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ON A NONLINEAR DEGENERATE ELLIPTIC EQUATION HAVING NATURAL GROWTH TERMS

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Abstract. In this work we are interested in the existence of a solution of the nonlinear degenerate elliptic equations $Lu(x) + H(x,u,\nabla u)\,\omega_2 = T$ in the setting of the weighted Sobolev spaces, where H is a nonlinear term with natural growth with respect to ∇u and $T \in [W_0^{1,p}(\Omega,\omega_1,\omega_2)]^* = W^{-1,p'}(\Omega,\omega_1,\omega_2)$.

1. Introduction

In this paper, we prove the existence of (weak) solutions in the weighted Sobolev space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ for the nonlinear degenerate elliptic problem with Dirichlet boundary conditions

$$(P)\begin{cases} Lu(x) + H(x,u,\nabla u)\,\omega_2 = T, \\ u \in W_0^{1,p}(\Omega,\omega_1,\omega_2), \\ H(x,u,\nabla u) \in L^1(\Omega,\omega_2), \\ H(x,u,\nabla u)\,u \in L^1(\Omega,\omega_2), \end{cases}$$

where L is the partial differential operator $Lu(x) = -\operatorname{div}(\omega_1\,\mathcal{A}(x,u,\nabla u))$ and $H(x,u,\nabla u)$ is a non linear lower order term having natural growth (of order p) with respect to $|\nabla u|$ (with respect to $|u|$ we do not assume any growth restrictions, but we assume the sign-condition $H(x,\eta,\xi)\,\eta \geq 0$), Ω is a bounded open set in $\mathbb{R}^n$, ω_1 and ω_2 are two weight functions, $1 < p < \infty$, $T \in [W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$ and the functions $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following conditions:

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(H1) $x \mapsto \mathcal{A}(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$, $(\eta,\xi) \mapsto \mathcal{A}(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x \in \Omega$.
(H2) $|\mathcal{A}(x,\eta,\xi)| \leq K(x) + h_1(x)\,|\eta|^{p-1} + h_2(x)\,|\xi|^{p-1}$, $K \in L^{p'}(\Omega,\omega_1)$, $K \geq 0$ (with $1/p + 1/p' = 1$), $h_1, h_2 \in L^\infty(\Omega)$, $h_1, h_2 \geq 0$.
(H3) $[\mathcal{A}(x,\eta,\xi) - \mathcal{A}(x,\eta',\xi')].(\xi - \xi') \geq 0$, whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, where $\mathcal{A}(x,\eta,\xi) = (\mathcal{A}_1(x,\eta,\xi), \cdots, \mathcal{A}_n(x,\eta,\xi))$ (where a dot denote here the Euclidian scalar product in $\mathbb{R}^n$).
(H4) $\mathcal{A}(x,\eta,\xi).\xi \geq \alpha\,|\xi|^p$, where α is a positive constant.
(H5) $x \mapsto H(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$, $(\eta,\xi) \mapsto H(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x \in \Omega$.
(H6) $|H(x,\eta,\xi)| \leq b(\eta)\big(|\xi|^p + h(x)\big)$, where $h \in L^1(\Omega,\omega_2)$, $h \geq 0$ and $0 \leq b(\eta) \leq \beta$ for all $\eta \in \mathbb{R}$.
(H7) $H(x,\eta,\xi)\,\eta \geq 0$.

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^n$ (and $\mu_i = \int_E \omega_i(x)\,dx$, $i = 1, 2$).

In general, the Sobolev spaces $\mathrm{W}^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, *i.e.*, equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [3], [4], [5], [8] and [12]).

In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. This bad behaviour can be caused by the coefficients of the corresponding differential operator as well as by the solution itself. The so-called p-Laplacian is a prototype of such an operator and its character can be interpreted as a degeneration or as a singularity of the classical (linear) Laplace operator (with $p = 2$). There are several very concrete problems from practice which lead to such differential equations, e.g. from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction, reaction-diffusion problems, etc.

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by Muckenhoupt (see [10]). These classes have found many useful applications in harmonic analysis (see [11]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [9]). There are, in fact, many interesting examples of weights (see [8] for p-admissible weights).

Note that, in the proof of our main results, many ideas have been adapted from [1], [2] and [3].

The following theorem will be proved in Section 3.

Theorem 1.1. *Assume* (H1)-(H7). *If* $\omega_1, \omega_2 \in A_p$ (*with* $1 < p < \infty$) *and* $\omega_2 \leq \omega_1$, *then there exist a solution* $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ *of problem* (P).

2. Preliminaries

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\right)^{p-1} \leq C_{p,\omega},$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [7], [8], [9] or [12] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x;2r)) \leq C\,\mu(B(x;r))$ for every ball $B = B(x;r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [8]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [11]).

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $1 \leq p < \infty$ we define the weighted Lebesgue space $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [12]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be open and let ω_1 and ω_2 be A_p-weights ($1 < p < \infty$). We define the weighted Sobolev space $W^{1,p}(\Omega, \omega_1, \omega_2)$ as the set of functions $u \in L^p(\Omega, \omega_2)$ with weak derivatives $D_j u \in L^p(\Omega, \omega_1)$ for $j = 1, \cdots, n$. The norm of u in $W^{1,p}(\Omega, \omega_1, \omega_2)$ is defined by

$$\|u\|_{W^{1,p}(\Omega,\omega_1,\omega_2)} = \left(\int_\Omega |u(x)|^p\,\omega_2(x)\,dx + \int_\Omega |\nabla u(x)|^p\,\omega_1(x)\,dx\right)^{1/p}. \quad (2.1)$$

We also define $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_{W^{1,p}(\Omega,\omega_1,\omega_2)}$.

