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ON PERTURBED QUASI-EQUILIBRIUM PROBLEMS WITH OPERATOR SOLUTIONS

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Abstract. In this paper, we introduce perturbed quasi equilibrium problems with operator solutions in Hausdorff topological vector spaces which contains operator quasi equilibrium problems, operator equilibrium problems and operator variational inequalities as special cases. We also established existence results for operator solutions of perturbed quasi equilibrium problems both under compactness and noncompactness assumptions.

1. Introduction

Quasi-equilibria constitute an extension of Nash equilibria, which are of fundamental importance in the theory of noncoperative games. By scalar equilibrium problem, Blume and Oettli [1] , we mean the problem of finding

$$x^* \in K \quad \text{such that} \quad f(x^*, y) \geq 0 \quad \text{for all} \quad y \in K,$$

where K is given set and $f : K \times K \to \mathbb{R}$ is a given bifunction.

In 2002, Domokos and Kolumban [7] introduced and studied a class of operator variational inequalities (In short, OVVI). These operator variational inequalities include not only scalar and vector variational inequalities as special cases, see for example [2]-[4], [8]-[12] but also have sufficent evidence for their importance to study, see [7]. Inspired by their work, in recent papers [13, 14],

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Kum and Kim developed the scheme of (OVVI) from single valued into general multi-valued settings.

Motivated and inspired by the work of Domokos and Kolumban [7], Kum and Kim [13], [14]. In this paper, we consider the perturbed quasi equilibrium problems with operator solutions and prove existence some results both under compact and noncompact assumptions.

2. Preliminaries

We begin with taking a brief look at the standard definition of continuities of multi-valued mappings. Let X and Y be nonempty topological spaces and $T : X \to 2^Y$ be a multi-valued mappings. A multi-valued map $T : X \to 2^Y$ is said to *upper semi continuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$. And a multi-valued map is said to be *lower semi continuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V = \Phi$, there exists an open neighborhood U of x in X such that$T(y) \cap V \neq \Phi$ for each $y \in U$. And T is said to be continuous if it is both *lower semi continuous*and *upper semi continuous.* It is also known that $T : X \to 2^Y$ is lower semi continuous if and only if for each closed set V in Y, the set $\{x \in X : T(x) \subset V\}$ is closed in X.

We define partial ordering $\preceq_{C_x}$ on Y by $y \preceq_{C_x} z$ if and only if $z - y \in C(x)$. So We shall write $y \prec_{C_x} z$ if and only if $z - y \in intC(x)$ in the case $intC(x) \neq \Phi$. A bi function $f : K \times K \to Y$ is said to be C_x-monotone if for each $x, y \in K$, $f(x,y) + f(y,x) \preceq_{C_x} 0$ and $g : K \to Y$ is said to be C_x- *convex*, if for each $x, y \in K$ and $\lambda \in [0,1]$, $g(\lambda y + (1-\lambda)x) \preceq_{C_x} \lambda g(y) + (1-\lambda) g(x)$ and *hemicontinuous,* if for each $x, y \in K$ and $\lambda \in [0,1]$, the mapping $\lambda \to g(\lambda y + (1-\lambda)x)$ is upper semi continuous at 0^+.

Let E be a Hausdorff topological vector space, X a nonempty convex subset of E, F is another Hausdorff topological vector space. A nonempty subset P of E is called *convex cone* if $\lambda P \subseteq P$, for all $\lambda > 0$ and $P + P = P$.

From now on, unless otherwise specified, we work under the following settings:

Let $L(E,F)$ be the space of all continuous linear operators from E to F, and X' a nonempty convex subset of $L(E,F)$, X a nonempty convex subset of E. Let $C : X' \to 2^F$ be a multivalued map such that for each $f \in X'$, $C(f)$ is convex cone in F with $0 \notin C(f)$. For a given continuous multivalued map $A : X' \to 2^{X'}$ and a bi-operator $T : X' \times X' \to F$, then *operator quasi-equilibrium problem* (in short, OQEP) is to find $f^* \in X'$ such that

$$f^* \in cl_{X'} A(f^*) \text{ and } T(f^*, g) \notin -int_F C(f^*) \text{ for all } g \in A(f^*), \tag{2.1}$$

where $cl_{X'} A(f^*)$ denotes the closure of $A(f^*)$ in X'.

