leq T, \qquad (3.12)$$

where $B = C_2^2(1+E|x_0|^2)$ and $M = 3^2C_1^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)$ is a constant depending only on α, T, C_1^2 and $E|x_0|^2$. Thus

$$\sup_{0\leq t\leq T} E[|x^{(k+1)}(t)-x^{(k)}(t)|^2] \leq \frac{BM^{k+1}t^{(k+1)}}{(k+1)!},\ k=0,1,2,\cdots. \qquad (3.13)$$

This implies the mean-square convergence of the successive approximations uniformly on J. That is,

$$\begin{aligned}
||x^{(m)}(t)-x^{(n)}(t)||^2_{L^2(P)} &\leq \sum_{k=n}^{m-1}||x^{(k+1)}-x^{(k)}||^2_{L^2(P)} \\
&\leq \sum_{k=n}^{m-1}\int_0^T\frac{BM^{k+1}t^{(k+1)}}{(k+1)!}dt \\
&= \sum_{k=n}^{m-1}\frac{BM^{k+1}T^{(k+2)}}{(k+2)!}\to 0, \quad \text{as} \quad m,n\to\infty.
\end{aligned}$$

Then, by applying the Chebyshev's inequality and summing up the resultant inequalities, we have

$$\sum_{k=1}^{\infty}P\left[\sup_{0\leq t\leq T}(|x^{(k+1)}(t)-x^{(k)}(t)|^2) > \frac{1}{k^2}\right] \leq \sum_{k=1}^{\infty}\frac{BM^{k+1}T^{(k+2)}k^4}{(k+2)!}, \qquad (3.14)$$

where the series on the right side converges by ratio test. Hence the series on the left side also converges, so by the Borel-Cantelli lemma, we conclude that

$\sup_{0\le t\le T}\left(|x^{(k+1)}(t)-x^{(k)}(t)|^2\right)$ converges to 0, almost surely, that is, the successive approximations $x^{(k)}(t)$ converge, almost surely, uniformly on J to a limit $x(t)$ defined by

$$\lim_{n\to\infty}\left(x^{(0)}(t)+\sum_{k=1}^{n}(x^{(k)}(t)-x^{(k-1)}(t))\right)=\lim_{n\to\infty}x^{(n)}(t)=x(t). \tag{3.15}$$

Since $x(t)$ is the limit of nonanticipating functions and the uniform limit of a sequence of continuous functions, it itself is nonanticipating and continuous. From (3.7), we have

$$\begin{aligned} x(t) = x_0 &+ \int_0^t b\left(s,x,\int_0^s f_1(s,\tau,x(\tau))d\tau\right)ds \\ &+ \int_0^t \sigma_1\left(s,x(s),\int_0^s f_2(s,\tau,x(\tau))d\tau\right)dW(s) \\ &+ \alpha\int_0^t (t-s)^{\alpha-1}\sigma_2\left(s,x(s),\int_0^s f_3(s,\tau,x(\tau))d\tau\right)ds, \end{aligned} \tag{3.16}$$

for all $t\in J$. This completes the proof of the existence of solutions of the stochastic fractional integrodifferential equation (3.5).

Uniqueness: The uniqueness follows from the Itô isometry, the Lipschitz conditions (3.2) and (3.3).
Let $x(t,\omega)$ and $y(t,\omega)$ be solution processes through the initial data $(0,x_0)$ and $(0,y_0)$ respectively, that is, $x(0,\omega)=x_0(\omega)$ and $y(0,\omega)=y_0(\omega)$, $\omega\in\Omega$. Let

$$\begin{aligned} a(s,\omega) &= b(s,x(s),\int_0^t f_1(s,\tau,x(s))d\tau)-b(s,y(s),\int_0^t f_1(s,\tau,y(s))d\tau), \\ \gamma_1(s,\omega) &= \sigma_1(s,x(s),\int_0^t f_2(s,\tau,x(s))d\tau)-\sigma_1(s,y(s),\int_0^t f_2(s,\tau,x(s))d\tau) \\ \text{and}\quad \gamma_2(s,\omega) &= \sigma_2(s,x(s),\int_0^t f_3(s,\tau,x(s))d\tau)-\sigma_2(s,y(s),\int_0^t f_3(s,\tau,x(s))d\tau). \end{aligned}$$