Remark 2.3. (a) If $\omega_2\leq\omega_1$ then $W_0^{1,p}(\Omega,\omega_1)\subset W_0^{1,p}(\Omega,\omega_1,\omega_2)\subset W_0^{1,p}(\Omega,\omega_2)$.
(b) If $\omega\in A_p$, then $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega,\omega)=W^{1,p}(\Omega,\omega,\omega)$ (see Corollary 2.1.6 in [12]).

The spaces $W^{1,p}(\Omega,\omega_1,\omega_2)$ and $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ are Banach spaces. The dual space of $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is the space $[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*=W^{-1,p'}(\Omega,\omega_1,\omega_2)$,

$$[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*=\Big\{T=f_0-\operatorname{div}(F),F=(f_1,\cdots,f_n):\\ \frac{f_0}{\omega_2}\in L^{p'}(\Omega,\omega_2),\ \frac{f_j}{\omega_1}\in L^{p'}(\Omega,\omega_1),j=,1,\cdots,n\Big\}.$$

If $T\in[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$ and $\varphi\in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, we denote

$$\langle T,\varphi\rangle=\int_\Omega f_0\,\varphi\,dx+\sum_{j=1}^n\int_\Omega f_j\,D_j\varphi\,dx,$$

$$\|T\|_*=\|f_0/\omega_2\|_{L^{p'}(\Omega,\omega_2)}+\sum_{j=1}^n\|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)},$$

$$|\langle T,\varphi\rangle|\leq\|T\|_*\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

In this article we use the following results.

Theorem 2.4. *Let $\omega\in A_p$, $1<p<\infty$ and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m\to u$ in $L^p(\Omega,\omega)$, then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi\in L^p(\Omega,\omega)$ such that*

(i) *$u_{m_k}(x)\to u(x)$, $m_k\to\infty$, μ-a.e. on Ω;*
(ii) *$|u_{m_k}(x)|\leq\Phi(x)$, μ-a.e. on Ω;*

where $\mu(E)=\int_E\omega(x)\,dx$.

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [6]. □

Theorem 2.5. (The weighted Sobolev inequality) *Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega\in A_p$ ($1<p<\infty$). There exist positive constants C_Ω and δ such that for all $u\in W_0^{1,p}(\Omega,\omega)$ and all θ satisfying $1\leq\theta\leq n/(n-1)+\delta$,*

$$\|u\|_{L^{\theta\,p}(\Omega,\omega)}\leq C_\Omega\|\nabla u\|_{L^p(\Omega,\omega)}. \tag{2.2}$$

Proof. Its suffices to prove the inequality for functions $u\in C_0^\infty(\Omega)$ (see Theorem 1.3 in [5]). To extend the estimates (2.2) to arbitrary $u\in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimates (2.2) to differences $u_{m_1}-u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{kp}(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2.2). □

Lemma 2.6. *If $\omega\in A_p$, then $\left(\dfrac{|E|}{|B|}\right)^p\le C_{p,\omega}\dfrac{\mu(E)}{\mu(B)}$, whenever B is a ball in $\mathbb{R}^N$ and E is a measurable subset of B.*

Proof. See Theorem 15.5, *Strong doubling of A_p-weights* in [8]. □

By Lemma 2.6, if $\mu(E)=0$, then $|E|=0$.

Lemma 2.7. *If $\omega\in A_p$, there are $0<q\le 1$ and $C>0$ depending only on n,p and $C_{p,\omega}$ such that $\dfrac{\mu(E)}{\mu(B)}\le C\left(\dfrac{|E|}{|B|}\right)^q$, and whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B.*

Proof. See Lemma 15.8 in [8]. □

By Lemma 2.7, if $|E|=0$ then $\mu(E)=0$.

Lemma 2.8. *Let ω_1 and ω_2 be A_p-weights, $1<p<\infty$ and a sequence $\{u_n\}$, $u_n\in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ satisfies*

(i) *$u_n\rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ and μ_1-a.e. in Ω, where $\mu_1(E)=\int_E\omega_1(x)dx$;*

(ii) *$\int_\Omega\langle\mathcal{A}(x,u_n,\nabla u_n)-\mathcal{A}(x,u_n,\nabla u),\nabla(u_n-u)\rangle\,\omega_1\,dx\to 0$ with $n\to\infty$.*

Then $u_n\to u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Proof. The proof of this lemma follows the line of Lemma 5 in [2]. □

Definition 2.9. We say that $u\in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is a solution of problem (P) is for any $\varphi\in W_0^{1,p}(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega)$ we have

$$\int_\Omega\omega_1\,\mathcal{A}(x,u,\nabla u).\nabla\varphi\,dx+\int_\Omega H(x,u,\nabla u)\,\varphi\,\omega_2\,dx=\langle T,\varphi\rangle,\tag{2.3}$$

$$H(x,u,\nabla u)\in L^1(\Omega,\omega_2),\tag{2.4}$$

$$H(x,u,\nabla u)\,u\in L^1(\Omega,\omega_2).\tag{2.5}$$

3. Main results

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. We split the proof of Theorem 1.1 in four steps.

