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# LINEAR APPROXIMATION AND ASYMPTOTIC EXPANSION ASSOCIATED TO THE ROBIN-DIRICHLET PROBLEM FOR A NONLINEAR WAVE EQUATION WITH THE SOURCE TERM CONTAINING AN UNKNOWN BOUNDARY VALUE

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**Abstract.** In this paper, we consider the Robin-Dirichlet problem for a nonlinear wave equation with the source term containing an unknown boundary value. Using the Faedo-Galerkin method and the linearization method for nonlinear term, the existence and uniqueness of a weak solution are proved. An asymptotic expansion of high order in a small parameter of a weak solution is also discussed.

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## 1. Introduction

In this paper, we consider the Robin-Dirichlet problem for a nonlinear wave equation with the source term containing an unknown boundary value as follows

$$u_{tt} - u_{xx} = f\left(x,t,u(x,t),u(0,t)\right), \quad 0<x<1, \quad 0<t<T, \tag{1.1}$$

$$u_x(0,t) - h_0 u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x), \qquad u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where $f$, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $h_0 \geq 0$ is a given constant.

Eq. (1.1) has a constructive relationship to a more general equation, namely

$$u_{tt} - \Delta u = F(x,t,u,u_x,u_t,u(0,t)), \quad 0<x<1, \quad 0<t<T. \tag{1.4}$$

In some special cases, when the nonlinear term has the simple forms, Eq. (1.4), with various boundary conditions, has been extensively studied by many authors, for example, we refer to [1]-[4], [6]-[13], [15] and the references given therein. In these works, many interesting results about existence, regularity, asymptotic behavior, asymptotic expansion, and decay of solutions were obtained.

In [6], Long and Diem has studied Prob. (1.3), (1.4) with the nonlinear term

$$F = f\left(x,t,u,u_x,u_t\right) + \varepsilon g\left(x,t,u,u_x,u_t\right), \tag{1.5}$$

associated with the mixed homogeneous boundary conditions

$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0. \tag{1.6}$$

In the case of $f \in C^2\left([0,1]\times[0,\infty)\times\mathbb{R}^3\right)$ and $g \in C^1\left([0,1]\times[0,\infty)\times\mathbb{R}^3\right)$, an asymptotic expansion of order 2 in $\varepsilon$ is obtained, for $\varepsilon$ sufficiently small.

In [4], Ficken and Fleishman established the unique global existence and stability of solutions for Prob. (1.3), (1.4) as follows

$$u_{xx} - u_{tt} - 2\alpha u_t - \beta u = \varepsilon u^3 + \gamma, \ \varepsilon > 0. \tag{1.7}$$

Rabinowitz [12] proved the existence of periodic solutions for the equation

$$u_{xx} - u_{tt} - 2\alpha u_t = \varepsilon f(x,t,u,u_x,u_t), \tag{1.8}$$

where $\varepsilon$ is a small parameter and $f$ is periodic in time.

In a paper of Caughey and Ellison [3], a unified approach to the previous cases was presented discussing the existence, uniqueness and asymptotic stability of classical solutions for a class of nonlinear continuous dynamical systems.

In the case $F = f\left(x,t,u,\int_0^1 g(u(y,t))dy\right)$ with $g(u) = u^2$, a high order iterative scheme was established in order to get a convergent sequence at a rate of order $N$ $(N \geq 1)$ to a local unique weak solution of a nonlinear wave equation

$$u_{tt} - u_{xx} = f(x, t, u, \|u\|^2), \quad 0 < x < 1, \quad 0 < t < T, \tag{1.9}$$

associated with the Dirichlet boundary conditions [8].

In [11], the authors considered a one dimentional nonlocal nonlinear strongly damped wave equation with dynamical boundary conditions. In other word, they looked to the following problem:

$$\begin{cases} u_{tt} - u_{xx} - \alpha u_{txx} + \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) = 0, \\ u(0,t) = 0, \\ u_{tt}(1,t) = -\varepsilon\left[u_x(1,t) + \alpha u_{tx}(1,t) + r u_t(1,t)\right] \\ \qquad\qquad -\varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right), \end{cases} \tag{1.10}$$

with $x \in (0,1), t > 0, \alpha, r > 0$ and $\varepsilon \geq 0$. Pro. (1.10) models a spring-mass-damper system, where the term $\varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right)$ represents a control acceleration at $x = 1$. By using the invariant manifold theory, the authors proved that for small values of the parameter $\varepsilon$, the solution of (1.10) attracted to a two dimensional invariant manifold.

The aforementioned works lead to the study of the existence, asymptotic expansion for Robin-Dirichlet problem for a nonlinear wave equation with the source term containing an unknown boundary value (1.1)-(1.3). The paper consists of four sections. In Section 2, we present some preliminaries. In Section 3, we associate with Prob. (1.1)-(1.3) a linear recurrent sequence which is bounded in a suitable space of functions. The existence of a local weak solution and the uniqueness are proved by using the Faedo-Galerkin method and the weak compact method. In Section 4, we establish an asymptotic expansion of a weak solution $u_\varepsilon(x,t)$ of order $N+1$ in a small parameter $\varepsilon$ for the equation

$$u_{tt} - u_{xx} = f\left(x, t, u(x,t), \varepsilon u(0,t)\right) + \varepsilon f_1\left(x, t, u(x,t), \varepsilon u(0,t)\right), \tag{1.11}$$

$0 < x < 1$, $0 < t < T$, associated to (1.2), (1.3). The results obtained here may be considered as a relative generalization of the results obtained in [6]-[13], [15].