Then, by virtue of the Schwarz inequality and the Itô isometry, we have

$$\begin{aligned} &E[|x(t)-y(t)|^2] \\ &\le 4E[|x_0-y_0|^2]+4tE\left[\int_0^t |a(s,\omega)|^2ds\right] \\ &\quad + 4E\left[\int_0^t |\gamma_1(s,\omega)|^2ds\right]+4\frac{\alpha^2t^{2\alpha-1}}{2\alpha-1}E\left[\int_0^t|\gamma_2(s,\omega)|^2ds\right] \\ &\le 4E[|x_0-y_0|^2]+4D^2\left(1+t+\frac{\alpha^2t^{2\alpha-1}}{2\alpha-1}\right)\int_0^t E[|x(s)-y(s)|^2]ds. \end{aligned}$$

We define $v(t) = E[|x(t) - y(t)|^2]$. Then the function v satisfies $v(t) \leq F + A\int_0^t v(s)ds$, where $F = 4E[|x_0 - y_0|^2]$ and $A = 4D^2\Bigg(1 + t + \frac{\alpha^2 t^{2\alpha-1}}{2\alpha-1}\Bigg)$. By the application of the Gronwall inequality, we conclude that$v(t) \leq F\exp(At)$. Now assume that $x_0 = y_0$. Then $F = 0$ and so $v(t) = 0$ for all $t \geq 0$. That is, $E[|x(t) - y(t)|^2] = 0$. Hence

$$P\Big\{|x(t) - y(t)| = 0, \quad \text{for all} \ \ t \in \mathbb{Q} \cap J\Big\} = 1,$$

where $\mathbb{Q}$ denotes the set of all rational numbers. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and since the solutions are continuous and coincide on a countably dense subset of $[0, T]$, they must coincide, almost surely, on the entire interval $[0, T]$. $P\Big\{|x(t,\omega) - y(t,\omega)| = 0 \quad \text{for all} \ \ t \in J\Big\} = 1$, that is, the solution of (3.5) is unique. This completes the proof of existence and uniqueness of solution of the stochastic fractional integrodifferential equation (3.5). □

4. Numerical Scheme

One of the most important numerical methods that has many advantages is the Galerkin method [3]. This method has been extensively used for stochastic differential equations [6, 9, 18, 20]. Here we apply this method to stochastic fractional integrodifferential equation to obtain the numerical solutions. In this section, we investigate the numerical scheme of stochastic fractional integrodifferential equation.

Let $T > 0$ and $(\Omega, \mathcal{F}, P)$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$ and $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined over this filtered probability space. Consider the following stochastic fractional integrodifferential equation:

$$D^\alpha x(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))\, dW(s),$$

with the initial condition $x(0) = x_0$, for $t \in [0, T]$ and $0 < \alpha < 1$, where D^α is the Caputo fractional derivative of order α.

4.1. **Settings and Assumptions.** We assume that $(\Omega, \mathcal{F}, P)$ is a probability space with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$ and $W = \{W(t) : t \geq 0\}$ is a Brownian motion defined over this space. Consider the following stochastic fractional integrodifferential equation (SFIDE):

$$\left.\begin{aligned} D^\alpha x(t) &= f(t, x(t)) + \int_0^t g(t, s, x(s))dW(s), \\ x(0) &= x_0 \end{aligned}\right\} \tag{4.1}$$

for $t \in [0, T]$. This equation can be written in the integral form as

$$x(t) = x_0 + I^{-\alpha} f(t, x(t)) + I^{-\alpha} \int_0^t g(t, s, x(s)) dW(s),$$

therefore

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(z, x(z))}{(t-z)^{1-\alpha}} dz + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^z \frac{g(z, s, x(s))}{(t-z)^{1-\alpha}} dW(s) dz. \tag{4.2}$$