In this paper, we established some existence results for the operator quasi-equilibrium problem in the case where

$$T(f, g) = G(f, g) + H(f, g).$$

That is, we consider the following perturbed operator quasi-equilibrium problem (in short, POQEP).

Find $f^* \in X'$ such that

$$f^* \in cl_{X'} A(f^*) \text{ and } G(f^*, g) + H(f^*, g) \notin -int_F C(f^*), \ \forall g \in A(f^*). \tag{2.2}$$

Now we need the following definitions:

The Graph of a multi-valued map $T : X' \subset L(E, F) \to 2^F$ denoted by $G(T)$ and is defined by

$$G(T) = \{(f, x) \in X' \times F : f \in X', \ x \in T(f)\}.$$

The inverse T^{-1} of T is the multi-valued map from $R(T)$, the range of T, to X' defined by

$$f \in T^{-1}(x) \quad \text{iff} \quad x \in T(f).$$

T is called *upper semi continuous* on X', if for each $f \in X'$ and any open set V in Y containing $T(f) \subseteq V$ for all $g \in U$.

Let $S, T : X' \to 2^{L(E,F)}$ be multivalued maps. Then the multivalued mappings clS, coS, $S \cap T : X' \to 2^F$ are defined respectively as

$$(clS)(f) = clS(f), \quad (coS)(f) = coS(f)$$

and

$$(S \cap T)(f) = S(f) \cap T(f)$$

for each $f \in X'$, where $coS(f)$ denotes the convex hull of $S(f)$.

Theorem 2.1. *Let $\Gamma = (X, A, P)$ be a 1-person game such that*

(i) *X is nonempty compact convex subset of a Hausdorff topological vector space E,*

(ii) *$A, cl_E(A) : X \to 2^X$ be a multi-valued mappings such that for each $x \in X, A(x)$ is nonempty convex set in X, for each $x \in X$, $A^{-1}(y)$ is open set in X and $cl_E A$ is upper semi continuous,*

(iii) *$P : X \to 2^X$ be a multi-valued mappings such that for each $x \in X$, $f \notin coP(x)$ and for each $y \in X, P^{-1}(y)$ is open set X.*

Then there exist $x^* \in X$ such that $x^* \in cl_X A(x^*)$ and $A(x^*) \cap P(x^*) = \Phi$.

Theorem 2.2. *Let $\Gamma = (X, A, P)$ be a 1-person game such that*

(i) *X is nonempty compact convex subset of a Hausdorff topological vector space E and X' be a nonempty compact subset of X,*
(ii) *$A : X \to 2^{X'}$ and $cl_E A : X \to 2^X$ be a multi-valued mappings such that for each $x \in X, A(x)$ is nonempty convex set, for each $y \in X$, $A^{-1}(y)$ is open set in X and $cl_E A$ is upper semi continuous,*
(iii) *$P : X \to 2^{X'}$ be a multi-valued mappings such that for each $x \in X$, $x \notin coP(x)$ and for each $y \in X'$, $P^{-1}(y)$ is open set X.*

Then there exist $x^* \in X$ such that $x^* \in cl_X A(x^*)$ and $A(x^*) \cap P(x^*) = \Phi$.

Remark 2.3. Theorem 2.1 is a special case of [[5], Theorem 2] and Theorem 2.2 is a special case of [[6], of Theorem 2].

3. Existence Results in Compact Setting

In this section, we establish some existence result for solution to the perturbed operator quasi-equilibrium problem (in short, POQEP) in compact setting.

Theorem 3.1. *Let X' is nonempty compact convex subset of $L(E, F)$. Let $H, G : X' \times X' \to F$ be vector valued bifunction and $C : X' \to 2^{X'}$ and $A : X' \to 2^{L(E,F)}$ be a multi-valued mappings. Assume that*

(i) *for each $f \in X'$, $G(f, f) = H(f, f) = 0$,*
(ii) *H is continuous in the first argument and $C(f)$-convex in the second argument,*
(iii) *G is C_f-monotone,*
(iv) *G is continuous in the first argument and $C(f)$-convex in the second argument,*
(v) *the mapping $W : X' \to 2^F$ defined by $W(f) = F \setminus (-int_F C(f))$ for each $f \in X'$, has a closed graph in $X' \times F$,*
(vi) *for each $f \in X'$, $C(f)$ is closed, convex and pointed cone in F such that $int_F C(f)$ is nonempty,*
(vii) *for each $f \in X'$, $A(f)$ is nonempty convex and for each $g \in X'$, A^{-1} is open in X'. Also $cl_{X'} A : X' \to 2^{X'}$ is upper semi continuous with compact values.*

