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A GENERALIZED CONTRACTION PRINCIPLE WITH CONTROL FUNCTION ON M-METRIC SPACES

Hossein Monfared[1], Mahdi Azhini[2] and Mehdi Asadi[3]

[1]Department of Mathematics,
Science and Research Branch, Islamic Azad University, Tehran, Iran
e-mail: `monfared.h@gmail.com`

[2]Department of Mathematics,
Science and Research Branch, Islamic Azad University, Tehran, Iran
e-mail: `m.azhini@srbiau.ac.ir`

[3]Department of Mathematics,
Zanjan Branch, Islamic Azad University, Zanjan, Iran
e-mail: `masadi@iauz.ac.ir`

Abstract. M-metric spaces were introduced by Asadi *et al.* [3] in 2014 as a part of the study of denotational semantics of data flow networks. In this paper, we prove a generalized contraction principle functions φ and ψ on M-metric spaces. The theorems we prove generalize many previously obtained results.

1. Introduction

The notion of metric space was introduced by Fréchet [8] in 1906. Later, many authors attempted to generalize the notion of metric space such as pseudo metric space, quasi metric space, semi metric spaces. In this paper, we consider another generalization of a metric space, so called M-metric space. This notion was introduced by Asadi *et al.* (see e.g. [3, 4, 5, 13]) to solve some difficulties in domain theory of computer science. The concept of the metric space was applied to domain theory problems by Khan [9] in 1974. By using Baire metric, Khan [9] modeled a parallel computation consisting of a set that sends unending streams of information. Basically, he modeled a

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[0]Corresponding Author: Mehdi Asadi

computation program that was based on an infinite sequence, but in computer science, an infinite sequence corresponding to unterminated programs. During the last decades many authors focused on a generalization of Banach contraction mapping principle. After the appearance of partial metric spaces as a place for distinct research work in to flow analysis, non-symmetric topology and domain theory [11, 12], some authors started to generalize this principle to these spaces [1, 2, 14, 16]. In this paper we prove fixed point theorem for a wide general type contractive mappings in M-metric spaces and hence generalize some previously obtained results such as those in [6, 7, 10, 15].

The following notations are useful in the sequel.

(1) $m_{xy} := \min\{m(x,x), m(y,y)\} = m(x,x) \vee m(y,y)$,
(2) $M_{xy} := \max\{m(x,x), m(y,y)\} = m(x,x) \wedge m(y,y)$.

Definition 1.1. ([3]) Let X be a non empty set. A function $m : X \times X \to \mathbb{R}^+$ is called a m-metric if the following conditions are satisfied:
(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$,
(m2) $m_{xy} \leq m(x,y)$,
(m3) $m(x,y) = m(y,x)$,
(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.
Then the pair (X, m) is called a M-metric space.

Remark 1.2. ([3]) For every $x, y \in X$
(1) $0 \leq M_{xy} + m_{xy} = m(x,x) + m(y,y)$,
(2) $0 \leq M_{xy} - m_{xy} = |m(x,x) - m(y,y)|$,
(3) $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

The next examples state that m^s and m^w are ordinary metric.

Example 1.3. ([3]) Let m be a m-metric. Put
(1) $m^w(x,y) = m(x,y) - 2m_{xy} + M_{xy}$,
(2) $m^s(x,y) = m(x,y) - m_{xy}$ when $x \neq y$ and $m^s(x,y) = 0$ if $x = y$.
Then m^w and m^s are ordinary metrics.

In the following example we present an example of a m-metric which is not p-metric.

Remark 1.4. ([3]) For every $x, y \in X$
(1) $m(x,y) - M_{xy} \leq m^w(x,y) \leq m(x,y) + M_{xy}$,
(2) $(m(x,y) - M_{xy}) \leq m^s(x,y) \leq m(x,y)$.

Example 1.5. ([3]) Let $X = \{1, 2, 3\}$. Define

$$m(1, 2) = m(2, 1) = m(1, 1) = 8,$$

$$m(1, 3) = m(3, 1) = m(3, 2) = m(2, 3) = 7, \quad m(2, 2) = 9, \quad m(3, 3) = 5,$$

so m is m-metric but m is not p-metric. Since $m(2, 2) \not\leq m(1, 2)$ means m is not p-metric. If $D(x, y) = m(x, y) - m_{x,y}$, then $m(1, 2) = m_{1,2} = 8$ but it means $D(1, 2) = 0$ while $1 \neq 2$ which means D is not metric.