We make the following hypotheses on the functions f and g :

Assume that f and g are L^2 measurable functions satisfying

$$\left.\begin{array}{lcl} |f(s,x) - f(s,y)| & \leq & C_1(|x-y|), \\ |g(t,s,x) - g(t,\tau,y)| & \leq & C_2(|s-\tau| + |x-y|), \end{array}\right\} \tag{4.3}$$

and

$$\left.\begin{array}{lcl} |f(s,x)| & \leq & C_3(1+|x|), \\ |g(t,s,x)| & \leq & C_4, \end{array}\right\} \tag{4.4}$$

for some positive constants C_1, C_2, C_3, C_4 and for every $x, y \in \mathbb{R}$ and $0 \leq s, \tau \leq t \leq T$. Also assume that x_0 is $\mathcal{F}_0$-measurable with $E(x_0) < \infty$.

Before we discuss the numerical scheme of equation (4.2), we should study the approximation of the stochastic term. The main part is approximating the term $\frac{dW(t)}{dt}$ which is called white noise and formally is known as the derivative of the Brownian motion $W(t)$ [20]. Let

$$t_0 = 0 < t_1 = \Delta t < \cdots < t_N = T,$$

with $t_i = i\Delta t$, for $i = 0, \cdots, N$, be a partition of $[0, T]$. Following the method introduced in [1], we approximate $\frac{dW(t)}{dt}$ by $\frac{d\hat{W}(t)}{dt}$ as

$$\frac{d\hat{W}(t)}{dt} = \frac{1}{\sqrt{\Delta t}} \sum_{i=1}^{N} \eta_i \xi_i(t), \tag{4.5}$$

where $\eta_i \sim N(0,1)$ is defined by

$$\eta_i = \frac{1}{\sqrt{\Delta t}} \int_{t_{i-1}}^{t_i} dW(t), \qquad i = 1, \cdots, N, \tag{4.6}$$

and

$$\xi_i(t) = \begin{cases} 1, & t_{i-1} \leq t \leq t_i, \\ 0, & \text{otherwise.} \end{cases}$$

Now, for numerical scheme of equation (4.2), we consider the approximation $d\hat{W}(t)$ in place of the white noise $dW(t)$ and obtain the following equation:

$$\hat{x}(t) = \hat{x}_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(z, \hat{x}(z))}{(t-z)^{1-\alpha}} dz + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^z \frac{g(z, s, \hat{x}(s))}{(t-z)^{1-\alpha}} d\hat{W}(s) dz. \tag{4.7}$$

In the forthcoming theorems, we prove that $\hat{x}(t)$, the solution of (4.7), is a good approximation of $x(t)$, the solution of stochastic fractional integrodifferential equation (4.2). It should be mentioned that, for simplicity of notation, in the following theorems C is a random constant which changes from line to line.

4.2. **Estimates and Solutions.** In order to prove the theorems, we need the well-known Gronwall's Lemma which is stated as follows [12]:

Lemma 4.1. *Let* $b, \beta : [0,\infty)$, $a \in [0,\infty)$ *and let* $e : [0,T] \to [0,\infty)$ *be measurable mappings which satisfy*

$$e(t) \leq a + b\int_0^t (t-s)^{\beta-1} e(s)ds,$$

for all $t \in [0,T]$. *Then we obtain*

$$e(t) \leq a.E_\beta\left(t(b\Gamma(\beta))^{\frac{1}{\beta}}\right),$$

for every $t \in [0,T]$, *where* $E_\beta : [0,\infty) \to [0,\infty)$ *is given by* $E_\beta(x) = \sum_{n=0}^{\infty}\frac{x^{n\beta}}{\Gamma(n\beta+1)}$ *for every* $x \in [0,\infty)$.