## 2. Preliminaries

Put $\Omega = (0,1)$. We will omit the definitions of the usual function spaces and denote them by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in $L^2$ or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in $L^2$ and we denote by $\|\cdot\|_X$ the norm in the Banach space $X$. We call $X'$ the dual space of $X$. We denote $L^p(0,T;X)$, $1 \leq p \leq \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \underset{0<t<T}{ess\sup} \|u(t)\|_X , & \text{if} \quad p = \infty. \end{cases}$$

With $f \in C^k([0,1] \times \mathbb{R}_+ \times \mathbb{R}^2)$, $f = f(x,t,y_1,y_2)$, we put

$$D_1 f = \frac{\partial f}{\partial x}, \quad D_2 f = \frac{\partial f}{\partial t}, \quad D_{i+2} f = \frac{\partial f}{\partial y_i}, \quad i = 1,2$$

and

$$D^\alpha f = D_1^{\alpha_1} \cdots D_4^{\alpha_4} f, \quad \alpha = (\alpha_1, \cdots, \ \alpha_4) \in \mathbb{Z}_+^4,$$
$$|\alpha| = \alpha_1 + \cdots + \alpha_4 = k, \quad D^{(0,\cdots,0)} f = f.$$

On $H^1$, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}.$$

We put

$$V = \{v \in H^1 : v(1) = 0\}, \tag{2.1}$$

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + h_0 u(0) v(0), \ u, v \in V. \tag{2.2}$$

$V$ is a closed subspace of $H^1$ and on $V$ three norms $\|v\|_{H^1}$, $\|v_x\|$ and $\|v\|_a = \sqrt{a(v,v)}$ are equivalent norms.

We have the following lemmas, the proofs of which are straightforward hence we omit the details.

**Lemma 2.1.** *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2} \|v\|_{H^1}, \quad \forall \, v \in H^1. \tag{2.3}$$

**Lemma 2.2.** *Let $h_0 \geq 0$. Then the imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\begin{cases} \|v\|_{C^0(\overline{\Omega})} \leq \|v_x\| \leq \|v\|_a, \\ \frac{1}{\sqrt{2}} \|v\|_{H^1} \leq \|v_x\| \leq \|v\|_a \leq \sqrt{1+h_0} \|v\|_{H^1}, \end{cases}$$

*for all $v \in V$.*

**Lemma 2.3.** *Let $h_0 \geq 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.2) *is continuous on $V \times V$ and coercive on $V$.*

**Lemma 2.4.** *Let $h_0 \geq 0$. Then there exists the Hilbert orthonormal base $\{\widetilde{w}_j\}$ of $L^2$ consisting of the eigenfunctions $\widetilde{w}_j$ corresponding to the eigenvalue $\lambda_j$ such that*

$$\begin{cases} 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \ \lim_{j\to+\infty} \lambda_j = +\infty, \\ a(\widetilde{w}_j, v) = \lambda_j \langle \widetilde{w}_j, v\rangle, \quad \forall\, v \in V,\ j = 1, 2, \cdots. \end{cases}$$

*Furthermore, the sequence $\{\widetilde{w}_j/\sqrt{\lambda_j}\}$ is also a Hilbert orthonormal base of $V$ with respect to the scalar product $a(\cdot,\cdot)$.*

*On the other hand, we also have $\widetilde{w}_j$ satisfying the following boundary value problem*

$$\begin{cases} -\Delta \widetilde{w}_j = \lambda_j \widetilde{w}_j \quad in \ (0,1), \\ \widetilde{w}_{jx}(0) - h_0 \widetilde{w}_j(0) = \widetilde{w}_j(1) = 0, \quad \widetilde{w}_j \in V \cap C^\infty(\overline{\Omega}). \end{cases}$$

The proof of Lemma 2.4 can be found in ([14], p.87, Theorem 7.7), with $H = L^2$ and $V$, $a(\cdot,\cdot)$ as defined by (2.1), (2.2).

**Remark 2.5.** The weak formulation of the initial-boundary value problem (1.1)–(1.3) can be given in the following manner:

Find

$$u \in \widetilde{W} = \{u \in L^\infty(0,T; V \cap H^2) : u_t \in L^\infty(0,T;V), \quad u_{tt} \in L^\infty(0,T;L^2)\},$$

such that $u$ satisfies the following variational equation

$$\langle u_{tt}(t), w\rangle + a(u(t), w) = \langle f\left(\cdot, t, u(t), u(0,t)\right), w\rangle, \tag{2.4}$$

for all $w \in V$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \quad u_t(0) = \tilde{u}_1. \tag{2.5}$$

## 3. The existence and uniqueness

We make the following assumptions:

$(H_1)$ $(\tilde{u}_0, \tilde{u}_1) \in \left(V \cap H^2\right) \times V$, $\tilde{u}_{0x}(0) - h_0 \tilde{u}_0(0) = 0$;

$(H_2)$ $f \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^2)$.

Fix $T^* > 0$. For each $M > 0$ given, we set the constant $K_M(f)$ as follows

$$K_M(f) = \sum_{i=1}^{4} K_0(M, D_i f),$$

where

$$\begin{cases} K_0(M,f) = \sup_{(x,t,y_1,y_2)\in A_1(M)} |f(x,t,y_1,y_2)|, \\ A_1(M) = [0,1] \times [0,T^*] \times [-M,M]^2. \end{cases}$$

For every $T \in (0, T^*]$ and $M > 0$, we put

$$\begin{cases} W(M,T) = \{v \in L^\infty(0,T; V \cap H^2) : v_t \in L^\infty(0,T;V),\ v_{tt} \in L^2(Q_T), \\ \qquad\qquad \text{with } \max\{\|v\|_{L^\infty(0,T;V\cap H^2)}, \|v_t\|_{L^\infty(0,T;V)}, \|v_{tt}\|_{L^2(Q_T)}\} \le M\}, \\ W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T;L^2)\}, \end{cases}$$

in which $Q_T = \Omega \times (0,T)$.

Now, we establish the recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv \tilde{u}_0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.1}$$

we associate Prob. (1.1) - (1.3) with the following problem.

Find $u_m \in W_1(M,T)$ $(m \ge 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + a(u_m(t), w) = \langle F_m(t), w\rangle, \ \forall w \in V, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.2}$$

where

$$F_m(x,t) = f[u_{m-1}](x,t) = f\left(x, t, u_{m-1}(x,t), u_{m-1}(0,t)\right). \tag{3.3}$$

Then, we have the following theorem.

**Theorem 3.1.** *Let* $(H_1)$, $(H_2)$ *hold. Then there exist positive constants* $M$, $T > 0$ *such that, for* $u_0 \equiv \tilde{u}_0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (3.1)-(3.3).

*Proof.* The proof consists of several steps.

