Nonlinear Functional Analysis and Applications
Vol. 22, No. 2 (2017), pp. 371-384
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

CLASSIFICATION OF COMPLETENESS OF QUASI METRIC SPACE AND SOME FIXED POINT RESULTS

Ishak Altun[1], Gülhan Mınak[2] and Murat Olgun[3]

[1]King Saud University, College of Science
Riyadh, Saudi Arabia
and
Department of Mathematics, Faculty of Science and Arts
Kirikkale University, 71540 Yahsihan, Kirikkale, Turkey
e-mail: altun@ksu.edu.sa, ishakaltun@yahoo.com

[2]Department of Mathematics, Faculty of Science and Arts
Kirikkale University, 71540 Yahsihan, Kirikkale, Turkey
e-mail: g.minak.28@gmail.com

[3]Department of Mathematics, Faculty of Science
Ankara University, 06100, Tandogan, Ankara, Turkey
e-mail: olgun@ankara.edu.tr

Abstract. In the present paper, considering some different type of Cauchy sequence on a quasi metric space, we classify the completeness of quasi metric spaces. By using some recent techniques we provide new fixed point results on some kind of complete quasi metric spaces.

1. Introduction and preliminaries

Fixed point theory is one of basic subjects in topology and analysis. There are many applications of this theory in the literature, which one of the most important of them is investigated the existence and uniqueness of solutions of differential and integral equations. Therefore, an operator is determined for differential or integral equations, and one to one correspondence is constructed between the existence (and the uniqueness) of fixed point of this operator and the existence (and uniqueness) of solution of differential or integral equations.

[0]Received July 31, 2016. Revised October 26, 2016.
[0]2010 Mathematics Subject Classification: 54H25, 47H10.
[0]Keywords: Quasi metric space, left K-Cauchy sequence, left $\mathcal{K}$-completeness, fixed point.

This theory, contains the research about whether the fixed point of an operator exist, whether the fixed point is unique, if it is unique, then how to find it.

Fundamentally, fixed point theory divides into three major subjects which are topological, discrete and metric. Especially, it has been intensively improving on the metric case because of useful to applications.

The purpose of this paper is that to investigate the feasibility of the studies of metrical fixed point theory on quasi metric space, which has a comprehensive structure space and more application on computer science and semantics.

On the metric space, definition of Cauchy sequence and completeness of space depend on metric. Hence, these are indispensable terms for fixed point theory. In the studies of metrical fixed point theory, it is essential method that shows the sequence which is obtained by an iteration is Cauchy in a complete metric space.

To obtain the quasi metric version of studies of metrical fixed point theory, we need to correspondence to quasi metric version of these two fundamental concepts. However, results of researches show that there are seven different definitions of Cauchy sequence on quasi metric spaces. On the other hand, taking into account the quasi metric, the conjugate quasi metric and the metric (which is obtained by quasi metric) topology, the convergence of a sequence in quasi metric spaces can be obtained in three different ways. In this case, when studying fixed point theory on quasi metric space, it should be analyzed which Cauchy sequence and type of convergence are more appropriate to use for completeness of quasi metric space. In this paper, we will examine this analyze by classifying the completeness of quasi metric spaces, and give some new fixed point results.

Now we recall some basic concepts.

A quasi-pseudo metric on a nonempty set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

(i) $d(x, x) = 0$,

(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If a quasi-pseudo metric d satisfies

(iii) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$,

then d is said to be quasi metric, in addition if a quasi metric d satisfies

(iv) $d(x, y) = 0 \Rightarrow x = y$,

then d is said to be T_1-quasi metric. It is clear that, every metric is a T_1-quasi metric, every T_1-quasi metric is a quasi metric and every quasi metric is a quasi-pseudo metric. In this case the pair (X, d) is said to be quasi-pseudo (resp. quasi, T_1-quasi) metric space.

Let (X, d) be a quasi-pseudo metric space. Given a point $x_0 \in X$ and a real constant $\varepsilon > 0$, the sets

$$B_d(x_0, \varepsilon) = \{y \in X : d(x_0, y) < \varepsilon\}$$

and

$$B_d[x_0, \varepsilon] = \{y \in X : d(x_0, y) \leq \varepsilon\}$$

are called open ball and closed ball, respectively, with center x_0 and radius ε.