Then there exist $f^* \in X'$ such that

$$f^* \in cl_{X'} A(f^*) \text{ and } G(f^*, g) + H(f^*, g) \notin -int_F C(f^*), \ \forall g \in A(f^*).$$

Proof. The proof follows directly follows from the following two lemmas for which the hypothesis of Theorem 3.1 remains the same. □

Lemma 3.2. *There exists $f^* \in X'$ such that*

$$f^* \in cl_{X'}A(f^*) \;\; and \;\; H(f^*, g) - G(g, f^*) \notin -int_F C(f^*), \;\; \forall\, g \in A(f^*).$$

Proof. Define a multi-valued mapping $P : X' \to 2^{X'}$ as

$$P(f) = \{g \in X' : \; H(f, g) - G(g, f) \subseteq -int_F C(f)\}, \;\; \forall\, f \in X'.$$

We show that $f \notin coP(f)$, for each $f \in X'$. In contrary we suppose that there exists there exists $f^* \in X'$ such that $f^* \in coP(f^*)$. This implies that f^* can be expressed as

$$f^* = \sum_{i \in I} \lambda_i g_i \quad \text{with} \quad \lambda_i \geq 0, \; \sum_{i \in I} \lambda_i = 1,$$

where $\{g_i : i \in N\}$ be a finite subset of $X^{'}$, $I \subset \mathbb{N}$ be arbitrary nonempty subset and $\mathbb{N}$ denotes the set of natural numbers. This follows that

$$H(f^*, g_i) - G(g_i, f^*) \in -int_F C(f^*), \quad \forall\, i = 1, 2, \cdots, n.$$

Hence

$$\sum_{i \in I} \lambda_i (H(f^*, g_i) - G(g_i, f^*)) \in -int_F C(f^*). \tag{3.1}$$

By assumption (i) and (ii), we have

$$0 = H(f^*, f^*) \; \preceq_{C(f^*)} \; \sum_{i \in I} \lambda_i H(f^*, g_i). \tag{3.2}$$

By assumption (iii) and (iv), we have

$$\begin{aligned} \sum_{i \in I} \lambda_i G(g_i, f^*) \quad & \preceq_{C(f^*)} \quad \sum_{i \in I} \sum_{j \in I} \lambda_j G(g_i, g_j) \\ & = \qquad \frac{1}{2} \sum_{i,j \in I} \lambda_i \lambda_j (G(g_i, g_j) + G(g_i, g_j)) \\ & \preceq_{C(f^*)} \quad 0. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), it follows that

$$\sum_{i \in I} \lambda_i G(g_i, f^*) \;\; \preceq_{C(f^*)} \;\; \sum_{i \in I} \lambda_i H(f^*, g_i).$$

Hence

$$\sum_{i \in I} \lambda_i (H(f^*, g_i) - G(g_i, f^*)) \in C(f^*). \tag{3.4}$$

Thus from (3.1) and (3.4), we have

$$\sum_{i \in I} \lambda_i \left(H(f^*, g_i) - G(g_i, f^*)\right) \in -int_F C(f^*) \cap (C(f^*)) \neq \Phi,$$

which is a contradiction. It remains to show that $P^{-1}(g)$ is open in X' which is equivalent to showing that $\left[P^{-1}(g)\right]^c = X' \setminus P^{-1}(g)$ is closed. Indeed we have

$$\begin{aligned} P^{-1}(g) &= \{f \in X' : g \in P(f)\} \\ &= \{f \in X' : H(f,g) - G(g,f) \in -int_F C(f^*)\}. \end{aligned}$$

Thus

$$\left[P^{-1}(g)\right]^c = \{f \in X' : H(f,g) - G(g,f) \notin -int_F C(f^*)\}.$$