Example 1.6. ([3]) Let (X, d) be a metric space and $\phi : [0, \infty) \to [\phi(0), \infty)$ be an one to one and nondecreasing or strictly increasing mapping with $\phi(0) \geq 0$ is defined, such that

$$\phi(x + y) \leq \phi(x) + \phi(y) - \phi(0), \quad \forall x, y \geq 0.$$

Then $m(x, y) = \phi(d(x, y))$ is a m-metric.

Example 1.7. ([3]) Let (X, d) be a metric space. Then $m(x, y) = ad(x, y) + b$ where $a, b > 0$ is a m-metric, because put $\phi(t) = at + b$.

Lemma 1.8. ([3]) *Every p-metric is an M-metric.*

2. Topology for M-metric space

It is clear that each m-metric p on X generates a T_0 topology τ_m on X. The set

$$\{B_m(x, \varepsilon) : x \in X, \;\; \varepsilon > 0\},$$

where

$$B_m(x, \varepsilon) = \{y \in X : m(x, y) < m_{x,y} + \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$, forms the base of τ_m.

Definition 2.1. ([3]) Let (X, m) be a M-metric space. Then:

(1) A sequence $\{x_n\}$ in a M-metric space (X, m) converges to a point $x \in X$ if and only if

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n,x}) = 0. \tag{2.1}$$

(2) A sequence $\{x_n\}$ in a M-metric space (X, m) is called a m-Cauchy sequence if

$$\lim_{n,m\to\infty} (m(x_n, x_m) - m_{x_n,x_m}) \quad \text{and} \quad \lim_{n,m\to\infty} (M_{x_n,x_m} - m_{x_n,x_m}) \tag{2.2}$$

are exist and finite.

(3) A M-metric space (X,m) is said to be complete if every m-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_m, to a point $x \in X$ such that

$$\lim_{n\to\infty}(m(x_n,x)-m_{x_n,x})=0 \quad \text{and} \quad \lim_{n\to\infty}(M_{x_n,x}-m_{x_n,x})=0.$$

Lemma 2.2. ([3]) *Let (X,m) be a M-metric space. Then:*

(1) *$\{x_n\}$ is a m-Cauchy sequence in (X,m) if and only if it is a Cauchy sequence in the metric space (X,m^w).*

(2) *A M-metric space (X,m) is complete if and only if the metric space (X,m^w) is complete. Furthermore,*

$$\lim_{n\to\infty} m^w(x_n,x)=0$$
$$\Longleftrightarrow \left[\lim_{n\to\infty}(m(x_n,x)-m_{x_n,x})=0 \ \ \textit{and} \ \ \lim_{n\to\infty}(M_{x_n,x}-m_{x_n,x})=0\right].$$

Likewise above definition holds also for m^s.

Lemma 2.3. ([3]) *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in a M-metric space (X,m). Then*

$$\lim_{n\to\infty}(m(x_n,y_n)-m_{x_n,y_n})=m(x,y)-m_{xy}.$$

Proof. We have

$$\begin{aligned}&|(m(x_n,y_n)-m_{x_n,y_n})-(m(x,y)-m_{x,y})|\\ &\leq (m(x_n,x)-m_{x_n,x})+(m(y,y_n)-m_{y,y_n}).\end{aligned}$$

□

From Lemma 2.3 we can deduce the following lemma.

Lemma 2.4. ([3]) *Assume that $x_n \to x$ as $n \to \infty$ in a M-metric space (X,m). Then*

$$\lim_{n\to\infty}(m(x_n,y)-m_{x_n,y})=m(x,y)-m_{x,y},$$

for all $y \in X$.