Each quasi-pseudo metric d on X generates a topology τ_d on X which has a base the family of open balls $\{B_d(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$. The closure of a subset A of X with respect to τ_d is denoted by $cl_{\tau_d}(A)$. If d is a quasi metric on X, then τ_d is a T_0 topology, and if d is a T_1-quasi metric, then τ_d is a T_1 topology on X.

If d is a quasi-pseudo metric on X, then the functions d^{-1}, d^s and d_+ defined by

$$d^{-1}(x, y) = d(y, x),$$
$$d^s(x, y) = \max\left\{d(x, y), d^{-1}(x, y)\right\}$$

and

$$d_+(x, y) = d(x, y) + d^{-1}(x, y)$$

are also quasi-pseudo metrics on X. If d is a quasi metric, then d^s and d_+ are (equivalent) metrics on X. Further, if a quasi-pseudo metric d satisfies

$$x \neq y \Rightarrow d(x, y) + d^{-1}(x, y) > 0,$$

then d_+ (and also d^s) is a metric on X.

Example 1.1. Let $X = \mathbb{R}$ and $d(x, y) = \max\left\{y^2 - x^2, 0\right\}$ for all $x, y \in X$. Then d is a quasi-pseudo metric, but not a quasi metric on X.

Example 1.2. Let $X = \mathbb{R}$ and $d(x, y) = \max\{y - x, 0\}$ for all $x, y \in X$. Then, d is a quasi metric but not a T_1-quasi metric on X. In this case τ_d is left order topology, $\tau_{d^{-1}}$ is right order topology and τ_{d^s} is usual topology on $\mathbb{R}$.

Example 1.3. Let $X = \mathbb{R}$ and

$$d(x, y) = \begin{cases} 0\,, & x = y, \\ |y|\,, & x \neq y, \end{cases}$$

for all $x, y \in X$. Then, d is a quasi metric but not a T_1-quasi metric on X. If we consider the subset $Y = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ of X, then (Y, τ_d) is cofinite topological space. Also $(Y, \tau_{d^{-1}})$ is discrete topological space.

Example 1.4. Let $X = \mathbb{R}$ and

$$d(x,y) = \begin{cases} y-x\,, & x \leq y, \\ \\ 1\,, & x > y, \end{cases}$$

for all $x, y \in X$. Then d is a T_1-quasi metric, but not a metric on X. In this case, τ_d is lower limit topology, $\tau_{d^{-1}}$ is upper limit topology and τ_{d^s} is discrete topology on $\mathbb{R}$.

Let (X, d) be a quasi metric space and $x \in X$. The convergence of a sequence $\{x_n\}$ to x with respect to τ_d called d-convergence and denoted by $x_n \stackrel{d}{\rightarrow} x$, is defined

$$x_n \stackrel{d}{\rightarrow} x \quad \Leftrightarrow \quad d(x, x_n) \rightarrow 0.$$

Similarly, the convergence of a sequence $\{x_n\}$ to x with respect to $\tau_{d^{-1}}$ called d^{-1}-convergence and denoted by $x_n \stackrel{d^{-1}}{\rightarrow} x$, is defined

$$x_n \stackrel{d^{-1}}{\rightarrow} x \quad \Leftrightarrow \quad d^{-1}(x_n, x) \rightarrow 0.$$

Finally, the convergence of a sequence $\{x_n\}$ to x with respect to τ_{d^s} called d^s-convergence and denoted by $x_n \stackrel{d^s}{\rightarrow} x$, is defined

$$x_n \stackrel{d^s}{\rightarrow} x \quad \Leftrightarrow \quad d^s(x_n, x) \rightarrow 0.$$

It is clear that $x_n \stackrel{d^s}{\rightarrow} x \Leftrightarrow x_n \stackrel{d}{\rightarrow} x$ and $x_n \stackrel{d^{-1}}{\rightarrow} x$. More and detailed information about some important properties of quasi metric spaces and their topological structures can be found in [8, 11, 12].

