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IMPROVED ROBUST SEMI-LOCAL CONVERGENCE ANALYSIS OF NEWTON'S METHOD FOR CONE INCLUSION PROBLEM IN BANACH SPACES UNDER RESTRICTED CONVERGENCE DOMAINS AND MAJORANT CONDITIONS

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Abstract. In this study, we consider Newton's method for solving the nonlinear inclusion problems in Banach space, where F is a Fréchet differentiable operator. Using restricted convergence domains we prove the convergence of the method with the following advantages: tighter error estimates on the distances involved and the information on the location of the solution is at least as precise. These advantages were obtained under the same computational cost using the idea of restricted convergence domains.

1. Introduction

In this study we consider the problem of approximately solving nonlinear inclusion problem of the form

$$\text{find } x \text{ such that } F(x) \in C, \tag{1.1}$$

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where C is a nonempty closed convex cone in a Banach space $\mathbb{Y}$ and $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is a nonlinear function between the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$. The nonlinear equations in Banach spaces are solved widely using Newton's method and its variant see [1]-[22] for details. Robinson in [20] generalized Newton's method for solving (1.1) in the special case in which C is the degenerate cone $\{0\} \subset \mathbb{Y}$. Lipschitz continuity or Lipschitz-like continuity of the derivative of the nonlinear operator in question is usually used in convergence analysis of Newton's type method. There are plethora of papers dealing with the convergence of analysis of Newton's like method by relaxing the assumption of Lipschitz type continuity of the derivative of the operator involved [1]-[22].

This work uses the idea of restricted convergence domains to present a convergence analysis of Newton's method for solving a nonlinear inclusion problems of the form (1.1). This analysis relaxes the Lipschitz type continuity of the derivative of the operator involved. The main these of the analysis is to find larger convergence domain for the Newton's method for solving (1.1). A finer convergence analysis is obtained using the restricted convergence domains, with the advantages (**A**): tighter error estimates on the distances involved and the information on the location of the solution is at least as precise. These advantages were obtained (with the same computational cost) using the same or weaker hypotheses as in [17]-[22].

The rest of the paper is organized as follows.

2. Semi-local analysis for Newton's method

Here we try to state and prove the improved semi-local affine invariant theorem for Newtons method to solve nonlinear equation of the form

$$F(x) \in C, \tag{2.1}$$

where $F : D \subseteq \mathbb{X} \to \mathbb{Y}$ is a non-linear function which is continuously differentiable, D is an open set and $C \subset \mathbb{Y}$ a non empty closed convex cone. Recall that [20], a nonlinear continuously Frechet differentiable function $F : D \to \mathbb{Y}$ satisfies Robinson's Condition at $\bar{x} \in D$, if

$$rge\ T_{\bar{x}} = \mathbb{Y}, \tag{2.2}$$

where $T_{\bar{x}} : \mathbb{X} \rightrightarrows \mathbb{Y}$, is sub linear mapping as defined in [20]. Let $R > 0$ a scalar constant. Set $\rho^* := sup\{t \in [0, R) : U(x_0, t) \subseteq D\}$. A continuously differentiable function $f_0 : [0, R) \to \mathbb{R}$ is a center-majorant function at a point $\bar{x} \in D$ for a continuously differentiable function $F : D \to Y$, if

$$\|T_{\bar{x}}^{-1}[F'(x) - F'(\bar{x})]\| \leq f_0'(\|x - \bar{x}\|) - f_0'(0) \tag{2.3}$$

for each $x \in U(\bar{x}, R)$ and satisfies the following conditions:

(h_1^0) $f_0(0) > 0$, $f_0'(0) = -1$;

(h_2^0) $f'(0)$ is convex and strictly increasing;
(h_3^0) $f_0(t) = 0$ for some $t \in (0, R)$.