Step 1. Let us define for $\varepsilon > 0$ the approximation

$$H_\varepsilon(x,\eta,\xi) = \frac{H(x,\eta,\xi)}{1+\varepsilon\,|H(x,\eta,\xi)|}.$$

We consider the approximate problem

$$(P_\varepsilon)\begin{cases} Lu_\varepsilon(x) + H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,\omega_2 = T, \\ u_\varepsilon \in W_0^{1,p}(\Omega,\omega_1,\omega_2), \end{cases}$$

which has a solution $u_\varepsilon \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ (see Step 1, Theorem 1.2 in [3]), that is,

$$\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla\varphi\,\omega_1\,dx + \int_\Omega H_\varepsilon(x,u_\varepsilon\,\nabla u_\varepsilon)\,\varphi\,\omega_2\,dx = \langle T,\varphi\rangle, \tag{3.1}$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega)$, and we also have $\|u_\varepsilon\|_{L^\infty(\Omega)} \le C_1$ (where C_1 is independent of ε, see Step 2, Theorem 1.2 in [3]). Using the function $\varphi = u_\varepsilon$ in (3.1), and by (H4) and (H7), we obtain

$$\begin{aligned}
&\alpha \int_\Omega |\nabla u_\varepsilon|^p\,\omega_1\,dx \\
&\le \int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla u_\varepsilon\,\omega_1\,dx \\
&\le \int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla u_\varepsilon\,\omega_1\,dx + \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,u_\varepsilon\,\omega_2\,dx \\
&= \langle T,u_\varepsilon\rangle \\
&\le \|T\|_*\|u_\varepsilon\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

Hence, by Theorem 2.5 (with $\theta = 1$) and $\omega_2 \le \omega_1$ we have

$$\begin{aligned}
\|u_\varepsilon\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} &= \int_\Omega |u_\varepsilon|^p\,\omega_2\,dx + \int_\Omega |\nabla u_\varepsilon|^p\,\omega_1\,dx \\
&\le (C_\Omega^p + 1)\int_\Omega |\nabla u_\varepsilon|^p\,\omega_1\,dx \\
&\le \frac{(C_\Omega^p+1)}{\alpha}\|T\|_*\|u_\varepsilon\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}$$

that is,

$$\|u_\varepsilon\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \le C_2, \tag{3.2}$$

where C_2 is independent of ε. Therefore, there exists a subsequence (still denoted by $\{u_\varepsilon\}$) and $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ such that

$$u_\varepsilon \rightharpoonup u \text{ in } W_0^{1,p}(\Omega,\omega_1,\omega_2), \tag{3.3}$$

$$u_\varepsilon \to u \quad \mu_2 - a.e., \tag{3.4}$$

and, by Lemma 2.6 and Lemma 2.7, $u_\varepsilon \to u$ μ_1-a.e and $a.e.$ (where $\mu_i = \int_E \omega_i(x)\,dx,\ i=1,2$).

Step 2. We will prove that $u_\varepsilon^+ \to u^+$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

For $k\in\mathbb{R},\ k>0$, we define $u_k^+ = \min\{u^+,k\}$. For k fixed, we define $g_\varepsilon = u_\varepsilon^+ - u_k^+$. We will study the behavior of g_ε^+ and of g_ε^-.

(2-I) Behavior of $g_\varepsilon^+ = (u_\varepsilon^+ - u_k^+)^+$.

Note that $g_\varepsilon \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, then $g_\varepsilon^+ \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ and

$$\|g_\varepsilon^+\|_{L^\infty(\Omega)} \le \|u_\varepsilon\|_{L^\infty(\Omega)} \le C_1,$$

(C_1 independent of ε). Hence, using g_ε^+ as test function in (3.1), we obtain

$$\int_\Omega \omega_1\,\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla g_\varepsilon^+\,dx + \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,g_\varepsilon^+\,\omega_2\,dx = \langle T, g_\varepsilon^+\rangle. \tag{3.5}$$

Note that where $g_\varepsilon^+ = (u_\varepsilon^+ - u_k^+)^+ > 0$, then $u_\varepsilon^+ > 0$, and we have $u_\varepsilon > 0$ and $H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\ge 0$ (by (H7)). Therefore, in (3.5) we get

$$\int_\Omega \omega_1\,\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla g_\varepsilon^+\,dx \le \langle T, g_\varepsilon^+\rangle.$$

Since $u_\varepsilon = u_\varepsilon^+$ on the set $\{x\in\Omega : g_\varepsilon^+(x) > 0\}$, we can also write

$$\int_\Omega \omega_1\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+).\nabla g_\varepsilon^+\,dx \le \langle T, g_\varepsilon^+\rangle. \tag{3.6}$$

Hence, by (3.6), we obtain

$$\begin{aligned}
&\int_\Omega \omega_1\big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\big).\nabla(u_\varepsilon^+ - u_k^+)^+\,dx\\
&= \int_\Omega \omega_1\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+).\nabla(u_\varepsilon^+ - u_k^+)^+\,dx\\
&\quad - \int_\Omega \omega_1\mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+ - u_k^+)^+\,dx\\
&\le -\int_\Omega \omega_1\,\mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+ - u_k^+)^+ + \langle T, g_\varepsilon^+\rangle.
\end{aligned} \tag{3.7}$$