Definition 1.5. ([17]) Let (X, d) be a quasi metric space. A sequence $\{x_n\}$ in X is called

(1) left d-Cauchy if for every $\varepsilon > 0$, there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0,\, d(x, x_n) < \varepsilon,$$

(2) right d-Cauchy if for every $\varepsilon > 0$, there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0,\, d(x_n, x) < \varepsilon,$$

(3) left K-Cauchy (or forward Cauchy) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k,\, n \geq k \geq n_0,\, d(x_k, x_n) < \varepsilon,$$

(4) right K-Cauchy (or backward Cauchy) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k,\, n \geq k \geq n_0,\, d(x_n, x_k) < \varepsilon,$$

(5) weakly left K-Cauchy (or weakly forward Cauchy) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0,\ d(x_{n_0}, x_n) < \varepsilon,$$

(6) weakly right K-Cauchy (or weakly backward Cauchy) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0,\ d(x_n, x_{n_0}) < \varepsilon,$$

(7) d^s-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k \geq n_0,\ d(x_n, x_k) < \varepsilon.$$

Remark 1.6. The following implications are hold:

$$d^s\text{-Cauchy} \Rightarrow \text{left } K\text{-Cauchy} \Rightarrow \text{weakly left } K\text{-Cauchy} \Rightarrow \text{left } d\text{-Cauchy},$$

$$d^s\text{-Cauchy} \Rightarrow \text{right } K\text{-Cauchy} \Rightarrow \text{weakly right } K\text{-Cauchy} \Rightarrow \text{right } d\text{-Cauchy}.$$

Example 1.7 ([17])**.** Consider the quasi metric space (X, d) such that $X = [0, 1]$ and

$$d(x, y) = \begin{cases} 0\,, & x \leq y, \\ 1\,, & x > y. \end{cases}$$

Let $\{x_n\}$ be the sequence in X defined as follows:

$$x_n = \begin{cases} \frac{1}{2} + \frac{1}{2^n}\,, & \text{if } n \text{ is odd}, \\ \frac{1}{3} + \frac{1}{3^n}\,, & \text{if } n \text{ is even}. \end{cases}$$

Then, $d(0, x_n) = 0$ for all $n \geq 1$, so that $\{x_n\}$ is d-convergent and also it is left d-Cauchy. But, it is not weakly left K-Cauchy, since

$$\max\{d(x_n, x_{n+1}), d(x_n, x_{n+2})\} = 1.$$

Therefore, from Remark 1.6, it is neither left K-Cauchy nor d^s-Cauchy. On the other hand, $d(x_n, x_1) = 0$ for all $n \geq 1$, so that $\{x_n\}$ is d^{-1}-convergent and so it is right d-Cauchy and also it is weakly right K-Cauchy. But, it is not right K-Cauchy, since $d(x_{2n+1}, x_{2n}) = 1$. Let $\{y_n\}$ be the sequence in X defined by $\{y_n\} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\}$. Then it is weakly left K-Cauchy but not left K-Cauchy, since $d(y_1, y_n) = 0$ for all $n \geq 1$ and $d(y_k, y_n) = 1$ for all $n > k > 1$.

If a sequence is left Cauchy (in the sense of d, K and weakly K, respectively) with respect to d, then it is right Cauchy (in the sense of d, K and weakly K, respectively) with respect to d^{-1}.

If a sequence is d^s-Cauchy if and only if it is both left K-Cauchy and right K-Cauchy.

It is well known that every convergent sequence is Cauchy in a metric space. This situation is not valid in a quasi metric space. That is, d-convergent or d^{-1}-convergent sequences may not be Cauchy (in the sense of d^s, K or weakly K) in a quasi metric space. But, if a sequence is d-convergent, then it is left d-Cauchy. Similarly, if a sequence is d^{-1}-convergent, then it is right d-Cauchy.

Example 1.8. Consider the quasi metric space (X, d) such that $X = [0, 1)$ and $d(x, y) = \max\{y - x, 0\}$. Let $x_n = \frac{n}{n+1}$, then it is d^s-Cauchy and so it is left d-Cauchy, but not d-convergent since

$$\lim_{n\to\infty} d(x, \frac{n}{n+1}) = \lim_{n\to\infty} \max\left\{\frac{n}{n+1} - x, 0\right\} = 1 - x \neq 0$$

for all $x \in X$.

2. Classification of completeness

It is well known that, a metric space is said to be complete if every Cauchy sequence is convergent. But, the completeness of a quasi metric space can not be uniquely defined. Taking into account the convergence of a sequence and Cauchyness, we can define most of completeness, which most of them are already available in the literature (see [1, 6, 7, 8, 12, 17]) with different notations.