Then, sequence $\{t_n^0\}$ generated for $t_0^0 = 0$,

$$t_{k+1}^0 = t_k^0 - \frac{f_0(t_k)}{f_0'(t_k)}, \; k = 0, 1, \cdots \tag{2.4}$$

is well defined strictly increasing, remains in $(0, t_0^*)$ and converged to t_0^*, where t_0^* is the smallest zero of function f_0 in $(0, R)$. Set $D_1 := \bar{U}(\bar{x}, \rho^*) \cap U(\bar{x}, t_0^*)$. Suppose that there exists $f_1 : [0, \rho_1) \to \mathbb{R}$, $\rho_1 = min\{\rho^*, t_0^*\}$ such that

$$\|T_{\bar{x}}^{-1}[F'(y) - F'(x)]\| \leq f_1'(\|y - x\| + \|x - \bar{x}\|) - f_1'(\|x - \bar{x}\|) \tag{2.5}$$

for each $x, y \in D_1$ and satisfies

(h_1) $f_1(0) > 0$, $f_1'(0) = -1$;
(h_2) f_1' is convex and strictly increasing;
(h_3) $f_1(t) = 0$ for some $t \in (0, R)$;
(h_4) $f_0(t) \leq f_1(t)$ and $f_0'(t) \leq f_1'(t)$ for each $t \in [0, \rho_1)$.

From now on we assume that the above "h" conditions hold. The following condition on the majorant condition f is required, which is considered valid only when explicitly stated.

(h_5) $f_1(t) < 0$ for some $t \in (0, R)$.

Remark 2.1. Since $f_1(0) > 0$ and f_1 is continuous then condition (h_4) implies condition (h_3).

Theorem 2.2. *Let $\mathbb{X}$, $\mathbb{Y}$ be Banach spaces and $\mathbb{X}$ is reflexive, $D \subseteq \mathbb{X}$ an open set, $F : D \to \mathbb{Y}$ a continuously Frechet differentiable function, $C \subset \mathbb{Y}$ a nonempty closed convex cone. Suppose $\bar{x} \in D$, F satisfies Robinson's condition at $\bar{x}$, f_0 is a center majorant function and f_1 is a majorant function for F at $\bar{x}$ and*

$$\|T_{\bar{x}}^{-1}(-F(\bar{x}))\| \leq f_1(0). \tag{2.6}$$

Then f_1 has the smallest zero $t_* \in (0, R)$, the sequence generated by Newtons Method for solving the inclusion $F(x) \in C$ and the equation $f(t) = 0$, with starting point $x_0 = \bar{x}$ and $t_0 = 0$, respectively.

$$\begin{aligned} x_{k+1} &\in x_k + argmin\{\|d\| : F(x_k) + F'(x_k)d \in C\}, \\ t_{k+1} &= t_k - \frac{f_1(t_k)}{f_1'(t(k))}, \quad k = 0, 1, \cdots \end{aligned} \tag{2.7}$$

are well defined $\{x_k\}$ *is a constrained in* $B(\bar{x}, t_*)$, $\{t_k\}$ *is strictly increasing, is constrained in* $[0, t_*)$ *and converges to* t_* *and satisfies the inequalities.*

$$\|x_{k+1} - x_k\| \le -\frac{f_1(t_k) - f_1(t_{k-1} - f_1'(t_{k-1})(t_k - t_{k-1})}{f_0'(t_k)} \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2$$
$$\le -\frac{f_1(t_k)}{f_1'(t_k)} \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2 ,$$
$$\|x_{k+1} - x_k\| \le t_{k+1} - t_k, \quad \|x_{k+1} - x_k\| \le \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2}\|x_k - x_{k-1}\|^2, \quad (2.8)$$

for all $k = 0, 1, \cdots$, *and* $k = 1, 2, \cdots$, *respectively. Moreover,* $\{x_k\}$ *converges* $x_* \in B[\bar{x}, t_*]$ *such that* $F(x_*) \in C$,

$$\|x_* - x_k\| \le t_* - t_k, \quad t_* - t_{k+1} \le \frac{1}{2}(t_* - t_k), \;\; k = 0, 1, \cdots \quad (2.9)$$