Theorem 4.2. *Assume that* $x(t)$ *is the solution of the stochastic fractional integrodifferential equation* (4.2) *and* $\hat{x}(t)$ *is the approximation of* $x(t)$ *given by* (4.7). *Then there exists a finite random constant* C *such that*

$$E\left[\int_0^T |\hat{x}(t)|^2 dt\right] \leq C(1 + E|\hat{x}(0)|^2). \tag{4.8}$$

Proof. Let $\tilde{t} \in (0,T]$. From (4.7), we have

$$\begin{aligned} &E\left[\int_0^{\tilde{t}} |\hat{x}(t)|^2 dt\right] \\ &\leq CE\left[\int_0^{\tilde{t}} |\hat{x}(0)|^2 dt\right] + \frac{C}{\Gamma^2(\alpha)}E\left[\int_0^{\tilde{t}}\left(\int_0^t \frac{f(z,\hat{x}(z))}{(t-z)^{1-\alpha}}dz\right)^2 dt\right] \\ &\quad + \frac{C}{\Gamma^2(\alpha)}E\left[\int_0^{\tilde{t}}\left(\int_0^t\int_0^z \frac{g(z,s,\hat{x}(s))}{(t-z)^{1-\alpha}}d\hat{W}(s)dz\right)^2 dt\right]. \end{aligned} \tag{4.9}$$

For the second term on the right hand side of (4.9) by putting $v = t - z$ and using the Minkowski integral inequality, we have

$$\begin{aligned} &E\left[\int_0^{\tilde{t}}\left(\int_0^t \frac{f(z,\hat{x}(z))}{(t-z)^{1-\alpha}}dz\right)^2 dt\right] \\ &\leq C\left[\int_0^{\tilde{t}} \frac{1}{v^{1-\alpha}}\left(E\int_v^{\tilde{t}} f^2(t-v,\hat{x}(t-v))dt\right)^{\frac{1}{2}} dv\right]^2 \end{aligned}$$

$$\leq C\left[\int_0^{\tilde{t}} \frac{1}{v^{1-\alpha}}\left(E\int_0^{\tilde{t}-v} f^2(z,\hat{x}(z))dz\right)^{\frac{1}{2}} dv\right]^2$$
$$\leq C\left[\int_0^{\tilde{t}} \frac{1}{(\tilde{t}-v)^{1-\alpha}}\left(E\int_0^{v} |1+\hat{x}(z)|^2 dz\right)^{\frac{1}{2}} dv\right]^2$$
$$\leq C\left(1+\left[\int_0^{\hat{t}} (\tilde{t}-v)^{\alpha-1}\left(E\int_0^{v} |\hat{x}(z)|^2 dz\right)^{\frac{1}{2}} dv\right]^2\right). \tag{4.10}$$

Now, for the last term in (4.9) again by putting $v = t - z$ and using the Minkowski integral inequality, we have

$$E\Bigg[\int_0^{\tilde{t}}\left(\int_0^t\int_0^z \frac{g(z,s,\hat{x}(s))}{(t-z)^{1-\alpha}} d\hat{W}(s)dz\right)^2 dt\Bigg]$$
$$\leq \Bigg[\int_0^{\tilde{t}} \frac{1}{v^{1-\alpha}}\bigg(\int_v^{\tilde{t}} E\bigg(\int_0^{t-v} g(t-v,s,\hat{x}(s))d\hat{W}(s)\bigg)^2 dt\bigg)^{\frac{1}{2}} dv\Bigg]^2. \tag{4.11}$$