Let $h \in \overline{[P^{-1}(g)]^c}$, the closure of $[P^{-1}(g)]^c$ in X'. We claim that $h \in \left[P^{-1}(g)\right]^c$. Indeed, let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a net in $\left[P^{-1}(g)\right]^c$ such that $f_\lambda \to h$. Then we have $H(f_\lambda, g) - G(g, f_\lambda) \notin -int_F C(f_\lambda)$ for each $g \in X^{'}$, that is,

$$H(f_\lambda, g) - G(g, f_\lambda) \in W(f_\lambda)$$

for all $\lambda \in \Lambda$. Since W has closed graph in $X' \times F$ and $G(.,.)$ and H(.,.) are continuous in the second and first arguments respectively, we have $H(h,g) - G(g,h) \in W(h)$, that is,

$$H(h,g) - G(g,h) \notin -int_F C(h).$$

Hence $h \in [P^{-1}(g)]^c$. From assumption(vii), it follows that all the hypothesis of Theorem 2.1 are satisfied. Hence there exists $f^* \in X'$ such that

$$f^* \in cl_{X'} A(f^*) \text{ and } A(f^*) \cap P(f^*) = \Phi.$$

Which implies that there exists $f^* \in X'$ such that

$$f^* \in cl_{X'} A(f^*) \text{ and } H(f^*, g) - G(g, f^*) \notin -int_F C(f^*), \ \forall g \in A(f^*).$$

□

Lemma 3.3. *The following statements are equivalent.*

(i) *There exists* $f^* \in X'$ *such that*

$$f^* \in cl_{X'} A(f^*) \text{ and } H(f^*, g) - G(g, f^*) \notin -int_F C(f^*), \ \forall g \in A(f^*).$$

(ii) *There exists* $f^* \in X'$ *such that*

$$f^* \in cl_{X'} A(f^*) \text{ and } G(f^*, g) + H(f^*, g) \notin -int_F C(f^*), \ \forall g \in A(f^*).$$

Proof. Let $f^* \in X'$ such that $f^* \in cl_{X'} A(f^*)$ and $H(f^*, g) - G(g, f^*) \notin -int_F C(f^*)$ holds for all $g \in A(f^*)$. Let $f_t = tg + (1-t)f^*,\ t \in (0,1]$, then

$$H(f^*, f_t) - G(f_t, f^*) \notin -int_F C(f^*). \tag{3.5}$$

Since $C(f^*)$ is a cone, we have

$$(1-t)H(f^*, f_t) - (1-t)G(f_t, f^*) \notin -int_F C(f^*).$$

Since G and H are C_f-convex in the second arguments, we have

$$tG(f_t, g) + (1-t)G(f_t, f^*) - G(f_t, f_t) \in C(f^*) \tag{3.6}$$

and

$$tH(f^*, g) + (1-t)H(f^*, f^*) - H(f^*, f_t) \in C(f^*).$$

$$t(1-t)H(f^*, g) + (1-t)(1-t)H(f^*, f^*) - (1-t)H(f^*, f_t) \in C(f^*). \tag{3.7}$$

Since $G(f, f) = 0$, $H(f, f) = 0$ for all f, thus a combination of (3.6) and (3.7) yields

$$\begin{aligned} &[tG(f_t, g) + t(1-t)H(f^*, g)] \\ &+ [(1-t)G(f_t, f^*) - (1-t)H(f^*, f_t)] \in C(f^*) \end{aligned} \tag{3.8}$$

Thus from (3.5) and (3.8), we have

$$tG(f_t, g) + t(1-t)H(f^*, g) \notin -intC(f^*).$$

Dividing by $t > 0$, we have

$$G(f_t, g) + (1-t)H(f^*, g) \notin -intC(f^*).$$

Since G is hemicontinuous in the first argument it follows that

$$G(f^*, g) + H(f^*, g) \notin -intC(f^*) \quad \text{as} \quad t \to 0^+.$$

Now let $f^* \in X'$ such that

$$f^* \in cl_{X'}A(f^*) \ \text{ and } \ G(f^*, g) + H(f^*, g) \notin -int_F C(f^*)$$

holds for all $g \in A(f^*)$. That is

$$-H(f^*, g) \ \ \preceq_{C(f^*)} \ G(f^*, g).$$

Now monotonicity of G yields,

$$G(f^*, g) + G(g, f^*) \ \ \preceq_{C(f^*)} \ \ 0.$$

Thus we have

$$G(g, f^*) - H(f^*, g) \ \ \preceq_{C(f^*)} \ \ G(f^*, g) + G(g, f^*) \ \ \preceq_{C(f^*)} \ \ 0.$$

Hence (i) holds. □

4. Existence results in noncompact setting

Now we prove the following existence results for the solution of perturbed operator quasi-equilibrium problem in noncompact setting. For this, we need the concept of the escaping sequence introduced in Border [2].