**Step 1.** *The Faedo–Galerkin approximation* (introduced by Lions [5]).

Consider the basis $\{w_j\}$ for $V$ as in Lemma 2.4. Put

$$u_m^{(k)}(t) = \sum_{j=1}^k c_{mj}^{(k)}(t) w_j, \tag{3.4}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + a(u_m^{(k)}(t), w_j) = \langle F_m(t), w_j\rangle, \quad 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k},\ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.5}$$

where

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^k \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } V \cap H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^k \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } V. \end{cases} \tag{3.6}$$

The system of the equations (3.5) can be rewritten in form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \lambda_j c_{mi}^{(k)}(t) = \langle F_m(t), w_j\rangle, \\ c_m^{(k)}(0) = \alpha_j^{(k)}, \quad \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \quad 1 \le j \le k. \end{cases} \tag{3.7}$$

It is not difficult to show that (3.7) has a unique solution $c_{mj}^{(k)}(t)$ in $[0,T]$ as follows

$$c_{mj}^{(k)}(t) = \alpha_j^{(k)} \cos(\sqrt{\lambda_j}t) + \beta_j^{(k)} \frac{\sin(\sqrt{\lambda_j}t)}{\sqrt{\lambda_j}} + \int_0^t \frac{\sin(\sqrt{\lambda_j}(t-s))}{\sqrt{\lambda_j}} \langle F_m(s), w_j \rangle ds, \quad 0 \le t \le T, \quad 1 \le j \le k. \quad (3.8)$$

Therefore, (3.5) has a unique solution $u_m^{(k)}(t)$ in $[0,T]$.

**Step 2.** *A priori estimates.*

We put

$$S_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_a^2 + \left\| u_m^{(k)}(t) \right\|_a^2 + ||\Delta u_m^{(k)}(t)||^2 + \int_0^t ||\ddot{u}_m^{(k)}(s)||^2 ds. \quad (3.9)$$

Then, it follows from (3.5) and (3.9) that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\langle F_m(0), \triangle \tilde{u}_{0k} \rangle + 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s) \rangle ds \\ &\quad - 2\langle F_m(t), \triangle u_m^{(k)}(t) \rangle + 2\int_0^t \langle F_m'(s), \triangle u_m^{(k)}(s) \rangle ds + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds \\ &\equiv S_m^{(k)}(0) + 2\langle F_m(0), \Delta \tilde{u}_{0k} \rangle + \sum_{j=1}^4 I_j. \end{aligned} \quad (3.10)$$

We can estimate without difficulty all terms on the right hand side of (3.10) and we obtain that

$$I_1 = 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s) \rangle ds \le 4TK_M^2(f) + \frac{1}{4}\int_0^t S_m^{(k)}(s) ds; \quad (3.11)$$

$$\begin{aligned} I_2 &= -2\langle F_m(t), \triangle u_m^{(k)}(t) \rangle \\ &\le 4\left( \|F_m(0)\|^2 + T^2(1+2M)^2 K_M^2(f) \right) + \frac{1}{2} S_m^{(k)}(t); \end{aligned} \quad (3.12)$$

$$\begin{aligned} I_3 &= 2\int_0^t \langle F_m'(s), \triangle u_m^{(k)}(s) \rangle ds \\ &\le 4T(1+2M)^2 K_M^2(f) + \frac{1}{4}\int_0^t S_m^{(k)}(s) ds; \end{aligned} \quad (3.13)$$

$$I_4 = \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds \le 2\int_0^t S_m^{(k)}(s) ds + 2TK_M^2(f). \quad (3.14)$$

It follows from (3.10)-(3.14) that

$$S_m^{(k)}(t) \leq D_0^{(k)}(f, \tilde{u}_{0k}, \tilde{u}_{1k}) + D_1(M,T) + 5\int_0^t S_m^{(k)}(s)ds, \tag{3.15}$$

where

$$\begin{cases} D_0^{(k)}(f, \tilde{u}_{0k}, \tilde{u}_{1k}) = 2S_m^{(k)}(0) + 4\left\langle F_m(0), \Delta\tilde{u}_{0k}\right\rangle + 8\left\|F_m(0)\right\|^2, \\ D_1(M,T) = 4T\left[3 + 2(1+2M)^2 + 2T(1+2M)^2\right] K_M^2(f). \end{cases} \tag{3.16}$$

By means of the convergences in (3.6), we can deduce the existence of a constant $M > 0$ independent of $k$ and $m$ such that

$$D_0^{(k)}(f, \tilde{u}_{0k}, \tilde{u}_{1k}) \leq \tfrac{1}{2}M^2, \quad \forall\, m, k \in \mathbb{N}. \tag{3.17}$$

We choose $T \in (0, T^*]$, such that

$$\left(\tfrac{1}{2}M^2 + D_1(M,T)\right)\exp(5T) \leq M^2 \tag{3.18}$$

and

$$k_T = 4\sqrt{Te^T}K_M(f) < 1. \tag{3.19}$$

Finally, it follows from (3.15), (3.17) and (3.18), that

$$S_m^{(k)}(t) \leq M^2\exp(-5T) + 5\int_0^t S_m^{(k)}(s)ds. \tag{3.20}$$

By using Gronwall's Lemma, we deduce from (3.20) that

$$S_m^{(k)}(t) \leq M^2\exp(-5T)\exp(5t) \leq M^2, \tag{3.21}$$

for all $t \in [0,T]$, for all $m$ and $k$. Therefore, we have

$$u_m^{(k)} \in W(M,T) \quad \text{for all } m \text{ and } k. \tag{3.22}$$

**Step 3.** *Limiting process.*

From (3.22), we deduce the existence of a subsequence of $\{u_m^{(k)}\}$ still so denoted, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T; V\cap H^2) \text{ weak*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T; V) \text{ weak*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^2(Q_T) \text{ weak}, \\ u_m \in W(M,T). \end{cases} \tag{3.23}$$

Passing to limit in (3.5), we have $u_m$ satisfying (3.2), (3.3) in $L^2(0,T)$.

On the other hand, it follows from $(3.2)_1$ and $(3.23)_4$ that $u_m'' = \Delta u_m + F_m \in L^\infty(0,T;L^2)$, hence $u_m \in W_1(M,T)$ and the proof of Theorem 3.1 is complete. □

We use the result given in Theorem 3.1 and the compact imbedding theorems to prove the existence and uniqueness of a weak solution of Prob. (1.1)-(1.3). Hence, we get the main result in this section as follows.