Definition 2.1. Let (X, d) be a quasi metric space. Then (X, d) is said to be

(1) left (right) ζ-complete if every left (right) d-Cauchy sequence is d-convergent,
(2) left (right) η-complete if every left (right) d-Cauchy sequence is d^{-1}-convergent,
(3) left (right) θ-complete if every left (right) d-Cauchy sequence is d^s-convergent,
(4) ζ-complete if every d^s-Cauchy sequence is d-convergent,
(5) η-complete if every d^s-Cauchy sequence is d^{-1}-convergent,
(6) θ-complete (or bicomplete) if every d^s-Cauchy sequence is d^s-convergent,
(7) (weakly) left (right) $\mathcal{K}$-complete if every (weakly) left (right) K-Cauchy sequence is d-convergent,
(8) (weakly) left (right) $\mathcal{M}$-complete if every (weakly) left (right) K-Cauchy sequence is d^{-1}-convergent,
(9) (weakly) left (right) Smyth complete if every (weakly) left (right) K-Cauchy sequence is d^s-convergent.

Remark 2.2. It is clear that the following implications are true:

$$\begin{aligned} \text{left } \zeta\text{-complete} &\Rightarrow \text{weakly left } \mathcal{K}\text{-complete} \\ &\Rightarrow \text{left } \mathcal{K}\text{-complete} \\ &\Rightarrow \zeta\text{-complete,} \end{aligned}$$

$$\begin{aligned} \text{right } \zeta\text{-complete} &\Rightarrow \text{weakly right } \mathcal{K}\text{-complete} \\ &\Rightarrow \text{right } \mathcal{K}\text{-complete} \\ &\Rightarrow \zeta\text{-complete.} \end{aligned}$$

Example 2.3. ([17]) Let $X = \mathbb{N}$ and

$$d(m,n) = \begin{cases} 0\,, & m = n, \\ \frac{1}{n}\,, & m > n,\ m \text{ is even, } n \text{ is odd,} \\ 1\,, & \text{otherwise.} \end{cases}$$

Since there are no non-trivial right K-Cauchy sequences, (X,d) is right $\mathcal{K}$-complete. However, the sequence $\{2,4,6,\cdots\}$ is right d-Cauchy but not d-convergent. Thus the space is not right ζ-complete. On the other hand, the same sequence is left d-Cauchy on (X,d^{-1}), but not weakly left K-Cauchy on (X,d^{-1}). Also, (X,d^{-1}) is left $\mathcal{K}$-complete, but not left ζ-complete.

Remark 2.4. ([8]) Let $\{x_n\}$ be a sequence in a quasi metric space (X,d) such that

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty,$$

then it is left K-Cauchy sequence.

In the light of the above definitions, we can give the following table:

	d-convergence	$\Leftarrow$	d^s-convergence	$\Rightarrow$	d^{-1}-convergence
left d-Cauchy	left ζ-complete	$\Leftarrow$	left θ-complete	$\Rightarrow$	left η-complete
$\Uparrow$	$\Downarrow$		$\Downarrow$		$\Downarrow$
w left K-Cauchy	w left K-complete	$\Leftarrow$	w left Smyth complete	$\Rightarrow$	w left M-complete
$\Uparrow$	$\Downarrow$		$\Downarrow$		$\Downarrow$
left K-Cauchy	left K-complete	$\Leftarrow$	left Smyth complete	$\Rightarrow$	left M-complete
$\Uparrow$	$\Downarrow$		$\Downarrow$		$\Downarrow$
d^s-Cauchy	ζ-complete	$\Leftarrow$	θ-complete (bicomplete)	$\Rightarrow$	η-complete
$\Downarrow$	$\Uparrow$		$\Uparrow$		$\Uparrow$
right K-Cauchy	right K-complete	$\Leftarrow$	right Smyth complete	$\Rightarrow$	right M-complete
$\Downarrow$	$\Uparrow$		$\Uparrow$		$\Uparrow$
w right K-Cauchy	w right K-complete	$\Leftarrow$	w right Smyth complete	$\Rightarrow$	w right M-complete
$\Downarrow$	$\Uparrow$		$\Uparrow$		$\Uparrow$
right d-Cauchy	right ζ-complete	$\Leftarrow$	right θ-complete	$\Rightarrow$	right η-complete

Remark 2.5. Note that weakly left K-Cauchy sequence may not be left K-Cauchy. But, according to the following result, weakly left $\mathcal{K}$-completeness and left $\mathcal{K}$-completeness are equivalent.