Now let $t_0^* = 0,\ t_1^* = \Delta t^*, \cdots,\ t_{n^*}^* = t - v = z$ be a partition of $[0,z]$, with $\Delta t^* = t_i^* - t_{i-1}^* \leq \Delta t$, for $i = 1,\cdots,n^*$. By using (4.5), (4.6) for the last term, we can write

$$\begin{aligned}
E\left[\int_0^z g(z,s,\hat{x}(s))d\hat{W}(s)\right]^2 &= E\left[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\left[\frac{1}{\Delta t^*}\int_{t_{i-1}^*}^{t_i^*} g(z,\tau,\hat{x}(\tau))d\tau\right]dW(s)\right]^2\\
&\leq \frac{1}{\Delta t^*}\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} |g(z,\tau,\hat{x}(\tau))|^2 d\tau ds\\
&\leq \frac{C_4^2}{\Delta t^*}\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} d\tau ds\\
&\leq C\left[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*} ds\right]\\
&\leq Cz.
\end{aligned} \tag{4.12}$$

Therefore, from (4.10), (4.11), we conclude that

$$E\left[\int_0^{\tilde{t}} |\hat{x}(t)|^2 dt\right]$$
$$\leq C\Bigg\{1+E|\hat{x}(0)|^2+\left(1+\left[\int_0^{\tilde{t}} (\tilde{t}-v)^{\alpha-1}\left(E\int_0^{v} |\hat{x}(z)|^2 dz\right)^{\frac{1}{2}} dv\right]^2\right)$$
$$+\left[\int_0^{\tilde{t}} \frac{1}{v^{1-\alpha}}\left(\int_v^{\tilde{t}} (t-v)dt\right)^{\frac{1}{2}} dv\right]^2\Bigg\}. \tag{4.13}$$

Thus

$$
\begin{aligned}
&E\bigg[\int_0^{\tilde t}|\hat x(t)|^2dt\bigg]\\
&\le C\bigg[1+E|\hat x(0)|^2+\bigg[\int_0^{\tilde t}(\tilde t-v)^{\alpha-1}\left(E\int_0^v|\hat x(z)|^2dz\right)^{\frac12}dv\bigg]^2+C\bigg]\\
&\le C\left(1+E|\hat x(0)|^2\right)+C\left[\int_0^{\tilde t}(\tilde t-v)^{\alpha-1}\left(E\int_0^v|\hat x(z)|^2dz\right)^{\frac12}dv\right]^2. \qquad (4.14)
\end{aligned}
$$

Now define $x_1:[0,T]\to[0,\infty)$ by

$$x_1(t)=\left[E\int_0^t|\hat x(s)|^2ds\right]^{\frac12}.$$

Then, from (4.14), we have

$$x_1(\tilde t)\le C(1+E|\hat x(0)|^2)^{\frac12}+C\int_0^{\tilde t}(\tilde t-v)^{\alpha-1}x_1(v)dv. \qquad (4.15)$$

By using Gronwall's Lemma 4.1, we conclude that

$$x_1(\tilde t)\le C(1+E|\hat x(0)|^2)^{\frac12}, \qquad (4.16)$$

and hence

$$E\left[\int_0^{\tilde t}|\hat x(t)|^2dt\right]\le C(1+E|\hat x(0)|^2). \qquad (4.17)$$

Now, since the equation (4.17) is satisfied for all $\tilde t\in(0,T]$, the result (4.8) is obtained by considering a special case $\tilde t=T$. □

Now we are ready to prove the main theorem which is related to the convergence of the method.

Theorem 4.3. *Let $x(t)$ be the solution of* (4.2) *and $\hat x(t)$ be the approximation of $x(t)$ given by* (4.7). *Then there exists a finite random constant C such that*

$$E\left[\int_0^{\tilde t}|x(t)-\hat x(t)|^2dt\right]\le C(\Delta t)^2. \qquad (4.18)$$

Proof. Let $e(t)=x(t)-\hat x(t)$. From (4.2) and (4.7), we have

$$
\begin{aligned}
e(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(z,x(z))-f(z,\hat x(z))}{(t-z)^{1-\alpha}}dz+\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^z\frac{g(z,s,x(s))}{(t-z)^{1-\alpha}}dW(s)dz\\
&-\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^z\frac{g(z,s,\hat x(s))}{(t-z)^{1-\alpha}}d\hat W(s)dz.
\end{aligned}
$$