**Theorem 3.2.** *Suppose that* $(H_1)$, $(H_2)$ *hold. Then, there exist the constants* $M > 0$ *and* $T > 0$ *such that the problem* (1.1)-(1.3) *has a unique weak solution* $u \in W_1(M,T)$. *Furthermore, the linear recurrent sequence* $\{u_m\}$ *defined by* (3.1)-(3.3) *converges to the solution* $u$ *strongly in the space*

$$W_1(T) = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2)\}$$

*with the estimation*

$$\|u_m - u\|_{W_1(T)} \le C_T k_T^m \quad \text{for all} \quad m \in \mathbb{N}, \tag{3.24}$$

*where the constant* $k_T \in [0,1)$ *is defined as in* (3.19) *and* $C_T$ *is a constant depending only on* $T$, $h_0$, $f$, $\tilde{u}_0$, $\tilde{u}_1$ *and* $k_T$.

*Proof.* **(a)** ***Existence of the solution.*** First, we note that $W_1(T)$ is a Banach space with respect to the norm (see Lions [5]).

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V)} + \|v'\|_{L^\infty(0,T;L^2)}.$$

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then $w_m$ satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + a(w_m(t), w) = \langle F_{m+1}(t) - F_m(t), w\rangle, \ \forall w \in V, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{3.25}$$

Taking $w = w_m'$ in $(3.25)_1$, after integrating in $t$, we get

$$\|w_m'(t)\|^2 + \|w_m(t)\|_a^2 = 2\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle\, ds. \tag{3.26}$$

By $(H_2)$ it is clear to see that

$$\begin{aligned} \|F_{m+1}(t) - F_m(t)\| &\le 2K_M(f)\,\|\nabla w_{m-1}(t)\| \\ &\le 2K_M(f)\,\|w_{m-1}\|_{W_1(T)}. \end{aligned} \tag{3.27}$$

Hence

$$\begin{aligned} &\|w_m'(t)\|^2 + \|w_m(t)\|_a^2 \\ &\le 4TK_M^2(f)\,\|w_{m-1}\|_{W_1(T)}^2 + \int_0^t \left(\|w_m'(s)\|^2 + \|w_m(s)\|_a^2\right) ds. \end{aligned} \tag{3.28}$$

Using Gronwall's Lemma, we deduce from (3.28) that

$$\|w_m\|_{W_1(T)} \le k_T\,\|w_{m-1}\|_{W_1(T)}, \quad \forall m \in \mathbb{N}, \tag{3.29}$$

where $k_T \in (0,1)$ is defined as in (3.19), which implies that

$$\|u_m - u_{m+p}\|_{W_1(T)} \le \|u_0 - u_1\|_{W_1(T)}\,(1 - k_T)^{-1} k_T^m, \ \forall m,\, p \in \mathbb{N}. \tag{3.30}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strongly in} \quad W_1(T). \tag{3.31}$$

Note that $u_m \in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weak*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T;V) \text{ weak*}, \\ u''_{m_j} \to u'' & \text{in} \quad L^2(Q_T) \text{ weak}, \\ u \in W(M,T). \end{cases} \tag{3.32}$$

We also note that

$$\begin{aligned} &\|F_m(t) - f(\cdot,t,u(x,t),u(0,t))\|_{L^\infty(0,T;L^2)} \\ &\le 2K_M(f)\,\|u_{m-1} - u\|_{W_1(T)}\,. \end{aligned} \tag{3.33}$$

Hence, from (3.31) and (3.33), we obtain

$$F_m(t) \to f(\cdot,t,u(t),u(0,t)) \quad \text{strongly in} \quad L^\infty(0,T;L^2). \tag{3.34}$$

Finally, passing to limit in (3.2)–(3.3) as $m = m_j \to \infty$, it implies from (3.31), $(3.32)_{1,3}$ and (3.34) that there exists $u \in W(M,T)$ satisfying (2.4), (2.5).

On the other hand, from the assumption $(H_2)$ we obtain from $(2.4)_1$, $(3.32)_4$ and (3.34) that

$$u'' = u_{xx} + f(\cdot,t,u(t),u(0,t)) \in L^\infty(0,T;L^2), \tag{3.35}$$

thus we have $u \in W_1(M,T)$. The existence proof is completed.

**(b) *Uniqueness of the solution.*** Let $u_1$, $u_2 \in W_1(M,T)$ be two weak solutions of Prob. (1.1)-(1.3). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u''(t), w\rangle + a(u(t),w) = \langle F_1(t) - F_2(t), w\rangle\,,\ \forall w \in V, \\ u(0) = u'(0) = 0, \end{cases} \tag{3.36}$$

where $F_i(x,t) = f(x,t,u_i(x,t),u_i(0,t))\,,\ i = 1,\ 2.$

We take $w = u'$ in $(3.36)_1$ and integrate in $t$ to get

$$\|u'(t)\|^2 + \|u(t)\|_a^2 \le K_M(f)\int_0^t \left(\|u'(s)\|^2 + \|u(s)\|_a^2\right) ds.$$

Using Gronwall's Lemma, it follows that $\|u'(t)\|^2 + \|u(t)\|_a^2 \equiv 0$, *i.e.*, $u_1 \equiv u_2$. So (i) is proved and (ii) follows. Theorem 3.2 is proved completely. □

## 4. Asymptotic expansion of the solution with respect to a small parameter

In this section, let $(H_1)$, $(H_2)$ hold. We make more the following assumptions:

$(H_2')$ $f_1 \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^2)$.