Theorem 2.6. ([18]) *A quasi metric space is weakly left $\mathcal{K}$ complete if and only if it is left $\mathcal{K}$-complete.*

Taking into account this theorem, we can reduce the table as follows:

	d-convergence	$\Leftarrow$	d^s-convergence	$\Rightarrow$	d^{-1}-convergence
left d-Cauchy	left ζ-complete	$\Leftarrow$	left θ-complete	$\Rightarrow$	left η-complete
$\Uparrow$	$\Downarrow$		$\Downarrow$		$\Downarrow$
left K-Cauchy	left $\mathcal{K}$-complete	$\Leftarrow$	left Smyth complete	$\Rightarrow$	left $\mathcal{M}$-complete
$\Uparrow$	$\Downarrow$		$\Downarrow$		$\Downarrow$
d^s-Cauchy	ζ-complete	$\Leftarrow$	θ-complete (bicomplete)	$\Rightarrow$	η-complete
$\Downarrow$	$\Uparrow$		$\Uparrow$		$\Uparrow$
right K-Cauchy	right $\mathcal{K}$-complete	$\Leftarrow$	right Smyth complete	$\Rightarrow$	right $\mathcal{M}$-complete
$\Downarrow$	$\Uparrow$		$\Uparrow$		$\Uparrow$
right d-Cauchy	right ζ-complete	$\Leftarrow$	right θ-complete	$\Rightarrow$	right η-complete

.

Remark 2.7. In spite of the fact that left d-Cauchyness is equivalent to right d^{-1}-Cauchyness, left ζ and right η-completeness are not equivalent. However, the quasi metric space (X, d) is left ζ-complete if and only if (X, d^{-1}) is right η-complete. Also, similar relations can be obtained other Cauchyness and completeness.

Remark 2.8. A quasi metric space is θ-complete (bicomplete) if and only if it is both ζ and η-complete.

Remark 2.9. If a quasi metric space (X, d) is left $\mathcal{M}$-complete, then it is η-complete. Also, if a quasi metric space (X, d) is left $\mathcal{K}$-complete, then it is ζ-complete. Therefore if (X, d) is both left $\mathcal{K}$-complete and left $\mathcal{M}$-complete, then it is θ-complete (bicomplete).

Example 2.10. Consider the quasi metric space (X, d) such that $X = [0, 1)$ and $d(x, y) = \max\{y - x, 0\}$. Then (X, d) is left (right) η-complete, η-complete and left (right) $\mathcal{M}$-complete since every sequence d^{-1}-converges to 0. On the other hand, since $d^s(x, y) = |x - y|$, then (X, d) is not θ-complete (bicomplete), therefore it is also not ζ-complete. Consider $Y = (0, 1]$ with the same quasi metric d. Then (Y, d) is ζ-complete but not η-complete.

Example 2.11. Consider the quasi metric space (X, d) such that $X = [0, 1]$ and

$$d(x, y) = \begin{cases} 0\,, & x \leq y, \\ \\ 1\,, & x > y. \end{cases}$$

Since every sequence d-converges to 0, then (X, d) is left (right) ζ-complete, ζ-complete and left (right) $\mathcal{K}$-complete.

Example 2.12. ([17]) Let X be the unit interval $(0,1)$ with the quasi metric d on X defined by

$$d(x,y) = \begin{cases} x-y\,, & x \geq y, \\ \\ 1\,, & x < y. \end{cases}$$

Consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n+1}$. Then, $d(x_k, x_n) < \frac{1}{k}$ for all $n > k$, so that $\{x_n\}$ is left K-Cauchy and hence left d-Cauchy. However, $\{x_n\}$ is not right d-Cauchy because for each point $x \in X$, $d(x_n, x) = 1$ after a certain stage. We also note that $\{x_n\}$ is not d-convergent. Similarly, the sequence $\{y_n\}$ where $y_n = 1 - \frac{1}{n+1}$ is right d-Cauchy. We observe that $\{y_n\}$ is in fact right K-Cauchy but not d^s-Cauchy, for if $n > k$ then $d(y_n, y_k) < \frac{1}{k}$ while if $n < k$ then $d(y_n, y_k) = 1$.

Let (A,d) be the subspace of (X,d), where $A = \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\}$. Then A is right ζ-complete, since every right d-Cauchy sequence in A is constant after a certain stage. On the other hand, A is not left ζ-complete, since the left d-Cauchy sequence $\left\{\frac{1}{n+1}\right\}$ is not d-convergent in A. We also observe that A is right $\mathcal{K}$-complete but not left $\mathcal{K}$-complete (since the left K-Cauchy sequence $\left\{\frac{1}{n+1}\right\}$ is not d-convergent in A). Similarly, the subspace (B,d) of (X,d), where $B = \left\{1 - \frac{1}{n+1} : n \in \mathbb{N}\right\}$, is left ζ-and $\mathcal{K}$-complete but neither right ζ- nor $\mathcal{K}$-complete since the sequence $\left\{1 - \frac{1}{n+1}\right\}$ is not d-convergent in B.