Definition 4.1. Let E be a topological vector space and X a subset of E such that $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact sets in the sense that $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$. A sequence $\{f_n\}_{n=1}^{\infty}$ in X is said to be *escaping sequence* from X (relative to $\{X_n\}_{n=1}^{\infty}$) if for each n there is an M such that $k \geq M$, $f_k \notin X_n$.

Theorem 4.2. *Let X' is nonempty subset of $L(E,F)$,and $X' = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subsets of X. Let G, H, C and A be the same as in Theorem* 3.1 *and satisfies all the conditions* (i)-(vii). *In additions, suppose that for each sequence $\{f_n\}_{n=1}^{\infty}$ in X' with $f_n \in X_n$, $n \in \mathbb{N}$ which is escaping from X' relative to $\{X_n\}_{n=1}^{\infty}$, there exists $m \in \mathbb{N}$ and $g_m \in X_m \bigcap A(f_m)$ such that for each $f_m \in cl_{X'} A(f_m)$,*

$$G(f_m, g_m) + H(f_m, g_m) \in -int_F C(f_m).$$

Then there exists $f^* \in X'$ such that

$$f^* \in cl_{X'} A(f^*) \ \text{ and } \ G(f^*, g) + H(f^*, g) \notin -int_F C(f^*), \ \ \forall\, g \in A(f^*).$$

Proof. Since for each $n \in \mathbb{N}$, X_n is compact and convex set in E, Theorem 3.1 implies that for all $n \in \mathbb{N}$, there is $f_n \in X_n$ such that

$$f_n \in cl_{X'} A(f_n) \ \text{ and } \ G(f_n, h) + H(f_n, h) \notin -int_F C(f_n), \ \ \forall\, h \in A(f_n). \quad (4.1)$$

Suppose that the sequence be $\{f_n\}_{n=1}^{\infty}$ escaping from X' relative to $\{f_n\}_{n=1}^{\infty}$. By assumption, there exists $m \in \mathbb{N}$ and $h_m \in X_m \bigcap A(f_n)$ such that for each $f_m \in cl_X A(f_m)$,

$$G(f_m, h_m) + H(f_m, h_m) \in -int_F C(f_m),$$

which contradicts (4.1). Hence $\{f_n\}_{n=1}^{\infty}$ is not an escaping sequence from X' relative to $\{X_n\}_{n=1}^{\infty}$. Therefore there exists $r \in \mathbb{N}$ and there is some subsequence $\{f_{j_n}\}$ of $\{f_n\}_{n=1}^{\infty}$ which must lies entire in X_r is compact, there is a subsequence $\{f_{i_n}\}_{i_n \in \Lambda}$ of $\{f_{j_n}\}$ in f_r such that $f_{i_n} \to f^*$, where $i_n \to \infty$. Since $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence, for all $g \in X$ there exists $i_0 \in \Lambda$ with $i_0 > r$ such that $g \in X_{i_0}$ for all $i_n \in \Lambda$ and $i_n > i_0$, we have $g \in X_{i_0} \subset X_{i_n}$ such that $f_{i_n} \in cl_{X'} A(f_{i_n})$ and $G(f_{i_n}, g) + H(f_{i_n}, g) \notin -int_F C f_{i_n})$. Since G and H are continuous in second and first argument, respectively. Using upper semicontinuity of $cl_X A$, assumption (v) and Theorem 2.2, we see that $f^* \in X'$ such that