As $\varepsilon\to 0$, we have $g_\varepsilon^+ = (u_\varepsilon^+ - u_k^+)^+ \to (u^+ - u_k^+)^+$ a.e., μ_1-a.e. and μ_2-a.e. (by Lemma 2.6, Lemma 2.7 and $\omega_2\le\omega_1$). Furthermore, since g_ε^+ is bounded in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ (by (3.2), we have $g_\varepsilon^+ \rightharpoonup (u^+ - u_k^+)^+$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$, as $\varepsilon\to 0$ and k fixed. We define

$$R_k = -\int_\Omega \omega_1\,\mathcal{A}(x,u,\nabla u_k^+).\nabla(u^+ - u_k^+)^+\,dx + \langle T, (u^+ - u_k^+)^+\rangle. \tag{3.8}$$

We have, by $(H2)$,

$$
\begin{aligned}
|R_k| &\leq \int_\Omega \omega_1 |\mathcal{A}(x,u,\nabla u_k^+)||\nabla(u^+ - u_k^+)^+|\,dx + |\langle T,(u^+ - u_k^+)^+\rangle| \\
&\leq \left(\int_\Omega |\mathcal{A}(x,u,\nabla u_k^+)|^{p'}\omega_1\,dx\right)^{1/p'} \left(\int_\Omega |\nabla(u^+ - u_k^+)^+|^p \omega_1\,dx\right)^{1/p} \\
&\quad + \|T\|_* \|(u^+ - u_k^+)\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \\
&\leq C\left(\int_\Omega (K^{p'} + h_1^{p'}|u|^p + h_2^{p'}|\nabla u_k^+|^p)\omega_1\,dx\right) \|(u^+ - u_k^+)^+\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \\
&\quad + \|T\|_* \|(u^+ - u_k^+)^+\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \\
&\leq C\Big(\|K\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{L^p(\Omega,\omega_1)} \\
&\quad + \|h_2\|^{p'}_{L^\infty(\Omega)}\|\nabla u_k^+\|^p_{L^p(\Omega,\omega_1)}\Big)\|(u^+ - u_k^+)^+\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \\
&\quad + \|T\|_* \|(u^+ - u_k^+)^+\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}
$$

Since $(u^+ - u_k^+)^+ \to 0$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ as $k\to\infty$, we obtain $R_k \to 0$ as $k\to\infty$. Now, passing to the limit in $\varepsilon \to 0$ (with k fixed) in (3.7) we obtain

$$
\overline{\lim_{\varepsilon\to 0}} \int_\Omega \omega_1 \left(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\right).\nabla(u_\varepsilon^+ - u_k^+)^+\,dx \leq R_k. \tag{3.9}
$$

(2-II) Behavior of $g_\varepsilon^- = (u_\varepsilon^+ - u_k^+)^-$.

We shall use as a test function in (3.1) the function $v_\varepsilon = \varphi_\lambda(g_\varepsilon^-)$, where $\varphi_\lambda(s) = s\,\mathrm{e}^{\lambda s^2}$, and $\lambda \in \mathbb{R}$ will be chosen later. We have $0 \leq g_\varepsilon^- \leq k$. Hence, $g_\varepsilon^- \in L^\infty(\Omega)$, and since $g_\varepsilon \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, then $v_\varepsilon \in W_0^{1,p}(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega)$. Therefore, using v_ε in (3.1) we get

$$
\int_\Omega \omega_1\,\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla v_\varepsilon\,dx + \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,v_\varepsilon\,\omega_2\,dx = \langle T, v_\varepsilon\rangle,
$$

that is,

$$
\begin{aligned}
&\int_\Omega \omega_1\,\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla g_\varepsilon^-\,\varphi_\lambda'(g_\varepsilon^-)\,dx \\
&+ \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_2\,dx = \langle T, \varphi_\lambda(g_\varepsilon^-)\rangle.
\end{aligned} \tag{3.10}
$$

Considering the sets

$$
\begin{aligned}
E_\varepsilon &= \{x\in\Omega : u_\varepsilon^+(x) \leq u_k^+(x)\}, \\
F_\varepsilon &= \{x\in\Omega : 0 \leq u_\varepsilon(x) \leq u_k^+(x)\}.
\end{aligned}
$$

If $x \notin E_\varepsilon$, then $g_\varepsilon^- = (u_\varepsilon^+ - u_k^+)^- = 0$ and $\varphi_\lambda(0) = 0$. Then we have

$$\int_\Omega H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\, \varphi_\lambda(g_\varepsilon^-)\, \omega_2\, dx = \int_{E_\varepsilon} H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\, \varphi_\lambda(g_\varepsilon^-)\, \omega_2\, dx. \tag{3.11}$$

If $u_\varepsilon \leq 0$, then $g_\varepsilon = u_\varepsilon^+ - u_k^+ \leq 0$, and $g_\varepsilon^- = (u_\varepsilon^+ - u_k^+)^- \geq 0$. Hence, $\varphi_\lambda(g_\varepsilon^-) \geq 0$, and since $H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\, u_\varepsilon \geq 0$, then $H_\varepsilon(x, u_\varepsilon\, \nabla u_\varepsilon) \leq 0$. Using (H3) and (H6), we obtain