3. Fixed point result

In this section, we provide some fixed point results considering appropriate completeness of quasi metric space. First, we will mention some recent fixed point results on quasi metric space.

In [10], considering a contractive condition depending on the quasi metric d, Gaba proved a fixed point result for single valued maps on left $\mathcal{K}$-complete quasi metric space. In the same paper, there are similar fixed point results for right $\mathcal{K}$-completeness and θ-completeness (bicompleteness) of (X,d).

Also, in [2, 13, 15], considering some contractive conditions with respect to q-function, which is introduced by Al-Homidan [2], the authors proved some fixed point results for multivalued mappings on quasi metric space. When the proof of Theorem 6.1 of [2] has been investigated, it is arise that the completeness of quasi metric space is left-η-completeness. In fact, it suffices η-completeness in this theorem. Although the completeness of quasi metric space is not clear in Theorem 2.1 of [13], Latif and Al-Mezel considered the

ζ-completeness of quasi metric space. On the other hand, it has been clearly indicated which Cauchyness and which type of convergence have been used in their fixed point results of [15]. Therefore, they used in our sense left-$\mathcal{M}$-completeness and η-completeness.

As understood from recent papers [2, 13, 14, 15], it is more suitable using the w-distance or q-function (a slight generalization of w-distance) instead of the quasi metric d in contractive condition of fixed point results.

A q-function on a quasi metric space (X, d) is a function $q : X \times X \to [0, \infty)$ satisfying the following conditions:

(Q1) $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$,
(Q2) if $x \in X$, $M > 0$, and $\{y_n\}$ is a sequence in X that $\tau_{d^{-1}}$-converges to a point $y \in X$ and satisfies $q(x, y_n) \leq M$ for all $n \in \mathbb{N}$, then $q(x, y) \leq M$,
(Q3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

Note that if $q(x, y) = 0$ and $q(x, z) = 0$, then $y = z$.

It is clear that if (X, d) is a metric space, then d is a q-function on (X, d). However, Example 3.2 (Example 1.3 above) of [2] shows that there exists a quasi metric space (X, d) such that d does not satisfy condition (Q3), and hence it is not a q-function on (X, d).

Remark 3.1. ([15]) Let q be a q-function on a quasi metric space (X, d). Then, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d^s(y, z) \leq \varepsilon$.

On the other hand, the following family of functions, which is introduced by Wardowski [20], has been considered recently for fixed point theory on metric spaces (see for example [3, 4, 5, 9, 16, 19, 21]).

Let $\mathcal{F}$ be the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, *i.e.*, for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.
(F2) For each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.
(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 3.2. Let q be a q-function on a quasi metric space (X, d), $T : X \to X$ be a mapping and $F \in \mathcal{F}$. Then T is said to be an F_q-contraction if

(i) $q(x, y) = 0 \;\Rightarrow\; q(Tx, Ty) = 0$,
(ii) there exists $\tau > 0$ such that

$$\tau + F(q(Tx, Ty)) \leq F(q(x, y)), \tag{3.1}$$

for all $x, y \in X$ with $q(Tx, Ty) > 0$.

Remark 3.3. It is clear from Definition 3.2 that if T is an F_q-contraction on a quasi metric space (X, d), then T is nonexpansive with respect to q, that is,

$$q(Tx, Ty) \leq q(x, y)$$

holds for all $x, y \in X$.

Our main fixed point result as follows:

Theorem 3.4. *Let (X, d) be a left $\mathcal{M}$-complete quasi metric space, q be a Q-function on X, $T : X \to X$ be a mapping and $F \in \mathcal{F}$. If T is an F_q-contraction, then T has a unique fixed point $z \in X$. Moreover, $q(z, z) = 0$.*

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Let $q_n = q(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ with $q(x_{k-1}, x_k) = 0$, then by (i) of Definition 3.2, we get $q(Tx_{k-1}, Tx_k) = 0$. That is, $q(x_k, x_{k+1}) = 0$. Thus, $q(x_{k-1}, x_{k+1}) \leq q(x_{k-1}, x_k) + q(x_k, x_{k+1}) = 0$. Since $q(x_{k-1}, x_k) = 0$ and $q(x_{k-1}, x_{k+1}) = 0$, we have $x_k = x_{k+1}$. and so T has a fixed point. Now assume that $q_n = q(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. Then, by (ii) of Definition 3.2, we get

$$F(q_n) \leq F(q_{n-1}) - \tau \leq F(q_{n-2}) - 2\tau \leq \cdots \leq F(q_0) - n\tau. \tag{3.2}$$