$$f^* \in cl_{X'}A(f^*) \text{ and } G(f^*, g) + H(f^*, g) \notin -intC(f^*).$$

Hence the result is proved. □

Theorem 4.3. *Let X' be a nonempty convex subset of a locally convex Hausdorff topological vector space E and X' be a nonempty compact subset of X. Let F be an ordered Hausdorff topological vector space. Let $G, H : X' \times X' \to F$ be a vector-valued bifunction, $T : X' \to 2^{L(E,F)}$ a multi-valued mapping with the compact values and $C : X' \to 2^F$ a multi-valued mapping such that for each $f \in X'$, $C(f)$ is closed, convex and pointed cone in F with $int_F C(f) \neq \Phi$. Let A, $cl_{X'}A : X' \to 2^F$ be a multi-valued mappings such that for each $f \in X'$, $A(f)$ is nonempty, for each $g \in X'$, $A^{-1}(g)$ is open in X' and $cl_{X'}A$ is upper semicontinuous. Suppose that conditions* (i)-(iii) *of Theorem* 3.1*. Then there exists $f^* \in X'$ such that for all $g \in A(f^*)$ there exists $t^* \in T(f^*)$ such that*

$$f^* \in cl_X A(f^*) \quad \textit{and} \quad \langle t^*, g - f^* \rangle + H(f^*, g) \notin -int_F C(f^*).$$

Proof. Define a multi-valued mapping $P : X' \to 2^{X'}$ as

$$P(f) = \{g \in X : \ G(f, g) + H(f, g) \in -int_F C(f)\}, \quad \forall f \in X'.$$

Then using the argument similar to those used in proving Theorem 3.1, we have $f \notin coP(f)$ for each $f \in X'$ and $P^{-1}(g)$ is open for each $g \in X'$. Thus all the conditions of Theorem 2.2 are satisfied. Hence there exists $f^* \in X'$ such that

$$f^* \in cl_X A(f^*) \quad \text{and} \quad A(f^*) \cap P(f^*) = \Phi.$$

Which implies that there exists $f^* \in X'$such that

$$f^* \in cl_{X'}A(f^*) \quad \text{and} \quad H(f^*, g) + H(f^*, g) \notin -int_F C(f^*), \ \ \forall g \in A(f^*).$$

Hence the proof is complete. □

References

[1] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud., **63** (1994), 123–145.

[2] K.C. Border, *Fix Point Theorems with Applications to Economics and game Theory*, Cambridge University Press, Cambridge (1985).

[3] G.Y. Chen, *Existence of solutions for a vector variational inequality: an extension of Hartman- Stampachhia theorem*, Journal of Optim. Theory and Appl., **74** (1992), 445–456.

[4] X.P. Ding, *The generalized vector quasi variational-like inequalities*, Computers Math. Appl., **37** (1999), 57–67.

[5] X.P. Ding, W.K. Kim and K.K. Tan, *Equilibria of generalized games with L- majorized preferences*, Int. J. Math. Math. Sci., **17** (1994), 783–790.
[6] X.P. Ding, W.K. Kim and K.K. Tan, *Equilibria of noncompact generalized games with L- majorized preferences*, J. Math. Anal. Appl., **164** (1992), 508–517.
[7] A. Domokos and J. Kolumban, *Variational inequalities with operator solutions*, J. Global Optim., **23** (2002), 99–110.
[8] F. Giannessi, *Theorems of alternatives, quadratic programms and complimentarity problems*, In Cottle, R.W., Giannessi, F and lions, J.L.(eds.), Variational and Complimentarity Problems, John Wiley and Sons, Chichester, England, pp.151–186.
[9] A. Khaliq, *Generalized nonlinear vector quasi variational-like inequalities*, Nonlinear Analysis Forum, **8** (2003), 115–126.
[10] A. Khaliq and S. Krishan, *Vector quasi equilibrium problems*, Bull. Aust. Math. Soc., **68** (2003), 295–302.
[11] P.Q. Khanh and L.M. Luu, *On the Existence of Solutions to vector quasi-variational inequalities and quasi compolimentarity with applications to traffic network equilibria*, J. Optim. Theory and Appl., **123** (2004), 533–548.
[12] I.V. Konnov and J.C. Yao, *On the generalized vector variational inequality problem*, J. Math. Anal. Appl., **206** (1997), 42–58.
[13] S. Kum and W.K. Kim, *Generalized variational and quasi-variational inequalities with operator solutions in a toplogical vector space*, J. Optim. Theory and Appl., **32** (2005), 581–595.
[14] S. Kum and W.K. Kim, *Applications of generalized vector variational and quasi-variational inequalities with operator solutions*, J. Global Optim., **133** (2007), 65–75.