$$\begin{aligned}
&\int_{E_\varepsilon} H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\, \varphi_\lambda(g_\varepsilon^-)\, \omega_2\, dx \\
&\leq \int_{F_\varepsilon} H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\, \varphi_\lambda(g_\varepsilon^-)\, \omega_2\, dx \\
&\leq \int_{F_\varepsilon} |H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|\, \varphi_\lambda(g_\varepsilon^-)\, \omega_2\, dx \\
&\leq \int_{F_\varepsilon} b(u_\varepsilon) \Big(|\nabla u_\varepsilon|^p + h(x) \Big) \varphi_\lambda(g_\varepsilon^-)\, \omega_2\, dx \\
&\leq \beta \int_{F_\varepsilon} |\nabla u_\varepsilon|^p \varphi_\lambda(g_\varepsilon^-)\, \omega_2\, dx + \beta \int_\Omega h(x) \varphi_\lambda(g_\varepsilon^-)\, \omega_2\, dx \\
&\leq \beta \int_{F_\varepsilon} |\nabla u_\varepsilon|^p \varphi_\lambda(g_\varepsilon^-)\, \omega_1\, dx + \beta \int_\Omega h(x) \varphi_\lambda(g_\varepsilon^-)\, \omega_2\, dx \\
&\leq \frac{\beta}{\alpha} \int_{F_\varepsilon} \big(\mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla u_\varepsilon \big)\, \varphi_\lambda(g_\varepsilon^-)\, \omega_1\, dx + \beta \int_\Omega h(x)\, \varphi_\lambda(g_\varepsilon^-)\, \omega_2\, dx.
\end{aligned} \tag{3.12}$$

Since $u_\varepsilon \leq 0$ implies $g_\varepsilon^- = (u_\varepsilon^+ - u_k^+)^- = u_k^+$, we have

$$\begin{aligned}
&\int_\Omega - \mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon^+).\nabla g_\varepsilon^-\, \varphi_\lambda'(g_\varepsilon^-)\, \omega_1\, dx \\
&= \int_\Omega - \mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla g_\varepsilon^-\, \varphi_\lambda'(g_\varepsilon^-)\, \omega_1\, dx \\
&\quad + \int_\Omega \mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla g_\varepsilon^-\, \varphi_\lambda'(g_\varepsilon^-)\, \omega_1\, dx \\
&\quad - \int_\Omega \mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon^+).\nabla g_\varepsilon^-\, \varphi_\lambda'(g_\varepsilon^-)\, \omega_1\, dx \\
&= \int_\Omega - \mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla g_\varepsilon^-\, \varphi_\lambda'(g_\varepsilon^-)\, \omega_1\, dx \\
&\quad + \int_\Omega \omega \Big(\mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon) - \mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon^+) \Big).\nabla g_\varepsilon^-\, \varphi_\lambda'(g_\varepsilon^-)\, \omega_1\, dx \\
&= - \int_\Omega \mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla g_\varepsilon^-\, \varphi_\lambda(g_\varepsilon^-)\, \omega_1\, dx \\
&\quad + \int_\Omega \Big(\mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon) - \mathcal{A}(x, u_\varepsilon, \nabla u_\varepsilon^+) \Big).\nabla u_k^+\, \varphi_\lambda'(u_k^+)\, \omega_1\, dx.
\end{aligned} \tag{3.13}$$

Using (3.10) and (3.13) we obtain

$$\begin{aligned}
&\int_\Omega -\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\Big).\nabla(u_\varepsilon^+-u_k^+)^-\,\varphi_\lambda'(g_\varepsilon^-)\,\omega_1\,dx\\
&=\int_\Omega -\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+).\nabla(u_\varepsilon^+-u_k^+)^-\,\varphi_\lambda'(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+-u_k^+)^-\,\varphi_\lambda'(g_\varepsilon^-)\,\omega_1\,dx\\
&=\int_\Omega -\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla(u_\varepsilon^+-u^+)^-\varphi_\lambda'(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)-\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)\Big).\nabla u_k^+\,\varphi_\lambda'(u_k^+)\,\omega_1\,dx\\
&\quad+\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+-u_k^+)^-\,\varphi_\lambda'(g_\varepsilon^-)\,\omega_1\,dx\\
&=\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_2\,dx+\langle -T,\varphi_\lambda(g_\varepsilon^-)\rangle\\
&\quad+\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)-\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)\Big).\nabla u_k^+\,\varphi_\lambda'(u_k^+)\,\omega_1\,dx\\
&\quad+\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+-u_k^+)^+\,\varphi_\lambda'(g_\varepsilon^-)\,\omega_1\,dx\\
&=I. \qquad (3.14)
\end{aligned}$$

Now, by (3.11) and (3.12) we obtain

$$\begin{aligned}
I &\le \int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)-\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)\Big).\nabla u_k^+\,\varphi_\lambda'(u_k^+)\,\omega_1\,dx\\
&\quad+\langle -T,\varphi_\lambda(g_\varepsilon^-)\rangle+\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+-u_k^+)^-\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\frac{\beta}{\alpha}\int_{F_\varepsilon}\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla u_\varepsilon\,\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx+\beta\int_\Omega h(x)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_2\,dx\\
&=\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)-\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)\Big).\nabla u_k^+\,\varphi_\lambda'(u_k^+)\,\omega_1\,dx\\
&\quad+\langle -T,\varphi_\lambda(g_\varepsilon^-)\rangle+\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+-u_k^+)^-\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\frac{\beta}{\alpha}\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\Big).\nabla(u_\varepsilon^+-u_k^+)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\frac{\beta}{\alpha}\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+).\nabla u_k^+\,\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\frac{\beta}{\alpha}\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+-u_k^+)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\beta\int_\Omega h(x)\varphi_\lambda(g_\varepsilon^-)\,\omega_2\,dx. \qquad (3.15)
\end{aligned}$$