We consider the following perturbed problem, where $\varepsilon$ is a small parameter such that, $|\varepsilon| \leq 1$ :

$$(P_\varepsilon)\begin{cases} u_{tt} - u_{xx} = F_\varepsilon[u](x,t), \quad 0 < x < 1, \quad 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where

$$\begin{cases} F_\varepsilon[u](x,t) = f_\varepsilon[u](x,t) + \varepsilon f_{1\varepsilon}[u](x,t), \\ f_\varepsilon[u](x,t) = f\left(x,t,u(x,t),\varepsilon u(0,t)\right), \\ f_{1\varepsilon}[u](x,t) = f_1\left(x,t,u(x,t),\varepsilon u(0,t)\right). \end{cases}$$

First, we note that if the functions $f$, $f_1$ satisfy $(H_2)$, $(H_2')$, then the a priori estimates of the Galerkin approximation sequence $\{u_m^{(k)}\}$ in the proof of Theorem 3.1 for Prob. (1.1)-(1.3) corresponding to $f = F_\varepsilon[u]$, $|\varepsilon| \leq 1$, satisfy $u_m^{(k)} \in W_1(M,T)$, where $M$, $T$ are constants independent of $\varepsilon$. We also note that the positive constants $M$ and $T$ are chosen as in (3.16)-(3.19) with $|f(\cdot,0,\tilde{u}_0,\tilde{u}_0(0))|$, $K_M(f)$, stand for $|f(\cdot,0,\tilde{u}_0,\tilde{u}_0(0))| + |f_1(\cdot,0,\tilde{u}_0,\tilde{u}_0(0))|$, $K_M(f)+K_M(f_1)$, respectively. Hence, the limit $u_\varepsilon$ in suitable function spaces of the sequence $\{u_m^{(k)}\}$ as $k \to +\infty$, after $m \to +\infty$, is a unique weak solution of the problem $(P_\varepsilon)$ satisfying $u_\varepsilon \in W_1(M,T)$. Then we can prove, in a manner similar to the proof of Theorem 3.2, that the limit $u_0$ in suitable function spaces of the family $\{u_\varepsilon\}$ as $\varepsilon \to 0$ is a unique weak solution of the problem $(P_0)$ (corresponding to $f = f_0[u_0](x,t) = f\left(x,t,u_0(x,t),0\right)$) satisfying $u_0 \in W_1(M,T)$.

We shall study the asymptotic expansion of the solution of the problem $(P_\varepsilon)$ with respect to a small parameter $\varepsilon$.

We use the following notations. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{Z}_+^N$, and $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, we put

$$\begin{cases} |\alpha| = \alpha_1 + \cdots + \alpha_N, \quad \alpha! = \alpha_1! \cdots \alpha_N!, \\ \alpha,\ \beta \in \mathbb{Z}_+^N, \quad \alpha \leq \beta \Longleftrightarrow \alpha_i \leq \beta_i, \quad \forall i = 1, \cdots, N, \\ x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}. \end{cases}$$

Next, we need the following lemma.

**Lemma 4.1.** *Let $m$, $N \in \mathbb{N}$ and $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, $\varepsilon \in \mathbb{R}$. Then*

$$\left(\sum\nolimits_{i=1}^{N} x_i \varepsilon^i\right)^m = \sum\nolimits_{k=m}^{mN} P_k^{(m)}[N,x]\varepsilon^k, \tag{4.1}$$

*where the coefficients* $P_k^{(m)}[N,x]$, $m \leq k \leq mN$ *depending on* $x = (x_1, \cdots, x_N)$ *defined by the formulas*

$$P_k^{(m)}[N,x] = \begin{cases} x_k, & 1 \leq k \leq N,\ m = 1, \\ \sum\limits_{\alpha \in A_k^{(m)}(N)} \dfrac{m!}{\alpha!} x^{\alpha}, & m \leq k \leq mN,\ m \geq 2, \end{cases} \tag{4.2}$$

*where* $A_k^{(m)}(N) = \{\alpha \in \mathbb{Z}_+^N : |\alpha| = m,\ \sum_{i=1}^N i\alpha_i = k\}$.

*Proof.* The proof of Lemma 4.1 is easy, hence we omit the details. □

Now, we assume that

$(H_2^{(N)})$ $\quad f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^2)$, $\ f_1 \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^2)$.

Let $u_0$ be a unique weak solution of the problem $(P_0)$ corresponding to $\varepsilon = 0$, *i.e.*,

$$(P_0) \begin{cases} u_0'' - \Delta u_0 = f\left(x,t,u_0(x,t),0\right) = f_0[u_0], \quad 0 < x < 1, \quad 0 < t < T, \\ u_{0x}(0,t) - h_0 u_0(0,t) = u_0(1,t) = 0, \\ u_0(x,0) = \tilde{u}_0(x), \quad u_0'(x,0) = \tilde{u}_1(x), \\ u_0 \in W_1(M,T). \end{cases}$$

Let us consider the sequence of the weak solutions $u_k$, $1 \leq k \leq N$, defined by the following problems:

$$(\tilde{P}_k) \begin{cases} u_k'' - \Delta u_k = F_k, \quad 0 < x < 1, \quad 0 < t < T, \\ u_{kx}(0,t) - h_0 u_k(0,t) = u_k(1,t) = 0, \\ u_k(x,0) = u_k'(x,0) = 0, \\ u_k \in W_1(M,T), \end{cases}$$

where $F_k$, $1 \leq k \leq N$, are defined by the formulas

$$F_k = \begin{cases} \bar{\Phi}_1[N,f] + f_1\left(x,t,u_0(x,t),0\right), & k = 1, \\ \bar{\Phi}_k[N,f] + \bar{\Phi}_{k-1}[N-1,f_1], & 2 \leq k \leq N, \end{cases} \tag{4.3}$$

with $\bar{\Phi}_k[N,f] = \bar{\Phi}_k[N,f,\vec{u}_*]$, $0 \leq k \leq N$, are defined by the formulas

$$\bar{\Phi}_k[N,f] = \begin{cases} f\left(x,t,u_0(x,t),0\right), & k = 0, \\ \sum\limits_{1 \leq |\gamma| \leq k} \frac{1}{\gamma!} D^{\gamma} f\left(x,t,u_0(x,t),0\right) \Psi_k[\gamma,N,\vec{u}_*], & 1 \leq k \leq N, \end{cases} \tag{4.4}$$

where

$$\Psi_k[\gamma,N,\vec{u}_*] = \sum_{\substack{(i,j) \in \widetilde{A}(\gamma,N), \\ i+j=k}} P_i^{(\gamma_1)}[N,\vec{u}(x,t)] P_j^{(\gamma_2)}[N+1,\vec{u}_*(0,t)] \tag{4.5}$$

with

$$\begin{aligned}&\widetilde{A}(\gamma, N)=\{(i,j)\in\mathbb{Z}_+^2:\gamma_1\le i\le N\gamma_1,\ \gamma_2\le j\le (N+1)\gamma_2\},\\&\gamma=(\gamma_1,\gamma_2)\in\mathbb{Z}_+^2,\ 1\le|\gamma|\le N,\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}\vec{u}(x,t)&=(u_1(x,t),\cdots,u_N(x,t)),\\\vec{u}_*(x,t)&=(u_0(x,t),u_1(x,t),\cdots,u_N(x,t)).\end{aligned}$$

Then, we have the following theorem.

