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APPROXIMATE CONTROLLABILITY OF SECOND ORDER SEMILINEAR DELAY CONTROL SYSTEMS

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Abstract. In this article, we investigate some sufficient conditions, for approximate controllability of an abstract second order semilinear delay control system of the form

$$x''(t) = Ax(t) + Bu(t) + f(t, x(t + \theta), u(t)),$$

where 0 < t ≤ T and $-h \leq \theta \leq 0$; using the theory of strongly continuous cosine families. An example is provided to illustrate the theory.

1. Introduction

Let U and V be the Hilbert spaces and $Y = L_2[0, T; U]$ and $Z = L_2[0, T; V]$ be the corresponding function spaces defined on [0, T], $0 \leq T < \infty$.

In this paper, we are interested in the study of approximate controllability of the second order semilinear delay control system;

$$\begin{aligned} y''(t) &= Ay(t) + Bv(t) + f(t, y_t, v(t)); \quad 0 < t \leq T, \\ y_0(\theta) &= \phi(\theta) \quad -h \leq \theta \leq 0, \\ y'(0) &= y_0. \end{aligned} \tag{1.1}$$

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Here the state $y(t)$ and control $v(t)$ take values in V and U respectively, for each $t \in [0,T]$. Further, we assume $A : D(A) \subseteq V \to V$ is a closed linear operator whose domain D(A) is dense in V and A is the infinitesimal generator of strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ defined on V. $B : U \to V$ is a bounded linear operator and the map $f : [0,T] \times C \times U \to V$ is a nonlinear operator. Let $h > 0$ be the delay time and $C = C([-h,0];V)$ be the Banach space of all continuous functions with the norm

$$||\phi|| = \sup\{|\phi(\theta)|, -h \leq \theta \leq 0\}.$$

If y : [-h, T] →V is a continuous function then y_t is an element in C defined by

$$y_t(\theta) = y(t+\theta); \quad \forall\, \theta \in [-h,0].$$

The mild solution of the above system is given by

$$\begin{aligned} y_t(0) = y(t) &= C(t)\phi(0) + S(t)y_0 + \int_0^t S(t-s)Bv(s)\mathrm{d}s \\ &\quad + \int_0^t S(t-s)f(s,y_s,v(s))\mathrm{d}s, \\ y_0(\theta) &= \phi(\theta); \quad -h \leq \theta \leq 0, \\ y'(0) &= y_0. \end{aligned} \tag{1.2}$$

Let $y(T;\phi(0),y_0,v)$ be the state value of the system (1.1) at time T, corresponding to the control $v \in Y$ and the initial conditions $(\phi(0),y_0)$. The system (1.1) is said to be approximate controllable in the time interval [0, T] if for every desired final state y_1 and $\epsilon > 0$, there exists a control function $v \in Y$ such that the solution of (1.1) satisfies

$$||y(T,\phi(0),y_0,v) - y_1|| < \epsilon.$$

The set defined by

$$K_T(f) = \{y(T,\phi(0),y_0,v); v \in Y\}; \quad 0 \leq t \leq T \tag{1.3}$$

is called the reachable set which consists of all possible final states.

Definition 1.1. *A control system is said to be approximate controllable on* $[0,T]$*, if* $K_T(f)$ *is dense in* V*.*

Definition 1.2. *A one parameter family* $\{C(t) : t \in \mathbb{R}\}$ *of bounded linear operators in the Banach space* V *is called a strongly continuous cosine family if*

(i) *C(0) = I;* *I is the identify operator on V.*
(ii) *C(s+t) + C(s-t) = 2 C(t)C(s);* $\forall s, t \in \mathbb{R}$.
(iii) *the map* $t \mapsto C(t)x$ *is strongly continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in V$.

Now, define the associated sine family $\{S(t) : t \in \mathbb{R}\}$ by

$$S(t)x = \int_0^t C(s)xds, \quad x \in V, t \in \mathbb{R}.$$

The infinitesimal generator $A : D(A) \subseteq V \to V$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}$$

where $D(A) = \{x \in V|C(t)x$ is twice continuously differentiable function of t$\}$. For more details on strongly continuous cosine and sine families, we can refer [4, 10].

In [5], Kalman(1963) introduced the concept of controllability for a finite-dimensional linear system. Controllability of linear and nonlinear systems represented by ordinary differential equations in finite dimensional space has been extensively studied. Several authors have extended the concept to infinite-dimensional systems in Banach Space [6, 8], also see [1] for an extensive review on controllability literature.