and, therefore, $\{t_k\}$ *converges* $Q-$*linearly to* t_* *and* $\{x_k\}$, *converges* $R-$*linearly to* x_*. *If, additionally* f_1 *satisfies* (h_4) *then the following inequalities hold:*

$$\begin{aligned} \|x_{k+1} - x_k\| &\le \frac{D^- f_1'(t^*)}{-2f_0'(t^*)}\|x_k - x_{k-1}\|^2, \\ t_{k+1} - t_k &\le \frac{D^- f_1'(t^*)}{-2f_0'(t^*)}(t_k - t_{k-1})^2, \;\; k = 1, 2, \cdots , \end{aligned} \quad (2.10)$$

and, as a consequence $\{x_k\}$ *and* $\{t_k\}$ *converges* $Q-$*quadratically to* x_* *and* t_*, *respectively, as follows*

$$\begin{aligned} \lim_{k\to\infty} sup\frac{\|x_* - x_{k+1}\|}{\|x_* - x_k\|^2} &\le \frac{D^- f_1'(t^*)}{-2f_0'(t_*)}, \\ t_* - t_{k+1} &\le \frac{D^- f_1'(t_*)}{-2f_0'(t_*)}(t_* - t_k)^2, \;\; k = 0, 1, \cdots . \end{aligned} \quad (2.11)$$

We will use the above result to prove a robust semi-local affine invariant theorem for Newtons method for solving non-linear inclusion of the form [18]. The statement of the main theorem is:

Theorem 2.3. *Let* $\mathbb{X}$, $\mathbb{Y}$ *be Banach spaces and* $\mathbb{X}$ *is reflexive,* $D \subseteq \mathbb{X}$ *an open set,* $F : D \to \mathbb{Y}$ *a continuously Frechet differentiable function,* $C \subset \mathbb{Y}$ *a nonempty closed convex cone,* $R > 0$ *and* $f : [0, R) \to R$ *a continuously differentiable function. Suppose* $\bar{x} \in D$, F *satisfies Robinson's condition at* $\bar{x}$, f_0 *is a center majorant function and* f_1 *is a majorant function for* F *at* $\bar{x}$ *satisfying* $(h5)$ *and*

$$\|T_{\bar{x}}^{-1}(-F(\bar{x}))\| \le f_1(0). \quad (2.12)$$

Define $\beta := sup\{-f(t) : t \in [0, R]\}$. *Let* $0 < \rho < \beta/2$ *and* $g : [0, R-\rho) \to \mathbb{R}$,

$$g(t) = \frac{-1}{f_0'(\rho)}[f(t+\rho) + 2\rho]. \tag{2.13}$$

Then g *has a smallest zero* $t_{*,\rho} \in (0, R-\rho)$, *the sequences generated by Newtons method for solving the inclusion* $F(x) \in C$ *and the equation* $g(t) = 0$, *with starting point* $x_0 = \cap x$ *for any* $\hat{x} \in B(\bar{x}, \rho)$ *and* $t_0 = 0$, *respectively,*

$$\begin{aligned} x_{k+1} &\in x_k + argmin\{\|d\| : F(x_k) + F'(x_k)d \in C\}, \\ t_{k+1} &= t_k - \frac{g(t_k)}{g'(t_k)}, \quad k = 0, 1, \cdots \end{aligned} \tag{2.14}$$

are well defined, $\{x_k\}$ *is constrained in* $B(\bar{x}, t_{*,\rho})$, $\{t_k\}$ *is strictly increasing, is contained in* $[0, t_{*,\rho})$ *and converges to* $t_{*,\rho}$ *and satisfies the inequalities*