From (3.2), we get $\lim_{n\to\infty} F(q_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} q_n = 0.$$

From (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} q_n^k F(q_n) = 0.$$

By (3.2), the following holds for all $n \in \mathbb{N}$,

$$q_n^k F(q_n) - q_n^k F(q_0) \leq -q_n^k n\tau \leq 0. \tag{3.3}$$

Letting $n \to \infty$ in (3.3), we obtain that

$$\lim_{n\to\infty} n q_n^k = 0. \tag{3.4}$$

From (3.4), there exits $n_1 \in \mathbb{N}$ such that $nq_n^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$q_n \leq \frac{1}{n^{1/k}}. \tag{3.5}$$

Therefore $\sum_{n=1}^{\infty} q_n < \infty$. Now let $\varepsilon > 0$ and $0 < \delta < \varepsilon$ for which condition (Q3) is satisfied. Thus there exists $n(\delta) \in \mathbb{N}$ such that

$$\sum_{n=n(\delta)}^{\infty} q_n < \delta.$$

Now, for all $n \geq n(\delta)$, we get, from (Q1)

$$\begin{aligned} q(x_{n(\delta)}, x_n) &\leq q(x_{n(\delta)}, x_{n(\delta)+1}) + q(x_{n(\delta)+1}, x_{n(\delta)+2}) + \cdots + q(x_{n-1}, x_n) \\ &= q_{n(\delta)} + q_{n(\delta)+1} + \cdots + q_{n-1} \\ &\leq \sum_{n=n(\delta)}^{\infty} q_n < \delta. \end{aligned}$$

Therefore, for all $m, n \geq n(\delta)$, we get $q(x_{n(\delta)}, x_n) < \delta$ and $q(x_{n(\delta)}, x_m) < \delta$. From Remark 3.1, we get $d^s(x_n, x_m) \leq \varepsilon$. Consequently, $\{x_n\}$ is a d^s-Cauchy and so it is left K-Cauchy sequence in the quasi metric space (X, d). Since (X, d) left $\mathcal{M}$-complete, there exists $z \in X$ such that $\{x_n\}$ is d^{-1}-convergent to z, that is, $d(x_n, z) \to 0$ as $n \to \infty$. On the other hand, for $m > n \geq n(\delta)$ we can get $q(x_n, x_m) < \delta$. Therefore by (Q2), we get $q(x_n, z) \leq \delta < \varepsilon$ and so $q(x_n, z) \to 0$ as $n \to \infty$. Hence, from Remark 3.3 we have $q(Tx_n, Tz) \to 0$ as $n \to \infty$. From Remark 3.1, we conclude that $d^s(z, Tz) = 0$, *i.e.*, $z = Tz$. Moreover, from (3.1) $q(z, z) = 0$.

Next, we show the uniqueness of the fixed point. Let w be another fixed point of T. If $q(z, w) > 0$, we have $\tau \leq F(q(z, w)) - F(q(Tz, Tw)) = 0$, a contradiction. Hence, $q(z, w) = 0$. Since $q(z, z) = 0$ and $q(z, w) = 0$, we obtain $z = w$, which is a contradiction. Hence, the fixed point of T is unique. □

Remark 3.5. Since the iterative sequence in the proof of Theorem 3.4 is d^s-Cauchy, from Remark 1.6 it is also left K-Cauchy, left d-Cauchy, right K-Cauchy and right d-Cauchy. Therefore, in the proof of Theorem 3.4, the condition that (X, d) is left $\mathcal{M}$-complete can be replaced by that (X, d) is η-complete, left η-complete, right η-complete or right $\mathcal{M}$-complete. Moreover, it can be considered the completeness which requires d^s-convergence in the proof, but it is stronger than the completeness which requires d^{-1}-convergence.

The following easy example shows that we can not consider the completeness which requires d-convergence with the same conditions in Theorem 3.4.