Hence, by (3.14) and (3.15) we have

$$
\begin{aligned}
&\int_\Omega -\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\Big).\nabla(u_\varepsilon^+-u_k^+)^-\,\varphi'_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\le \int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)-\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)\Big).\nabla u_k^+\,\varphi'_\lambda(u_k^+)\,\omega_1\,dx\\
&\quad+\langle -T,\varphi_\lambda(g_\varepsilon^-)\rangle+\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+-u_k^+)^-\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\frac{\beta}{\alpha}\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\Big).\nabla(u_\varepsilon^+-u_k^+)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\frac{\beta}{\alpha}\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+).\nabla u_k^+\,\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\frac{\beta}{\alpha}\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+-u_k^+)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\beta\int_\Omega h(x)\varphi_\lambda(g_\varepsilon^-)\,\omega_2\,dx.
\end{aligned}
\tag{3.16}
$$

Now we choose $\lambda=\dfrac{\beta^2}{4\alpha^2}$, we have $\varphi'_\lambda-\dfrac{1}{\alpha}\varphi_\lambda\ge\dfrac{1}{2}$. Thus, in (3.16) we obtain

$$
\begin{aligned}
&-\frac{1}{2}\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\Big).\nabla(u_\varepsilon^+-u_k^+)^-\,\omega_1\,dx\\
&\le \int_\Omega -\Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\Big).\nabla(u_\varepsilon^+-u_k^+)\,\varphi'_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad-\frac{\beta}{\alpha}\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\Big).\nabla(u_\varepsilon^+-u_k^+)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&=\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)-\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)\Big).\nabla u_k^+\,\varphi'_\lambda(u_k^+)\,\omega_1\,dx\\
&\quad+\langle -T,\varphi_\lambda(g_\varepsilon^-)+\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+-u_k^+)^-\,\varphi'_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\frac{\beta}{\alpha}\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+).\nabla u_k^+\,\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\frac{\beta}{\alpha}\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_k^+).\nabla(u_\varepsilon^+-u_k^+)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_1\,dx\\
&\quad+\beta\int_\Omega h(x)\,\varphi_\lambda(g_\varepsilon^-)\,\omega_2\,dx.
\end{aligned}
\tag{3.17}
$$

By $(H2)$ and (3.2) we have

$$
\|\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)\|^{p'}_{L^{p'}(\Omega,\omega_1)}=\int_\Omega|\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)|^{p'}\,\omega_1\,dx
$$

$$\leq C_p\bigg[\int_\Omega |K|^{p'}\omega_1\,dx + \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^p\,\omega_1\,dx + \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^p\,\omega_1\,dx\bigg]$$
$$\leq C_p\bigg[\|K\|^{p'}_{L^{p'}(\Omega,\omega_1)} + (\|h_1\|^{p'}_{L^\infty(\Omega)} + \|h_2\|^{p'}_{L^\infty(\Omega)})\|u_\varepsilon\|^p_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\bigg]$$
$$\leq C_p\bigg[\|K\|^{p'}_{L^{p'}(\Omega,\omega_1)} + C_2^p\,\big(\|h_1\|^{p'}_{L^\infty(\Omega)} + \|h_2\|^{p'}_{L^\infty(\Omega)}\big)\bigg],$$

that is, $\|\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon)\|_{L^{p'}(\Omega,\omega_1)} \leq C_4$ (where C_4 independent of ε). And, analogously, $\|\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)\|_{L^{p'}(\Omega,\omega_1)} \leq C_4$. Then, by extracting a subsequence of $\{u_\varepsilon\}$ (which is still denoted by $\{u_\varepsilon\}$) such that

$$\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon) \rightharpoonup \mathcal{F} \quad\text{in}\quad (L^{p'}(\Omega,\omega_1))^n,$$
$$\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) \rightharpoonup \mathcal{G} \quad\text{in}\quad (L^{p'}(\Omega,\omega_1))^n,$$

and using the Dominated Convergence Theorem, we can pass the limit $\varepsilon\to 0$ (with k fixed) in the right hand of (3.17), we obtain

$$\int_\Omega (\mathcal{F}(x)-\mathcal{G}(x)).\nabla u_k^+\,\varphi'_\lambda(u_k^+)\,\omega_1\,dx + \langle -T, \varphi_\lambda((u^+-u_k^+)^-)\rangle$$
$$+\int_\Omega \mathcal{A}(x,u,\nabla u_k^+).\nabla(u^+-u_k^+)^-\,\varphi'_\lambda((u^+-u_k^+)^-)\,\omega_1\,dx$$
$$+\frac{\beta}{\alpha}\int_\Omega \mathcal{G}(x).\nabla u_k^+\,\varphi_\lambda((u^+-u_k^+)^-)\,\omega_1\,dx$$
$$+\frac{\beta}{\alpha}\int_\Omega \mathcal{A}(x,u,\nabla u_k^+).\nabla(u^+-u_k^+)\,\varphi_\lambda((u^+-u_k^+)^-)\,\omega_1\,dx$$
$$+\beta\int_\Omega h(x)\,\varphi_\lambda((u^+-u_k^+)^-)\,\omega_2\,dx$$
$$=\int_\Omega (\mathcal{F}(x)-\mathcal{G}(x)).\nabla u_k^+\,\varphi'_\lambda(u_k^+)\,\omega_1\,dx,$$

since $(u^+-u_k^+)^- = 0$ and $\varphi_\lambda(0)=0$. Moreover, since

$$\bigg(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon) - \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)\bigg).\nabla(u_\varepsilon)_k^+ = 0 \ \ a.e.$$

where $(u_\varepsilon)_k^+ = \min\{u_\varepsilon,k\}$, then $(\mathcal{F}(x)-\mathcal{G}(x)).\nabla u_k^+ = 0$ a.e and μ_1-a.e.. Hence, $\displaystyle\int_\Omega \bigg(\mathcal{F}(x)-\mathcal{G}(x)\bigg).\nabla u_k^+\,\varphi'_\lambda(u_k^+)\,\omega_1\,dx = 0$. Thus, passing to the limit $\varepsilon\to 0$ (with k fixed) in (3.17) we obtain

$$\overline{\lim_{\varepsilon\to 0}} - \int_\Omega \bigg(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+) - \mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\bigg).\nabla(u_\varepsilon^+-u_k^+)^-\omega_1\,dx \leq 0. \quad (3.18)$$