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \quad k = 0, 1, \cdots \tag{2.15}$$

$$\begin{aligned} \|x_{k+1} - x_k\| &\leq \frac{t_{k+1} - t_k}{t_k - t_{k-1}^2}\|x_k - x_{k-1}\|^2 \\ &\leq \frac{D^- g'(t_{*,\rho})}{-2g'(t_{*,\rho})}\|x_k - x_{k-1}\|^2, \quad k = 1, 2, \cdots . \end{aligned} \tag{2.16}$$

Moreover, $\{x_k\}$ *converges to* $x_* \in B[\bar{x}, t_{*,\rho})$ *such that* $F(x_*) \in C$, *satisfies the inequalities*

$$\|x_* - x_k\| \leq t_{*,\rho} - t_k, t_{*,\rho} \leq \frac{1}{2}(t_{*,\rho} - t_k), \quad k = 0, 1, \cdots \tag{2.17}$$

and the convergence of $\{x_k\}$ *and* $\{t_k\}$ *to* x_* *and* $t_{*,\rho}$, *respectively is quadratic as follows*

$$\begin{aligned} \lim_{k \to infty} sup \frac{\|x_* - x_{k+1}\|}{\|x_* - x_k\|^2} &\leq \frac{D^- g'(t_{*,\rho})}{-2g'(t_{*,\rho})}, \\ t_{*,\rho} - t_{k+1} &\leq \frac{D^- g'(t_{*,\rho})}{-2g'(t_{*,\rho})}(t_{*,\rho} - t_k)^2, \quad k = 0, 1, \cdots . \end{aligned} \tag{2.18}$$

Remark 2.4. It is easy to see that the inequalities (2.3) and (2.12) are well defined.

Remark 2.5. The definitions of the sequence $\{x_k\}$in (2.7) is equivalent to the conditions

$$x_{k+1} - x_k \in T_{x_k}^{-1}(-F(x_k))$$

and

$$\|x_{k+1} - x_k\| = \|T_{x_k}^{-1}(-F(x_k))\|, \quad k = 0, 1, \cdots .$$

Remark 2.6. Theorems 2.2 and 2.3 are affine invariant in the following sense. Letting $A:\mathbb{Y}\to\mathbb{Y}$ be an invertible continuous linear mapping, $F := AoF$ and the set $\bar{C} := A(C)$, the corresponding inclusion problem (2.1) is given by

$$\bar{F}(x)\in\bar{C},$$

and the convex process associated is denoted by $\bar{T}_{\bar{x}}d = \bar{F}(\bar{x})d - \bar{C}$. Then $\bar{T}_{\bar{x}} = AoT_{\bar{x}}$ and $\bar{T}_{\bar{x}}^{-1} = T_{\bar{x}}^{-1}oA^{-1}$. Moreover, we have the conditions rge $\bar{T}_{\bar{x}} = Y$,

$$\|\bar{T}_{\bar{x}}^{-1}(-\bar{F}(\bar{x}))\| \le f_1(0),$$

and the affine majorant condition (Lipschitz-like condition)

$$\|\bar{T}_{\bar{x}}^{-1}\left[\bar{F}'(y)-\bar{F}'(x)\right]\| \le f_1'(\|y-x\|+\|x-\bar{x}\|)-f_1'(\|x-\bar{x}\|),$$

for $x,y\in B(\bar{x},R)$, $\|x-\bar{x}\|+\|y-x\|<R$. Therefore the assumption in Theorem 2.2 and 2.3 are insensitive with respect to invertible continuous linear mappings. Note that such property does not hold in [20].

Remark 2.7. (a) Suppose that

$$\|T_{\bar{x}}^{-1}[F'(y)-F'(x)]\| \le f'(\|y-x\|+\|x-\bar{x}\|)-f'(\|x-\bar{x}\|) \tag{2.19}$$

holds for each $x,y\in\bar{U}(\bar{x}-\rho^*)$ and some continuously differentiable function $f:[0,R]\to\mathbb{R}$ satisfying conditions (h_1)-(h_3) and (h_5). Then if $f_0(t)=f(t)=f_1(t)$ for each $t\in[0,R)$, Theorem 2.2 and Theorem 2.3 reduced to Theorem 7 and Theorem 8, respectively in [18]. Notice however that for each $t\in[0,R)$,

$$f_0' \le f'(t), \tag{2.20}$$

$$f_1'(t) \le f'(t), \tag{2.21}$$

holds in general. Therefore, if strict inequality holds in (2.20) or (2.21) then the advantages as stated in he abstract of this study holds (see the numerical section also).