Lemma 2.5. ([3]) *Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in a M-metric space (X,m). Then $m(x,y)=m_{xy}$. Further if $m(x,x)=m(y,y)$, then $x=y$.*

Proof. By Lemma 2.3, we have

$$0=\lim_{n\to\infty}(m(x_n,x_n)-m_{x_n,x_n})=m(x,y)-m_{xy}.$$

□

3. Main result

Theorem 3.1. *Let (X, m) be a complete M-metric space and $T : X \to X$ be a self-mapping satisfying*

$$\psi(m(Tx, Ty)) \leq \psi(m(x, y)) - \phi(m(x, y)), \quad \forall\, x, y \in X, \tag{3.1}$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing function with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof. Fix $x_0 \in X$ and define $x_n = T^n x_0$ for every $n = 1, 2, 3, \cdots$. We shall prove that

$$m(x_n, x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty.$$

We have

$$\begin{aligned} \psi(m(x_n, x_{n+1})) &= \psi(m(Tx_{n-1}, Tx_n)) \\ &\leq \psi(m(x_{n-1}, x_n)) - \phi(m(x_{n-1}, x_n)). \end{aligned} \tag{3.2}$$

So we have

$$\psi(m(x_n, x_{n+1})) \leq \psi(m(x_{n-1}, x_n)).$$

Together with that ψ is nondecreasing implies that the sequence $\{m(x_n, x_{n+1})\}$ is monotone decreasing and hence there is an $m \geq 0$ such that

$$m(x_n, x_{n+1}) \to m \quad \text{as} \quad n \to \infty.$$

Letting $n \to \infty$ in (3.2) and by continuity of ψ and ϕ we obtain

$$\psi(m) \leq \psi(m) - \phi(m).$$

Which is a contradiction unless $m = 0$. Hence,

$$m(x_n, x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty.$$

Now we want to show that $\{x_n\}$ is an m-Cauchy sequence in (X, m) so by Lemma 2.2 we will prove that $\{x_n\}$ is a Cauchy sequence in (X, m^w). But we have

(1) $\lim_{n\to\infty} m(x_n, x_{n+1}) = 0$,
(2) $0 \leq m_{x_n, x_{n+1}} \leq m(x_n, x_{n+1}) \Rightarrow \lim_{n\to\infty} m_{x_n, x_{n+1}} = 0$,
(3) $m_{x_n, x_{n+1}} = \min\{m(x_n, x_n), m(x_{n+1}, x_{n+1})\} \Rightarrow \lim_{n\to\infty} m(x_n, x_n) = 0$.

On the other hand

$$m_{x_n, x_m} = \min\{m(x_n, x_n), m(x_m, x_m)\} \quad \Rightarrow \quad \lim_{n,m\to\infty} m_{x_n, x_m} = 0$$

and

$$M_{x_n, x_m} = \max\{m(x_n, x_n), m(x_m, x_m)\} \quad \Rightarrow \quad \lim_{n,m\to\infty} M_{x_n, x_m} = 0.$$

Assume that $\{x_n\}$ is not Cauchy in (X, m^w). Then there exist some $\epsilon > 0$ for sub sequences $\{x_{l_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > l_k > k$ such that

$$m^w(x_{l_k}, x_{n_k}) \geq \epsilon.$$

Now corresponding to l_k, we can choose n_k such that it is the smallest integer with $n_k > l_k$ and satisfying above inequality. Hence

$$m^w(x_{l_k}, x_{n_k-1}) < \epsilon.$$

So we have

$$\begin{aligned}\epsilon &\leq m^w(x_{l_k}, x_{n_k}) \\ &\leq m^w(x_{l_k}, x_{n_k-1}) + m^w(x_{n_k-1}, x_{n_k}) \\ &< \epsilon + m^w(x_{n_k-1}, x_{n_k}). \end{aligned} \tag{3.3}$$

We know that

$$m^w(x_{n_k-1}, x_{n_k}) = m(x_{n_k-1}, x_{n_k}) - 2m_{x_{n_k-1}, x_{n_k}} + M_{x_{n_k-1}, x_{n_k}}. \tag{3.4}$$

Now by (3.3) and (3.4), we have

$$\lim_{k\to\infty} m^w(x_{n_k-1}, x_{n_k}) = 0 \quad \text{and} \quad \lim_{k\to\infty} m^w(x_{l_k}, x_{n_k}) = \epsilon. \tag{3.5}$$