In many cases, it is advantageous to treat the second order abstract differential equation directly rather than to convert them to first order systems. For example, Fitzgebbon [3] used the second order abstract differential equations for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. A useful tool for the study of abstract second order equation is the theory of strongly continuous cosine families. C.C. Travis and G.F. Webb [9] studied the existence, uniqueness, continuous dependence and smoothness of solution of the system

$$\begin{aligned} y''(t) &= Ay(t) + Bv(t) + f(t, y(t)); \quad 0 < t \leq T, \\ y(0) &= x_0, \\ y'(0) &= y_0, \end{aligned} \tag{1.4}$$

with $B = 0$. Park and Han [7] studied the controllability for the nonlinear second order control system (1.4) $[B \neq 0]$, by using the theory of strongly continuous cosine family and under the following assumptions

(i) The associated sine family $S(t)$ is compact.

(ii) $Bv(t)$ is continuous differentiable.

(iii) $f(t, y(t))$ is continuously differentiable function such that

$$||f(t, y(t))|| \leq K.$$

Balachandran et.al. [2] have discussed the controllability of second order nonlinear system with delay in Banach space with the help of schauder fixed point theorem.

In this paper we show the approximate controllability of the system (1.1) under simple sufficient conditions as given in the next section. The assumptions do not include the compactness of the sine family $S(t)$ or the uniform boundedness of the nonlinear function f, which are some of the main assumptions found in earlier papers [2, 7]. Moreover the system (1.4), which was considered by many authors is a particular case of the system (1.1) considered in this paper. In subsection 2.1 the controllability result is proved for the case when the control operator B is identity operator and in subsection 2.2, we consider the general case.

2. Controllability Results

2.1. Controllability of Second Order Semilinear System with Delay when $B = I$.

In this section, it is proved that under certain conditions on the nonlinear term, the approximate controllability of the linear system implies that of the semilinear system. Here, obviously $U = V$.

Consider the linear system

$$\begin{aligned} x''(t) &= Ax(t) + u(t); \quad 0 < t \leq T, \\ x_0(\theta) &= \phi(\theta); \quad -h \leq \theta \leq 0, \\ x'(0) &= y_0 \end{aligned} \tag{2.1}$$

and the semilinear system

$$\begin{aligned} y''(t) &= Ay(t) + v(t) + f(t, y_t, v(t)); \quad 0 < t \leq T, \\ y_0(\theta) &= \phi(\theta); \quad -h \leq \theta \leq 0, \\ y'(0) &= y_0. \end{aligned} \tag{2.2}$$

Assumption [2A]

(i) Linear system (2.1) is approximate controllable.

(ii) $f(t, x, u)$ satisfies the Lipschitz condition in x and u, that is,

$$||f(t, x_t, u) - f(t, y_t, v)||_V \leq l(||x_t - y_t||_C + ||u - v)||_V)$$

for some constant $l > 0$, for all $x_t, y_t \in C$, $u, v \in V$ and $t \in [0, T]$.

Theorem 2.1. *Under assumption [2A], the semilinear control system* (2.2) *is approximate controllable if the constant* l *satisfies the condition* $l < 1$.

Proof. Let $x(t)$ be a mild solution of (2.1) corresponding to a control u. Consider the following semilinear system:

$$\begin{aligned} y''(t) &= Ay(t) + f(t, y_t, v(t)) + u(t) - f(t, x_t, v(t)); \quad 0 < t \leq T, \\ y_0(\theta) &= \phi(\theta); \quad -h \leq \theta \leq 0, \\ y'(0) &= y_0. \end{aligned} \tag{2.3}$$

Comparing (2.2) and (2.3), it can be seen that the control function v is chosen such that

$$v(t) = u(t) - f(t, x_t, v(t)). \tag{2.4}$$

Let us now assume that for a given $u(t)$ and $x(t)$ there exist unique $v(t)$ satisfying the equation (2.4). We know that the mild solution of the linear system (2.1) is

$$\begin{aligned} x_t(0) &= C(t)\phi_o + S(t)y_o + \int_0^t S(t-s)u(s)ds, \\ x_0(\theta) &= \phi(\theta); \quad -h \leq \theta \leq 0, \\ x'(0) &= y_0 \end{aligned} \tag{2.5}$$