Therefore

$$e^2(t) \leq \frac{C}{\Gamma^2(\alpha)}\left(\int_0^t \frac{f(z,x(z))-f(z,\hat{x}(z))}{(t-z)^{1-\alpha}}dz\right)^2 + \frac{C}{\Gamma^2(\alpha)}\left[\int_0^t\int_0^z \frac{(g(z,s,x(s))}{(t-z)^{1-\alpha}}dW(s)dz - \int_0^t\int_0^z \frac{(g(z,s,\hat{x}(s))}{(t-z)^{1-\alpha}}d\hat{W}(s)dz\right]^2. \tag{4.19}$$

Now let $\tilde{t} \in (0,T]$, then

$$E\left(\int_0^{\tilde{t}} e^2(t)dt\right) \leq CE\left[\int_0^{\tilde{t}}\left(\int_0^t \frac{f(z,x(z))-f(z,\hat{x}(z))}{(t-z)^{1-\alpha}}dz\right)^2 dt\right] + CE\int_0^{\tilde{t}}\left[\int_0^t\int_0^z \frac{(g(z,s,x(s))}{(t-z)^{1-\alpha}}dW(s) - \int_0^t\int_0^z \frac{(g(z,s,\hat{x}(s))}{(t-z)^{1-\alpha}}d\hat{W}(s))dz\right]^2 dt. \tag{4.20}$$

For the first term in (4.20) by putting $v = t - z$ and using the Minkowski integral inequality, we have

$$\begin{aligned}
&E\left[\int_0^{\tilde{t}}\left(\int_0^t \frac{f(z,x(z))-f(z,\hat{x}(z))}{(t-z)^{1-\alpha}}dz\right)^2 dt\right] \\
&\leq C\left[\int_0^{\tilde{t}} \frac{1}{v^{1-\alpha}}\left(E\int_v^{\tilde{t}} e^2(t-v)dt\right)^{\frac{1}{2}} dv\right]^2 \\
&\leq C\left[\int_0^{\tilde{t}} \frac{1}{v^{1-\alpha}}\left(E\int_0^{\tilde{t}-v} e^2(z)dz\right)^{\frac{1}{2}} dv\right]^2 \\
&\leq C\left[\int_0^{\tilde{t}} \frac{1}{(\tilde{t}-v)^{1-\alpha}}\left(E\int_0^{v} e^2(z)dz\right)^{\frac{1}{2}} dv\right]^2.
\end{aligned} \tag{4.21}$$

For the last term in (4.20) again by putting $v = t - z$ and using the Minkowski integral inequality, we derive

$$E\int_0^{\tilde{t}}\left[\int_0^t\int_0^z \frac{1}{(t-z)^{1-\alpha}}(g(z,s,x(s))dW(s) - g(z,s,\hat{x}(s))d\hat{W}(s))dz\right]^2 dt$$

$$\leq E\int_0^{\tilde{t}}\left[\int_0^t \frac{1}{v^{1-\alpha}}\left[\int_0^{t-v}\Big(g(t-v,s,x(s))dW(s)-g(t-v,s,\hat{x}(s))d\hat{W}(s)\Big)\right]dv\right]^2 dt$$
$$\leq \left[\int_0^{\tilde{t}}\frac{1}{v^{1-\alpha}}\left[\int_v^{\tilde{t}} E\left(\int_0^{t-v}\Big(g(t-v,s,x(s))dW(s)-g(z,s,\hat{x}(s))d\hat{W}(s)\Big)\right)^2 dt\right]^{\frac{1}{2}} dv\right]^2. \tag{4.22}$$