Using (3.9) and (3.18) we have

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}}\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla(u_\varepsilon^+-u^+)\,\omega_1\,dx\\
&\le \overline{\lim_{\varepsilon\to 0}}\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\Big).\nabla(u_\varepsilon^+-u_k^+)^+\,\omega_1\,dx\\
&\quad+\overline{\lim_{\varepsilon\to 0}}-\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\Big).\nabla(u_\varepsilon^+-u^+)^-\,\omega_1\,dx\\
&\quad+\overline{\lim_{\varepsilon\to 0}}\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)\Big).\nabla(u_k^+-u^+)\,\omega_1\,dx\\
&\quad+\overline{\lim_{\varepsilon\to 0}}\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_k^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla(u_\varepsilon^+-u^+)\,\omega_1\,dx\\
&\le R_k+\int_\Omega \Big(\mathcal{G}(x)-\mathcal{A}(x,u,\nabla u^+)\Big).\nabla(u_k^+-u^+)\,\omega_1\,dx. \qquad (3.19)
\end{aligned}$$

Hence, using (H2) and passing to the limit $k\to\infty$ we obtain

$$R_k+\int_\Omega \Big(\mathcal{G}(x)-\mathcal{A}(x,u,\nabla u^+)\Big).\nabla(u_k^+-u^+)\,\omega_1\,dx\to 0.$$

Therefore,

$$\int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon^+)-\mathcal{A}(x,u_\varepsilon,\nabla u^+)\Big).\nabla(u_\varepsilon^+-u^+)\,\omega_1\,dx\to 0.$$

Thus, by (3.3), (3.4) and Lemma 2.7, we have

$$u_\varepsilon^+\to u^+ \quad\text{in}\quad W_0^{1,p}(\Omega,\omega_1,\omega_2). \qquad (3.20)$$

Step 3. We will prove that $u_\varepsilon^-\to u^-$ in $W_0^{1,p}(\Omega,\omega)$.

We define $u_k^-=\min\{u^-,k\}$ and $f_\varepsilon=u_\varepsilon^- - u_k^-$. We have

$$f_\varepsilon^+\in W_0^{1,p}(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega).$$

Hence, using f_ε^+ as a test function in (3.1) we obtain

$$\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla f_\varepsilon^+\,\omega_1\,dx+\int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,f_\varepsilon^+\,\omega_2\,dx=\langle T,f_\varepsilon^+\rangle.$$

Note that $H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,f_\varepsilon^+\le 0$ a.e. and μ_2-a.e.. Thus, analogously to (3.9) we obtain (with k fixed)

$$\overline{\lim_{\varepsilon\to 0}}\int_\Omega -\Big(\mathcal{A}(x,u_\varepsilon,-\nabla u_\varepsilon^-)-\mathcal{A}(x,u_\varepsilon,-\nabla u_k^-)\Big).\nabla(u_\varepsilon^- - u_k^-)^+\,\omega_1\,dx\le S_k, \qquad (3.21)$$

where $\lim_{k\to\infty} S_k = 0$. Now, considering the test function $v_\varepsilon = \varphi_\lambda(f_\varepsilon^-)$, we obtain as in (3.18) that

$$\overline{\lim_{\varepsilon\to 0}} \int_\Omega \Big(\mathcal{A}(x,u_\varepsilon,-\nabla u_\varepsilon^-) - \mathcal{A}(x,u_\varepsilon,-\nabla u_k^-)\Big).\nabla(u_\varepsilon^- - u_k^-)^-\,\omega_1\,dx \le 0. \tag{3.22}$$

Therefore, using (3.21) and (3.22), we have as in (3.20) that

$$u_\varepsilon^- \to u^- \quad \text{em} \quad W_0^{1,p}(\Omega,\omega_1,\omega_2). \tag{3.23}$$

Step 4. Convergence.

Using (3.20) and (3.23) we obtain a subsequence of $\{u_\varepsilon\}$ satisfying

$$u_\varepsilon \to u \;\text{ in }\; L^p(\Omega,\omega_2),\; \mu_2 - a.e. \text{ and } a.e., \tag{3.24}$$
$$D_j u_\varepsilon \to D_j u \;\text{ in }\; L^p(\Omega,\omega_1),\; \mu_1 - a.e. \text{ and } a.e.. \tag{3.25}$$