(b) Let $t^* := sup\{t\in[0,R) : f_0'(t)<0\}$. Define $D_1^* = \bar{U}(\bar{x},\rho^*)\cap U(\bar{x},\bar{t})$. Then the conclusion of the two proceeding theorems hold with the set D_1^*, t^* replacing D_1, t_0^*, respectively. In this case we do not have to define sequence $\{t_k^0\}$. Notice however that the function f_1^* satisfying (2.5) will be different in general from f_1.

We need two auxiliary results needed for the proofs of the proceeding Theorems.

Lemma 2.8. *If $\|x-\bar{x}\|\le t<\bar{t}$, then dom $(T_x^{-1}F'(\bar{x}))=\mathbb{X}$ and there holds*

$$\|T_x^{-1}F'(\bar{x})\| \le -1/f_0'(t). \tag{2.22}$$

As a consequence, rge $T_x=\mathbb{Y}$.

Proof. Take $0 \le t < \bar{t}$ and $x \in B[\bar{x}, t]$. Since f_0 is a center majorant function for F at $\bar{x}$, using (2.3), (h_2) and (h_1) we obtain

$$\|T_{\bar{x}}^{-1}[F'(x) - F'(\bar{x})]\| \le f_0'(\|x - \bar{x}\|) - f'(0) \tag{2.23}$$
$$\le f_0'(t) - f_0'(0) \tag{2.24}$$
$$\le f_0'(t) + 1 < 1. \tag{2.25}$$

To simplify the notation define $S = T_{\bar{x}}^{-1}[F'(x) - F'(\bar{x})]$. Since $[F'(x) - F'(\bar{x})]$ is a continuous linear mapping and $T_{\bar{x}}^{-1}$ is a sub-linear mapping with closed graph, it is easy to see that S is a sub-linear mapping with closed graph. Moreover, as by assumption $rgeT_{\bar{x}} = \mathbb{Y}$ we have dom $S = \mathbb{X}$. Because S is a closed graph, it is easy to see that $(S + I)(x)$ has also a closed graph for all $x \in X$, where I is an identity mapping on $\mathbb{X}$. We conclude that $rge(T_{\bar{x}}^{-1}[F'(x) - F'(\bar{x})] + I) = \mathbb{X}$ and

$$\|(T_{\bar{x}}^{-1}[F'(x) - F'(\bar{x})] + I)^{-1}\| \le \frac{1}{1 - (f_0'(t) + 1)} = \frac{1}{-f_0'(t)}. \tag{2.26}$$

The rest of the proof is similar to Proposition 12 in [18] is omitted. □

Remark 2.9. Newtons iteration at a point $x \in D$ happens to be a solution of the linearizion of the inclusion $F(y) \in C$ at such a point, namely, a solution of the linear inclusion $F(x) + F'(x)(x - y) \in C$. So, we study the linearizion error of F at a point in D

$$E(x, y) := F(y) - [F(x) + F'(x)(y - x)], \quad y, x \in D. \tag{2.27}$$

We will bound this error by the error in the linearizion of the majorant function f

$$e(t, s) := f(s) - [f(t) + f'(t)(s - t)], \quad t, s \in [0, R). \tag{2.28}$$

Lemma 2.10. *Let $R > 0$ and $f : [0, R) \to R$ a continuously differentiable function. Suppose that $\bar{x} \in D$, f is a majorant function for F at $\bar{x}$ and satisfies (h_4). If $0 \le \rho \le \beta/2$, where $\beta := sup\{-f_1(t) : t \in [0, R)\}$ then for any $\hat{x} \in B(\bar{x}, \rho)$ the scalar function $g : [0, R - \rho) \to \mathbb{R}$,*

$$g(t) = \frac{-1}{f_0'(\rho)}[f_1(t + \rho) + 2\rho],$$

is a majorant function for F at $\hat{x}$ and also satisfies condition (h_4).