Example 3.6. Let $X = (0, 1]$ and $d(x, y) = \max\{y - x, 0\}$, $q(x, y) = \max\{x, y\}$, $Tx = \frac{x}{2}$ and $F(\alpha) = \ln \alpha$. Then (X, d) is a quasi metric space, q is a q-function on X, and T is F_q-contraction with $\tau = \ln 2$. Also, since every sequence d-converges to 1, then (X, d) is ζ-complete, left (right) $\mathcal{K}$-complete and left (right) ζ-complete. But T has no fixed point.

By taking into account Remark 3.5 and Example 3.6, the following questions may come to mind:

Problem 3.7. *Under what conditions, we can use one of the completeness which requires d-convergence in Theorem* 3.4*?*

Problem 3.8. *Is it possible to find a new contractive condition that the iterative sequence is left* (*right*) *K-Cauchy or left* (*right*) *d-Cauchy, but not d^s-Cauchy in Theorem* 3.4*?*

References

[1] E. Alemany and S. Romaguera, *On half-completion and bicompletion of quasi-metric spaces*, Comment. Math. Univ. Carolin., **37**(4) (1996), 749–756.

[2] S. Al-Homidan, Q.H. Ansari and J.C. Yao, *Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory*, Nonlinear Anal., **69**(1) (2008), 126–139.

[3] I. Altun, G. Mınak and H. Dağ, *Multivalued F-contractions on complete metric space*, J. Nonlinear Convex Anal., **16**(4) (2015), 659–666.

[4] I. Altun, M. Olgun and G. Mınak, *On a new class of multivalued weakly Picard operators on complete metric spaces*, Taiwanese J. Math., **19**(3) (2015), 659–672.

[5] I. Altun, M. Olgun and G. Mınak, *A new approach to the Assad-Kirk fixed point theorem*, J. Fixed Point Theory Appl., **18** (2016), 201–212.

[6] A. Andrikopoulos, *A solution to the completion problem for quasi-pseudometric spaces*, Int. J. Math. Math. Sci., 2013, Article ID 278381, 12 pp.

[7] S. Cobzaş, *Completeness in quasi-metric spaces and Ekeland Variational Principle*, Topol. Appl., **158** (2011), 1073–1084.

[8] S. Cobzaş, *Functional analysis in asymmetric normed spaces*, Springer, Basel (2013).

[9] M. Cosentino and P. Vetro, *Fixed point results for F-contractive mappings of Hardy-Rogers-type*, Filomat, **28**(4) (2014), 715–722.

[10] Y.U. Gaba, *Startpoints and (α-γ)-contractions in quasi-pseudometric spaces*, J. Math., 2014, Article ID 709253, 8 pp.

[11] H.P.A. Künzi and V. Vajner, *Weighted quasi-metrics*, Ann. New York Acad. Sci., **728** (1994), 64–67.

[12] H.P.A. Künzi, *Nonsymmetric distances and their associated topologies: about the origins of basic ideas in the area of asymmetric topology*. In: Aull, CE, Lowen, R (eds.) Handbook of the History of General Topology, vol. 3, pp. 853-968. Kluwer Academic, Dordrecht (2001).

[13] A. Latif and S.A. Al-Mezel, *Fixed point results in quasimetric spaces*, Fixed Point Theory Appl., 2011, Article ID 178306, 8 pp.

[14] J. Marín, S. Romaguera and P. Tirado, *Generalized contractive set-valued maps on complete preordered quasi-metric spaces*, J. Funct. Spaces Appl., 2013, Article. ID 269246, 6 pp.

[15] J. Marín, S. Romaguera and P. Tirado, *Q-functions on quasi-metric spaces and fixed points for multivalued maps*, Fixed Point Theory Appl., 2011, Article ID 603861, 10 pp.
[16] G. Mınak, M. Olgun and I. Altun, *A new approach to fixed point theorems for multivalued contractive maps*, Carpathian J. Math., **31**(2) (2015), 241–248.
[17] I.L. Reilly, P.V. Subrahmanyam and M.K. Vamanamurthy, *Cauchy sequences in quasi-pseudo-metric spaces*, Monatsh. Math., **93** (1982), 127–140.
[18] S. Romaguera, *Left K-completeness in quasi-metric spaces*, Math. Nachr., **157** (1992), 15–23.
[19] M. Sgrio and C. Vetro, *Multi-valued F-contractions and the solution of certain functional and integral equations*, Filomat, **27**(7) (2013), 1259–1268.
[20] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012** 2012:94, 6 pp.
[21] D. Wardowski and N. Van Dung, *Fixed points of F-weak contractions on complete metric spaces*, Demonstr. Math., **47**(1) (2014), 146–155.