Using $(H5)$ we obtain

$$\begin{aligned} &H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) \to H(x,u,\nabla u) \;\; a.e. \text{ and } \;\mu_2 - a.e., \\ &H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,u_\varepsilon \to H(x,u,\nabla u)\,u \;\; a.e. \text{ and } \;\mu_2 - a.e., \end{aligned} \tag{3.26}$$

and using $(H4)$, we have $\displaystyle\int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla u_\varepsilon\,\omega_1\,dx \ge \alpha\,\|\nabla u_\varepsilon\|^p_{L^p(\Omega,\omega_1)} \ge 0$. We obtain, as in (3.2),

$$\begin{aligned} 0 &\le \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,u_\varepsilon\,\omega_2\,dx \\ &\le \int_\Omega \mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla u_\varepsilon\,\omega_1\,dx + \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,u_\varepsilon\,\omega_2\,dx \\ &= \langle T, u_\varepsilon\rangle \\ &\le \|T\|_*\|u_\varepsilon\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \\ &\le \|T\|_* C_2 = C_3. \end{aligned} \tag{3.27}$$

For $m > 0$, we define

$$X_m^\varepsilon = \{x\in\Omega : |u_\varepsilon(x)| \le m\} \;\text{ and }\; Y_m = \{x\in\Omega : |u_\varepsilon(x)| > m\}.$$

For any measurable subset $E\subset\Omega$, by $(H6)$, we have

$$\begin{aligned} &\int_E |H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx \\ &= \int_{E\cap X_m^\varepsilon} |H_\varepsilon(x,u_\varepsilon\,\nabla u_\varepsilon)|\,\omega_2\,dx + \int_{E\cap Y_m^\varepsilon} |H_\varepsilon(x,u_\varepsilon\,\nabla u_\varepsilon)|\,\omega_2\,dx \end{aligned}$$

$$\leq \int_{E\cap X_m^\varepsilon} |H_\varepsilon(x,u_\varepsilon\,\nabla u_\varepsilon)|\,\omega_2\,dx + \frac{1}{m}\int_{E\cap Y_m^\varepsilon} H_\varepsilon(x,u_\varepsilon\,\nabla u_\varepsilon)\,u_\varepsilon\,\omega_2\,dx$$
$$\leq \int_E |H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,\omega_2\,dx + \frac{C_3}{m}$$
$$\leq \beta\int_\Omega \Big(|\nabla u_\varepsilon|^p + h\Big)\,\omega_2\,dx + \frac{C_3}{m}$$
$$\leq \beta\int_\Omega |\nabla u_\varepsilon|^p\omega_1\,dx + \beta\int_\Omega h\,\omega_2\,dx + \frac{C_3}{m}. \tag{3.28}$$

Since the sequence $\{\nabla u_\varepsilon\}$ strongly converges in $(L^p(\Omega,\omega_1))^n$, then (3.28) implies the equi-integrability of $H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)$. Using (3.26) and the Vitali's Theorem (see [13]), we obtain

$$H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\to H(x,u,\nabla u) \quad \text{in} \quad L^1(\Omega,\omega_2). \tag{3.29}$$

Now, using (3.24) and (3.27), and passing to the limit $\varepsilon\to 0$ in

$$\int_\Omega \omega_1\,\mathcal{A}(x,u_\varepsilon,\nabla u_\varepsilon).\varphi\,dx + \int_\Omega H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,\varphi\,\omega_2\,dx = \langle T,\varphi\rangle,$$

for all $\varphi\in W_0^{1,p}(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega)$, we obtain

$$\int_\Omega \omega_1\,\mathcal{A}(x,u,\nabla u).\nabla\varphi\,dx + \int_\Omega H(x,u,\nabla u)\,\varphi\,\omega_2\,dx = \langle T,\varphi\rangle. \tag{3.30}$$

Moreover, since $H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,u_\varepsilon\geq 0$ a.e. and μ_2-a.e., using (3.26), (3.27) and Fatou's Lemma we have

$$H(x,u,\nabla u)\,u\in L^1(\Omega,\omega_2). \tag{3.31}$$

Therefore, by (3.29), (3.30) and (3.31), u is a solution of problem (P). □

Example 3.1. Let $\Omega = \{(x,y)\in\mathbb{R}^2 \ : x^2+y^2<1\}$, and consider the weight functions $\omega_1(x,y) = (x^2+y^2)^{-1/2}$ and $\omega_2(x,y) = (x^2+y^2)^{1/2}$ ($\omega_1,\omega_2\in A_2$), the functions $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$, and $H:\Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ defined by

$$\mathcal{A}((x,y),\eta,\xi) = g(x,y)\,\xi,$$
$$H((x,y),\eta,\xi) = |\xi|^2\sin^2(xy)\,\frac{\eta^2}{\eta^2+1}\arctan(\eta) + h(x,y)\,\frac{\eta^2}{\eta^2+1}\arctan(\eta),$$

where $g(x,y) = \mathrm{e}^{x^2+y^2}$ and $h(x,y) = (x^2+y^2)^{1/2}\cos^2(xy)$. Let us consider the partial differential operator $Lu(x,y) = -\mathrm{div}\Big[\omega_1(x,y)\mathcal{A}((x,y),u,\nabla u)\Big]$ and $T = f_0(x,y) = (x^2+y^2)^{-1/5}\cos(1/(x^2+y^2))$. Therefore, by Theorem 1.1, the problem

$$(P)\begin{cases} Lu(x,y) + H(x,u,\nabla u)\,\omega_2 = T, \\ u\in W_0^{1,2}(\Omega,\omega_1,\omega_2), \\ H(x,u,\nabla u)\in L^1(\Omega,\omega_2), \\ H(x,u,\nabla u)\,u\in L^1(\Omega,\omega_2), \end{cases}$$

has a solution $u\in W_0^{1,2}(\Omega,\omega_1,\omega_2)$.

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