and the mild solution of the semilinear system (2.2) is

$$\begin{aligned} y_t(0) &= C(t)\phi_o + S(t)y_o + \int_0^t S(t-s)f(s, y_s, v(s))ds \\ &\quad + \int_0^t S(t-s)u(s)ds - \int_0^t S(t-s)f(s, x_s, v(s))ds \\ y_0(\theta) &= \phi(\theta); \quad -h \leq \theta \leq 0, \\ y'(0) &= y_0. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we get

$$x_t(0) - y_t(0) = \int_0^t S(t-s)\{f(s, x_s, v(s)) - f(s, y_s, v(s))\}ds \tag{2.7}$$

and

$$||x_t(0) - y_t(0)||_V \leq M \int_0^t ||f(s, x_s, v(s)) - f(s, y_s, v(s))||_V ds,$$

where M is a constant such that $||S(t)|| \leq M$ for all $t \in [0, T]$.
Applying Lipschitz condition (2A) - (ii) in the right hand side, we get

$$||x_t(0) - y_t(0)||_V \leq Ml \int_0^t ||x_s - y_s||_C ds. \tag{2.8}$$

Thus, from (2.5), (2.6) and (2.8), we get

$$||x_t(\theta) - y_t(\theta)||_V \leq Ml \int_0^t ||x_s - y_s||_C ds. \quad \forall \theta \in [-h, 0].$$

Hence,

$$||x_t - y_t||_C \leq Ml \int_0^t ||x_s - y_s||_C ds.$$

Now Gronwall's inequality implies that $x_t(0) = y_t(0)$ for all $t \in [0, T]$. Therefore every solution of the linear system with control u is also a solution of the semilinear system with control v, i.e. $K_T(f) \supset K_T(0)$. Since $K_T(0)$ is dense in V [due to condition 2A-(i)], $K_T(f)$ is also dense in V. This prove the approximate controllability of the system (2.2).

Now, we shall show that there exists a $v(t) \in V$ such that

$$v(t) = u(t) - f(t, x_t, v(t)).$$

Let $v_o \in V$ and $v_{n+1} = u - f(t, x_t, v_n) : \quad n = 0, 1, 2, \cdots$. Then, we have

$$v_{n+1} - v_n = f(t, x_t, v_{n-1}) - f(t, x_t, v_n)$$

Therefore, by condition [2A] - (ii)

$$||v_{n+1} - v_n||_V \leq l||v_n - v_{n-1}||_V \leq l^n ||v_1 - v_0||_V.$$

The R.H.S. of the above inequality tends to zero as $n \to \infty$ (because $l < 1$). Hence the $\{v_n\}$ is a Cauchy sequence in V and it converges to an element $v \in V$. Now,

$$||(u - v_{n+1}) - f(t, x_t, v)||_V = ||f(t, x_t, v_n) - f(t, x_t, v)||_V \leq l||v_n - v||_V.$$

Since R.H.S. of the above inequality tends to zero as $n \to 0$, we get

$$f(t, x_t, v) = \lim_{n\to\infty} (u - v_n) = u - v \implies v = u - f(t, x_t, v).$$

Moreover, it can be proved easily that v is unique. □

2.2. **Controllability of Second Order Semilinear System with Delay when $B \neq I$.**

In this section, we investigate the approximate controllability of the semilinear system under the simple sufficient condition on the system operator B and f.

Consider the linear system

$$\begin{aligned} x''(t) &= Ax(t) + Bu(t); \quad 0 < t \leq T, \\ x_0(\theta) &= \phi(\theta); \quad -h \leq \theta \leq 0, \\ x'(0) &= y_0 \end{aligned} \tag{2.9}$$

and the semilinear system

$$\begin{aligned} y''(t) &= Ay(t) + Bv(t) + f(t, y_t, v(t)); \quad 0 < t \leq T, \\ y_0(\theta) &= \phi(\theta); \quad -h \leq \theta \leq 0, \\ y'(0) &= y_0. \end{aligned} \tag{2.10}$$

Assumption [2B]

(i) Linear system (2.9) is approximate controllable.
(ii) Condition [2A]-(ii) is satisfied.
(iii) $R(f) \subseteq R(B)$.
(iv) There exist a constant $\beta > 0$ such that $||Bv|| \geq \beta||v|| \ \forall \ v \in U$.