For the last term in (4.22), again like as estimating (4.12), let $t_0^* = 0$, $t_1^* = \Delta t^*, \cdots, t_{n^*}^* = t-v = z$ be a partition of $[0,z]$ with $\Delta t^* = t_i^* - t_{i-1}^* \leq \Delta t$, for $i = 1,\cdots,n^*$. Then, from (4.5) and (4.6), we have

$$E\bigg[\int_0^z \Big[g(z,s,x(s))dW(s)-g(z,s,\hat{x}(s))d\hat{W}(s)\Big]\bigg]^2$$
$$= E\left[\int_0^z g(z,s,x(s))dW(s)-\int_0^z g(z,s,\hat{x}(s))d\hat{W}(s)\right]^2$$
$$= E\left[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\left[g(z,s,x(s))-\frac{1}{\Delta t^*}\int_{t_{i-1}^*}^{t_i^*} g(z,\tau,\hat{x}(\tau))d\tau\right]dW(s)\right]^2$$
$$= E\left[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\left[\frac{1}{\Delta t^*}\int_{t_{i-1}^*}^{t_i^*} g(z,s,x(s))d\tau-\frac{1}{\Delta t^*}\int_{t_{i-1}^*}^{t_i^*} g(z,\tau,\hat{x}(\tau))d\tau\right]dW(s)\right]^2$$
$$\leq E\left[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\left[\frac{1}{\Delta t^*}\int_{t_{i-1}^*}^{t_i^*}\left[g(z,s,x(s))-g(z,\tau,\hat{x}(\tau))\right]d\tau\right]^2 ds\right]. \tag{4.23}$$

Thus, from (4.3), we conclude that

$$E\bigg[\int_0^z \Big[g(z,s,x(s))dW(s)-g(z,s,\hat{x}(s))d\hat{W}(s)\Big]\bigg]^2$$
$$\leq \frac{C}{(\Delta t^*)^2}E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*}\Big(|s-\tau|^2+|x(s)-\hat{x}(\tau)|^2\Big)d\tau ds\bigg]$$
$$\leq \frac{C}{(\Delta t^*)^2}E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*}|s-\tau|^2 d\tau ds\bigg]$$
$$+\frac{C}{(\Delta t^*)^2}E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*}|x(s)-\hat{x}(\tau)|^2 d\tau ds\bigg]. \tag{4.24}$$

Now we are estimating the second term in (4.24) as follows:

$$\frac{C}{(\Delta t^*)^2}E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*}|x(s)-\hat{x}(\tau)|^2 d\tau ds\bigg]$$

$$= \frac{C}{(\Delta t^*)^2} E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} |(x(s)-\hat{x}(s))+(\hat{x}(s)-\hat{x}(\tau))|^2 d\tau ds\bigg]$$
$$\le \frac{C}{(\Delta t^*)^2} E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} e^2(s) d\tau ds\bigg]$$
$$+ \frac{C}{(\Delta t^*)^2} E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} |\hat{x}(s)-\hat{x}(\tau)|^2 d\tau ds\bigg].$$

Then the equation (4.24) becomes

$$E\bigg[\int_0^z \Big[g(z,s,x(s))dW(s) - g(z,s,\hat{x}(s))d\hat{W}(s)\Big]\bigg]^2$$
$$\le C(\Delta t^*)^2 + \frac{C}{(\Delta t^*)^2} E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} e^2(s) d\tau ds\bigg]$$
$$+ \frac{C}{(\Delta t^*)^2} E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} |\hat{x}(s)-\hat{x}(\tau)|^2 d\tau ds\bigg].$$