**Theorem 4.2.** *Let* $(H_1)$ *and* $(H_2^{(N)})$ *hold. Then there exist constants* $M>0$ *and* $T>0$ *such that, for every* $\varepsilon\in[-1,1]$, *the problem* $(P_\varepsilon)$ *has a unique weak solution* $u_\varepsilon\in W_1(M,T)$ *satisfying the asymptotic estimation up to order* $N+1$ *as follows*

$$\left\|u_\varepsilon-\sum\nolimits_{k=0}^{N}u_k\varepsilon^k\right\|_{W_1(T)}\le C_T\left|\varepsilon\right|^{N+1},\tag{4.7}$$

*where the functions* $u_k$, $0\le k\le N$ *are the weak solutions of the problems* $(P_0)$, $(\tilde{P}_k)$, $1\le k\le N$, *respectively, and* $C_T$ *is a constant depending only on* $N$, $T$, $f$, $f_1$, $u_k$, $0\le k\le N$.

In order to prove Theorem 4.2, we need the following Lemmas.

**Lemma 4.3.** *Let* $\bar{\Phi}_k[N,f]$, $1\le k\le N$, *be the functions are defined by the formulas* (4.4)–(4.6). *Put* $h=\sum_{k=0}^{N}u_k\varepsilon^k$, *then we have*

$$f_\varepsilon[h]=f_0[u_0]+\sum\nolimits_{k=1}^{N}\bar{\Phi}_k[N,f]\varepsilon^k+\left|\varepsilon\right|^{N+1}\hat{R}_N[f,\vec{u}_*,\varepsilon]\tag{4.8}$$

*with* $\left\|\hat{R}_N[f,\vec{u}_*,\varepsilon]\right\|_{L^\infty(0,T;L^2)}\le C$, *where* $C$ *is a constant depending only on* $N$, $T$, $f$, $u_k$, $0\le k\le N$.

*Proof.* (i) In the case of $N=1$, the proof of (4.8) is easy, hence we omit the details, which we only prove with $N\ge 2$. Put $h=u_0+\sum_{k=1}^{N}u_k\varepsilon^k\equiv u_0+h_1$, we rewrite as follows

$$f_\varepsilon[h]=f\left(x,t,h(x,t),\varepsilon h(0,t)\right)=f(x,t,u_0(x,t)+h_1(x,t),\varepsilon h(0,t)).\tag{4.9}$$

By using Taylor's expansion of the function

$$f_\varepsilon[h]=f(x,t,u_0(x,t)+h_1(x,t),\varepsilon h(0,t))$$

around the point $[u_0] \equiv (x, t, u_0(x,t), 0)$ up to order $N+1$, we obtain

$$f_\varepsilon[h] = f_0[u_0] + \sum_{1\leq|\gamma|\leq N} \frac{1}{\gamma!} D^\gamma f_0[u_0] h_1^{\gamma_1}(x,t) \left(\varepsilon h(0,t)\right)^{\gamma_2} + R_N[f, \vec{u}_*, \varepsilon], \tag{4.10}$$

where

$$\begin{aligned}
&R_N[f, \vec{u}_*, \varepsilon] \\
&= \sum_{|\gamma|=N+1} \frac{N+1}{\gamma!} \int_0^1 (1-\theta)^N D^\gamma f\left(x,t,\varepsilon,\theta\right) h_1^{\gamma_1}(x,t) \left(\varepsilon h(0,t)\right)^{\gamma_2} d\theta \\
&= |\varepsilon|^{N+1} R_N^{(1)}[f, \vec{u}_*, \varepsilon], \qquad (4.11)\\
&D^\gamma f\left(x,t,\varepsilon,\theta\right) = D^\gamma f\left(x,t,u_0(x,t) + \theta h_1(x,t), \theta\varepsilon h(0,t)\right), \\
&f_0[u_0] = f\left(x,t,u_0(x,t),0\right), \quad D^\gamma f_0[u_0] = D^\gamma f\left(x,t,u_0(x,t),0\right), \\
&\gamma = (\gamma_1,\gamma_2) \in \mathbb{Z}_+^2, \quad |\gamma| = \gamma_1+\gamma_2, \quad \gamma! = \gamma_1!\gamma_2!, \quad D^\gamma f = D_3^{\gamma_1} D_4^{\gamma_2} f.
\end{aligned}$$

By the formula (4.1), we get

$$\begin{aligned}
h_1^{\gamma_1}(x,t) &= \left(\sum\nolimits_{k=1}^{N} u_k(x,t)\varepsilon^k\right)^{\gamma_1} = \sum\nolimits_{k=\gamma_1}^{N\gamma_1} P_k^{(\gamma_1)}[N, \vec{u}(x,t)]\varepsilon^k, \\
\left(\varepsilon h(0,t)\right)^{\gamma_2} &= \left(\sum\nolimits_{k=1}^{N+1} u_{k-1}(0,t)\varepsilon^k\right)^{\gamma_2} \\
&= \sum\nolimits_{k=\gamma_2}^{(N+1)\gamma_2} P_k^{(\gamma_2)}[N+1, \vec{u}_*(0,t)]\varepsilon^k,
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
\vec{u}(x,t) &= (u_1(x,t), \cdots, u_N(x,t)), \\
\vec{u}_*(x,t) &= (u_0(x,t), u_1(x,t), \cdots, u_N(x,t)).
\end{aligned}$$