Theorem 2.2. *Under assumption [2B], the semilinear control system* (2.10) *is approximate controllable if the constant l satisfies the condition $l < \beta$.*

Proof. Let $x(t)$ be a mild solution of (2.9) corresponding to a control u, then consider the following semilinear system:

$$\begin{aligned} y''(t) &= Ay(t) + f(t, y_t, v(t)) + Bu(t) - f(t, x_t, v(t)); \quad 0 < t \leq T, \\ y_0(\theta) &= \phi(\theta); \quad -h \leq \theta \leq 0, \\ y'(0) &= y_0. \end{aligned} \tag{2.11}$$

Here the control function is v which satisfies the equation $Bv(t) = Bu(t) - f(t, x_t, v(t))$, and condition [2B]-(iii) implies that this type of control is valid. By applying condition [2B]-(iv) and the method as in Theorem 2.1, it can be shown that if $l < \beta$, there exists a $v(t) \in U$ such that

$$Bv(t) = Bu(t) - f(t, x_t, v(t)).$$

The mild solutions of (2.9) and (2.11), respectively, can be written as

$$\begin{aligned} x_t(0) &= C(t)\phi_o + S(t)y_o + \int_0^t S(t-s)Bu(s)ds, \\ x_0(\theta) &= \phi(\theta); \quad -h \le \theta \le 0, \\ x'(0) &= y_0. \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} y_t(0) &= C(t)\phi_o + S(t)y_o + \int_0^t S(t-s)f(s, y_s, v(s))ds \\ &\quad + \int_0^t S(t-s)Bu(s)ds - \int_0^t S(t-s)f(s, x_s, v(s))ds, \\ y_0(\theta) &= \phi(\theta); \quad -h \le \theta \le 0, \\ y'(0) &= y_0. \end{aligned} \tag{2.13}$$

From (2.12) and (2.13), we get

$$x_t(0) - y_t(0) = \int_0^t S(t-s)\{f(s, x_s, v(s)) - f(s, y_s, v(s))\}ds,$$

which is same as equation (2.7). Now as in theorem (2.1) it can be shown that $x_t(0) = y_t(0)$ for all $t \in [0, T]$. Hence the result. □

3. EXAMPLE

Consider the following partial differential equation

$$\frac{\partial^2}{\partial t^2} y(t,x) = y_{xx}(t,x) + \mu(t,x) + f(t, y(t-h,x), u(t,x)), \tag{3.1}$$

$$\begin{array}{lll} y(t,0) = y(t,\pi) = 0 & \text{for} & t > 0, \\ y(t,x) = \phi(t,x) & \text{for} & -h \le t \le 0, \\ \frac{\partial y}{\partial t}(0,x) = y_0(x) & \text{for} & t \in [0,T],\ 0 < x < \pi. \end{array}$$

Here one can take arbitrary nonliner function f satisfying the assumption [2B]. Let $V = L^2[0, \pi]$ and $C = C([-h, 0], V)$ be as in section 1 .

Let control function $\mu : [0, T] \times (0, \pi) \to (0, \pi)$ be continuous in t and it is related to the control $v(t, x)$ as in (3.2).
Define $A : D(A) \subseteq V \to V$ by

$$Aw = w'' \qquad w \in D(A)$$

where,

$$D(A) = \{w \in V : w, w' \text{ are absolutely continuous}, w'' \in V, w(0) = w(\pi) = 0\}.$$

Then A has the spectral representation

$$Aw = \sum_{n=1}^{+\infty} -n^2 (w, w_n) w_n, \qquad w \in D(A),$$

where $w_n(s) = \sqrt{\frac{2}{\pi}} \sin ns$, $n = 1, 2, 3 \cdots$ is the orthogonal set of eigenfunctions of A. Further, it can be shown that A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ defined on V which is given by

$$C(t)w = \sum_{n=1}^{+\infty} \cos nt (w, w_n) w_n; \qquad w \in V$$

and the associated sine family is given by

$$S(t)w = \sum_{n=1}^{+\infty} \frac{1}{n} \sin nt (w, w_n) w_n, \quad w \in V.$$

Let the control operator $B : U \to V$ defined by

$$(Bv(t))(x) = \mu(t, x), x \in (0, \pi) \tag{3.2}$$

Now, the PDE (3.1) can be represented in form (1.1). Hence by section 2, the system (3.1) is approximately controllable. In [7] the approximate controllability of the system (3.1) was proved with the assumption that the nonlinear function f is uniformly bounded and $S(t)$ is compact, which are now relaxed by Theorem 2.1 and Theorem 2.2 for a class of nonlinear functions satisfying assumptions [2A] and [2B] respectively.

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