Therefore

$$E\int_0^{\tilde{t}}\bigg[\int_0^t\int_0^z \frac{1}{(t-z)^{1-\alpha}}(g(z,s,x(s))dW(s) - g(z,s,\hat{x}(s))d\hat{W}(s))dz\bigg]^2 dt$$
$$\le \Bigg[\int_0^{\tilde{t}} \frac{1}{v^{1-\alpha}}\Bigg\{\int_v^{\tilde{t}}\Bigg(C(\Delta t^*)^2 + \frac{C}{(\Delta t^*)^2} E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} e^2(s) d\tau ds\bigg]$$
$$+ \frac{C}{(\Delta t^*)^2} E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} |\hat{x}(s)-\hat{x}(\tau)|^2 d\tau ds\bigg]\Bigg)dt\Bigg\}^{\frac{1}{2}} dv\Bigg]^2$$
$$\le \Bigg[\int_0^{\tilde{t}} \frac{1}{v^{1-\alpha}}\Bigg\{\int_v^{\tilde{t}} C(\Delta t^*)^2 dt + \frac{C}{(\Delta t^*)^2}\int_v^{\tilde{t}} E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} e^2(s) d\tau ds\bigg]dt$$
$$+ \frac{C}{(\Delta t^*)^2}\int_v^{\tilde{t}} E\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*}\int_{t_{i-1}^*}^{t_i^*} |\hat{x}(s)-\hat{x}(\tau)|^2 d\tau ds\bigg]dt\Bigg\}^{\frac{1}{2}} dv\Bigg]^2$$
$$\le C(\Delta t^*)^2 + CTE\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*} e^2(s) ds\bigg] + CE\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*} |\hat{x}(s)|^2 ds\bigg]$$
$$+ CE\bigg[\sum_{i=1}^{n^*}\int_{t_{i-1}^*}^{t_i^*} |\hat{x}(\tau)|^2 d\tau\bigg].$$

By applying Theorem 4.2, we get

$$E\int_0^{\tilde{t}}\left[\int_0^t\int_0^z \frac{1}{(t-z)^{1-\alpha}}(g(z,s,x(s))dW(s)-g(z,s,\hat{x}(s))d\hat{W}(s))dz\right]^2 dt$$
$$\leq C(\Delta t^*)^2 + CTE\bigg[\int_0^{\tilde{t}} e^2(s)ds\bigg]. \tag{4.25}$$

So, from (4.20) and (4.25), we get

$$E\left[\int_0^{\tilde{t}} e^2(t)dt\right] \leq C\left[\int_0^{\tilde{t}}(\tilde{t}-v)^{\alpha-1}\left(E\int_0^v e^2(z)dz\right)^{\frac{1}{2}}dv\right]^2 + C(\Delta t^*)^2$$
$$+ CTE\bigg[\int_0^{\tilde{t}} e^2(s)ds\bigg]. \tag{4.26}$$

Then

$$(1-CT)E\left[\int_0^{\tilde{t}} e^2(t)dt\right] \leq C\left[\int_0^{\tilde{t}}(\tilde{t}-v)^{\alpha-1}\left(E\int_0^v e^2(z)dz\right)^{\frac{1}{2}}dv\right]^2 + C(\Delta t^*)^2.$$

Taking T sufficiently small, we have

$$E\left[\int_0^{\tilde{t}} e^2(t)dt\right] \leq C\left[\int_0^{\tilde{t}}(\tilde{t}-v)^{\alpha-1}\left(E\int_0^v e^2(z)dz\right)^{\frac{1}{2}}dv\right]^2 + C(\Delta t^*)^2.$$

Define $e_1 : [0,T] \to [0,\infty)$ by

$$e_1(t) = \left[E\left(\int_0^{\tilde{t}} e^2(t)dt\right)\right]^{\frac{1}{2}}.$$

Then, from (4.26), we derive

$$e_1(\tilde{t}) \leq C\left(\int_0^{\tilde{t}}(\tilde{t}-v)^{\alpha-1}e_1(v)dv\right) + C(\Delta t^*).$$

By using Gronwall's Lemma 3.1, we conclude that

$$E\left(\int_0^{\tilde{t}} e^2(t)dt\right) \leq C(\Delta t^*)^2 \leq C(\Delta t)^2. \tag{4.27}$$

Hence the theorem is proved. □

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