Hence, we deduce from (4.12), that

$$\begin{aligned}
h_1^{\gamma_1}(x,t) \left(\varepsilon h(0,t)\right)^{\gamma_2} = &\sum\nolimits_{k=|\gamma|}^{N} \Psi_k[\gamma, N, \vec{u}_*]\varepsilon^k \\
&+ \sum\nolimits_{k=N+1}^{N|\gamma|+\gamma_2} \Psi_k[\gamma, N, \vec{u}_*]\varepsilon^k,
\end{aligned} \tag{4.13}$$

where

$$\begin{cases}
\Psi_k[\gamma, N, \vec{u}_*] = \sum\limits_{\substack{(i,j)\in\widetilde{A}(\gamma,N),\\ i+j=k}} P_i^{(\gamma_1)}[N, \vec{u}(x,t)] P_j^{(\gamma_2)}[N+1, \vec{u}_*(0,t)], \\
\widetilde{A}(\gamma,N) = \{(i,j) \in \mathbb{Z}_+^2 : \gamma_1 \leq i \leq N\gamma_1,\ \gamma_2 \leq j \leq (N+1)\gamma_2\}.
\end{cases} \tag{4.14}$$

We deduce from (4.10), (4.13) that

$$f_\varepsilon[h] = f_0[u_0] + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f_0[u_0] \sum\nolimits_{k=|\gamma|}^{N} \Psi_k[\gamma, N, \vec{u}_*]\varepsilon^k$$
$$+ |\varepsilon|^{N+1}\, \hat{R}_N[f, \vec{u}_*, \varepsilon]$$
$$= f_0[u_0] + \sum_{k=1}^{N} \left( \sum_{1\le|\gamma|\le k} \frac{1}{\gamma!} D^\gamma f_0[u_0] \Psi_k[\gamma, N, \vec{u}_*] \right) \varepsilon^k$$
$$+ |\varepsilon|^{N+1}\, \hat{R}_N[f, \vec{u}_*, \varepsilon]$$
$$= f[u_0] + \sum_{k=1}^{N} \bar{\Phi}_k[N, f]\varepsilon^k + |\varepsilon|^{N+1}\, \hat{R}_N[f, \vec{u}_*, \varepsilon], \tag{4.15}$$

where $\bar{\Phi}_k[N, f]$, $0 \le k \le N$, are defined by (4.4)–(4.6) and

$$|\varepsilon|^{N+1}\, \hat{R}_N[f, \vec{u}_*, \varepsilon] = \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f_0[u_0] \sum\nolimits_{k=N+1}^{N|\gamma|+\gamma_2} \Psi_k[\gamma, N, \vec{u}_*]\varepsilon^k$$
$$+ |\varepsilon|^{N+1}\, R_N^{(1)}[f, \vec{u}_*, \varepsilon]. \tag{4.16}$$

By the boundedness of the functions $u_k$, $0 \le k \le N$ in the function space $L^\infty(0, T; V)$, we obtain from (4.11), (4.13) and (4.16) that

$$\left\| \hat{R}_N[f, \vec{u}_*, \varepsilon] \right\|_{L^\infty(0,T;L^2)} \le C,$$

where $C$ is a constant depending only on $N$, $T$, $f$, $u_k$, $0 \le k \le N$. Thus, the Lemma 4.3 is proved. □

**Remark 4.4.** Lemma 4.3 is the key to establish the asymptotic expansion of the weak solution $u_\varepsilon$ of order $N+1$ in a small parameter $\varepsilon$ as below.

Let $u = u_\varepsilon \in W_1(M, T)$ be the unique weak solution of the problem $(P_\varepsilon)$. Then $v = u_\varepsilon - \sum_{k=0}^{N} u_k \varepsilon^k \equiv u_\varepsilon - h$ satisfies the problem

$$\begin{cases} v'' - \Delta v = f_\varepsilon[v+h] - f_\varepsilon[h] + \varepsilon\left(f_{1\varepsilon}[v+h] - f_{1\varepsilon}[h]\right) + E_\varepsilon(x,t), \\ 0 < x < 1, \quad 0 < t < T, \\ v_x(0,t) - h_0 v(0,t) = v(1,t) = 0, \\ v(x,0) = v'(x,0) = 0, \end{cases} \tag{4.17}$$

where

$$E_\varepsilon(x,t) = f_\varepsilon[h] - f_0[u_0] + \varepsilon f_{1\varepsilon}[h] - \sum\nolimits_{k=1}^{N} F_k \varepsilon^k. \tag{4.18}$$

Then, we have the following lemma.

**Lemma 4.5.** *Let* $(H_1)$ *and* $(H_2^{(N)})$ *hold. Then there exists a constant* $C_*$ *such that*

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \leq C_* |\varepsilon|^{N+1}, \tag{4.19}$$

*where* $C_*$ *is a constant depending only on* $N$, $T$, $f$, $f_1$, $u_k$, $0 \leq k \leq N$.

*Proof.* In the case of $N = 1$, the proof of Lemma 4.5 is easy, hence we omit the details, which we only prove with $N \geq 2$.

By using the formula (4.8) for the function $f_{1\varepsilon}[h]$ we obtain

$$f_{1\varepsilon}[h] = f_{10}[u_0] + \sum_{k=1}^{N-1} \bar{\Phi}_k[N-1, f_1]\varepsilon^k + |\varepsilon|^N \hat{R}_{N-1}[f_1, \vec{u}_*, \varepsilon], \tag{4.20}$$

where $\left\| \hat{R}_{N-1}[f, \vec{u}_*, \varepsilon] \right\|_{L^\infty(0,T;L^2)} \leq C$, with $C$ is a constant depending only on $N$, $T$, $f_1$, $u_k$, $0 \leq k \leq N$. By (4.20), we rewrite $\varepsilon f_1[h]$ as follows

$$\varepsilon f_{1\varepsilon}[h] = \varepsilon f_{10}[u_0] + \sum_{k=2}^{N} \bar{\Phi}_{k-1}[N-1, f_1]\varepsilon^k + \varepsilon |\varepsilon|^N \hat{R}_{N-1}[f, \vec{u}_*, \varepsilon], \tag{4.21}$$

Combining (4.3), (4.8), (4.18) and (4.21) lead to

$$E_\varepsilon(x,t) = |\varepsilon|^{N+1} \hat{R}_N[f, \vec{u}_*, \varepsilon] + \varepsilon |\varepsilon|^N \hat{R}_{N-1}[f_1, \vec{u}_*, \varepsilon]. \tag{4.22}$$