Again,

$$m^w(x_{n_k}, x_{l_k}) \leq m^w(x_{n_k}, x_{n_k-1}) + m^w(x_{n_k-1}, x_{l_k-1}) + m^w(x_{l_k-1}, x_{l_k}), \tag{3.6}$$

$$m^w(x_{n_k-1}, x_{l_k-1}) \leq m^w(x_{n_k-1}, x_{n_k}) + m^w(x_{n_k}, x_{l_k}) + m^w(x_{l_k}, x_{l_k-1}). \tag{3.7}$$

Letting $k \to \infty$ in the above inequalities (3.6) and (3.7), we have

$$\lim_{k\to\infty} m^w(x_{n_k-1}, x_{l_k-1}) = \epsilon,$$

so we have

$$\begin{aligned}&\lim_{k\to\infty} m(x_{n_k-1}, x_{l_k-1}) \\ &= \lim_{k\to\infty} (m(x_{n_k-1}, x_{n_k-1} - 2m_{x_{n_k-1}, x_{n_k-1}} + M_{x_{n_k-1}, x_{n_k-1}}) \\ &= \lim_{k\to\infty} m^w(x_{n_k-1}, x_{n_k-1}) = \epsilon.\end{aligned}$$

Now by (3.1), we have

$$\begin{aligned}\psi(\epsilon) &= \lim_{k\to\infty} \psi(m(x_{n_k}, x_{l_k})) \\ &\leq \lim_{k\to\infty} [\psi(m(x_{n_k-1}, x_{l_k-1})) - \phi(m(x_{n_k-1}, x_{l_k-1}))],\end{aligned}$$

therefore

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon).$$

Which is a contradiction, thus $\{x_n\}$ is a Cauchy sequence in complete metric space (X, m^w) and so $\{x_n\}$ is an m-Cauchy sequence in complete M-metric space (X, m). Hence there exist some $v \in X$ such that

$$\lim_{n\to\infty} (m(x_n, v) - m_{x_n,v}) = 0.$$

But we have $\lim_{n\to\infty} m_{x_n,v} = 0$, hence $\lim_{n\to\infty} m(x_n, v) = 0$ and by Remark 1.2, $m(v, v) = 0$. Now we want to show that v is the fixed point of T. By (3.1) we have

$$0 \le \psi(m(Tv, Tv)) \le \psi(m(v, v)) - \phi(m(v, v)) = \psi(0) - \phi(0) = 0.$$

And hence

$$\psi(m(Tv, Tv)) = 0 \quad \Rightarrow \quad m(Tv, Tv) = 0.$$

On the other hand

$$\psi(m(x_n, Tv)) \le \psi(m(x_{n-1}, v)) - \phi(m(x_{n-1}, v)).$$

Then letting $n \to \infty$ above and making use of Lemma 2.4 and continuity of functions ψ and ϕ we have

$$m(v, Tv) = 0.$$

Hence, we have

$$m(v, v) = m(Tv, Tv) = m(v, Tv) = 0,$$

so by (m1) we have $Tv = v$. Now let $u, v \in X$ and both of them are the fixed points of T. We have $m(u, u) = m(v, v) = 0$, because if $m(v, v) > 0$, by (3.1)

$$\psi(m(v, v)) = \psi(m(Tv, Tv)) \le \psi(m(v, v)) - \phi(m(v, v)) < \psi(m(v, v))$$

and since ψ is monotone nondecreasing we get $m(v, v) < m(v, v)$, which is contradiction. Similarly we obtain $m(u, u) = 0$.

On the other hand, if $m(v, u) > 0$ then by (3.1) we have

$$\psi(m(v, u)) = \psi(m(Tv, Tu)) \le \psi(m(v, u)) - \phi(m(v, u)) < \psi(m(v, u),$$

again by monotone nondecreasing of ψ we have $m(v, u) < m(v, u)$, which is contradiction. Thus $m(v, u) = 0$. Now we get

$$m(v, v) = m(u, u) = m(v, u) = 0.$$

Hence by (m1) $u = v$. □

Remark 3.2. If we take $\phi(t) = (1 - k)\psi(t)$, where $0 < k < 1$, then we obtain the M-metric generalization of the result in [10].

Remark 3.3. If we take $\psi(t) = t$, then we obtain the M-metric generalization for the weakly contractive fixed point theorem in [15].

Remark 3.4. All of Theorems and Lemmas are satisfying in p-metric and metric spaces, since by Lemma 1.8 every p-metric and metric spaces are M-metric space.

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