By the boundedness of the functions $u_k$, $0 \leq k \leq N$ in the function space $L^\infty(0,T;V)$, we obtain from (4.8), (4.20) and (4.22) that

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \leq C_* |\varepsilon|^{N+1}, \tag{4.23}$$

where $C_*$ is a constant depending only on $N$, $T$, $f$, $f_1$, $u_k$, $0 \leq k \leq N$. The proof of Lemma 4.5 is complete. □

*Proof of Theorem 4.2.* Consider the sequence $\{v_m\}$ defined by

$$\begin{cases} v_0 \equiv 0, \\ v_m'' - \Delta v_m = f_\varepsilon[v_{m-1} + h] - f_\varepsilon[h] + \varepsilon \left( f_{1\varepsilon}[v_{m-1} + h] - f_{1\varepsilon}[h] \right) \\ \qquad\qquad + E_\varepsilon(x,t), \quad 0 < x < 1, \quad 0 < t < T, \\ v_{mx}(0,t) - h_0 v_m(0,t) = v_m(1,t) = 0, \\ v_m(x,0) = v_m'(x,0) = 0, \quad m \geq 1. \end{cases} \tag{4.24}$$

By multiplying two sides of (4.24) with $v_m'$ and after integration in $t$, we have

$$\begin{aligned} Z_m(t) &= 2\int_0^t \langle E_\varepsilon(s), v_m'(s) \rangle ds + 2\int_0^t \langle f_\varepsilon[v_{m-1} + h] - f_\varepsilon[h], v_m'(s) \rangle ds \\ &\quad + 2\varepsilon \int_0^t \langle f_{1\varepsilon}[v_{m-1} + h] - f_{1\varepsilon}[h], v_m'(s) \rangle ds \\ &= \bar{J}_1 + \bar{J}_2 + \bar{J}_3, \end{aligned} \tag{4.25}$$

where $Z_m(t) = \|v_m'(t)\|^2 + \|v_m(t)\|_a^2$. We estimate the integrals on the right-hand side of (4.25) as follows.

*Estimating* $\bar{J}_1$. By using Lemma 4.5, we deduce from (4.19) that

$$\bar{J}_1 = 2\int_0^t \langle E_\varepsilon(s), v_m'(s)\rangle ds \leq TC_*^2 |\varepsilon|^{2N+2} + \int_0^t Z_m(s)ds. \tag{4.26}$$

*Estimating* $\bar{J}_2$. We note that

$$\|f_\varepsilon[v_{m-1}+h] - f_\varepsilon[h]\| \leq 2K_{M_*}(f)\|v_{m-1}\|_{W_1(T)} \tag{4.27}$$

with $M_* = (N+2)M$. It follows from (4.27), that

$$\begin{aligned} \bar{J}_2 &= 2\int_0^t \langle f_\varepsilon[v_{m-1}+h] - f_\varepsilon[h], v_m'(s)\rangle ds \\ &\leq 4TK_{M_*}^2(f)\|v_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s)ds. \end{aligned} \tag{4.28}$$

*Estimating* $\bar{J}_3$. Similarly

$$\begin{aligned} \bar{J}_3 &= 2\varepsilon\int_0^t \langle f_{1\varepsilon}[v_{m-1}+h] - f_{1\varepsilon}[h], v_m'(s)\rangle ds \\ &\leq 4TK_{M_*}^2(f_1)\|v_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s)ds. \end{aligned} \tag{4.29}$$

Combining (4.26), (4.28), (4.29), it leads to

$$\begin{aligned} Z_m(t) \leq\ & 4T\left[K_{M_*}^2(f) + K_{M_*}^2(f_1)\right]\|v_{m-1}\|_{W_1(T)}^2 + TC_*^2|\varepsilon|^{2N+2} \\ & + 3\int_0^t Z_m(s)ds. \end{aligned} \tag{4.30}$$

By using Gronwall's lemma, we deduce from (4.30) that

$$\|v_m\|_{W_1(T)} \leq \sigma_T\|v_{m-1}\|_{W_1(T)} + \delta_T(\varepsilon), \quad \forall\, m \geq 1, \tag{4.31}$$

where $\sigma_T = 4\left[K_{M_*}(f) + K_{M_*}(f_1)\right]\sqrt{Te^{3T}}$, $\delta_T(\varepsilon) = 2C_*\sqrt{Te^{3T}}\,|\varepsilon|^{N+1}$.

We can assume that

$$\sigma_T < 1 \quad \text{with the suitable constant} \quad T > 0. \tag{4.32}$$

We require the following lemma whose proof is immediate.

**Lemma 4.6.** *Let the sequence* $\{\gamma_m\}$ *satisfy*

$$\gamma_m \leq \sigma\gamma_{m-1} + \delta, \quad \forall\, m \geq 1, \;\; \gamma_0 = 0, \tag{4.33}$$

*where* $0 \leq \sigma < 1$, $\delta \geq 0$ *are the given constants. Then*

$$\gamma_m \leq \delta/(1-\sigma), \quad \forall\, m \geq 1. \tag{4.34}$$

Applying Lemma 4.6 with $\gamma_m = \|v_m\|_{W_1(T)}$, $\sigma = \sigma_T < 1$, $\delta = \delta_T(\varepsilon) = 2C_*\sqrt{Te^{3T}}\,|\varepsilon|^{N+1}$, it follows from (4.34) that

$$\|v_m\|_{W_1(T)} \leq \frac{\delta_T(\varepsilon)}{1-\sigma_T} = C_T|\varepsilon|^{N+1}, \tag{4.35}$$

where $C_T = \frac{2C_*\sqrt{Te^{3T}}}{1-4[K_{M_*}(f)+K_{M_*}(f_1)]\sqrt{Te^{3T}}}$.

On the other hand, the linear recurrent sequence $\{v_m\}$ defined by (4.24) converges strongly in the space $W_1(T)$ to the solution $v$ of the problem (4.17). Hence, letting $m \to +\infty$ in (4.35), we get

$$\|v\|_{W_1(T)} \leq C_T \, |\varepsilon|^{N+1} . \tag{4.36}$$

This implies (4.7). The proof of Theorem 4.2 is complete. □

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