Proof. Since the domain of f_1 is $[0, R)$ and $f_1'(\rho) \ne 0$ we conclude that g is well defined. First we will prove that function g satisfies conditions (h_1), (h_2), (h_3) and (h_4). We trivially have that $g'(0) = 1$. Since f_1 is convex, combining this with (h_1) we have $f_1(t) + t \ge f_1(0) > 0$, for all $0 \le t < R$, which implies

$g(0) = [f_1(\rho)+2\rho]/f_0'(\rho) > 0$, hence g satisfies (h_1) and (h_2). Now, as $\rho < \beta/2$, we have

$$\lim_{t\to\bar{t}-\rho} = \frac{-1}{f_0'(\rho)}(2\rho - \beta) < 0, \tag{2.29}$$

which implies that g satisfies (h_4) and, as g is continuous and $g(0) > 0$, it also satisfies (h3). To complete the proof, it remains to prove that g satisfies (h_2). First of all, note that for any $\hat{x} \in B(\bar{x}, \rho)$, we have $\|\hat{x} - \bar{x}\| < \rho < \bar{t}$ and by using Lemma 2.8 we obtain that

$$\|T_{\hat{x}}^{-1} F'(\bar{x})\| \leq \frac{-1}{f_0'(\rho)}. \tag{2.30}$$

Because $B(\bar{x}, R) \subseteq D$, for any $\hat{x} \in B(\bar{x}, \rho)$ we trivially have $B(\hat{x}, R - \rho) \subset D$. Now, take $x, y \in \mathbb{X}$ such that $x, y \in B(\bar{x}, R - \rho)$, $\|x - \hat{x}\| + \|y - x| < R - \rho$. Hence $x, y \in B(\bar{x}, R)$ and $\|x - \bar{x}\| + |y - x| < R$. Thus, we have

$$\begin{aligned}\|T_{\hat{x}}^{-1}[F'(y) - F'(x)]\| &\leq \|T_{\hat{x}}^{-1}F'(\bar{x})T_{\bar{x}}^{-1}[F'(y) - F'(x)]\| \\ &\leq \|T_{\hat{x}}^{-1}F'(x)\|\|T_{\bar{x}}^{-1}[F'(y) - F'(x)]\| \\ &\leq \frac{-1}{f_0'(\rho)}[f'(\|y - x\| + \|x - \bar{x}\|) - f'(\|x - \bar{x}\|)].\end{aligned}$$

On the other hand, since f' is convex, the function $s \mapsto f_1'(t + s) - f_1'(s)$ is increasing for $t \geq 0$ and as $\|x - \bar{x}\| \leq \|x - \hat{x}\| + \rho$ we conclude that

$$\begin{aligned}&f_1'(\|y - x\| + \|x - \bar{x}\|) - f_1'(\|x - \bar{x}\|) \\ &\leq f_1'(\|y - x\| + \|x - \hat{x}\| + \rho) - f_1'(\|x - \hat{x}\| + \rho).\end{aligned}$$

Hence, combining the two above inequalities with the definition of the function g we obtain

$$\|T_{\hat{x}}^{-1}[F'(y) - F'(x)]\| \leq g'(\|y - x\| + \|x - \hat{x}\|) - g'(\|x - \hat{x}\|),$$

implying that the function g satisfies (2.5). □

Remark 2.11. If $f_0(t) = f(t) = f_1(t)$ for each $t \in [0, R)$, then the last two Lemmas reduce to Proposition 12 and Proposition 17, respectively in [18]. Otherwise, in view of (2.20) and (2.21), the new Lemmas constitute an improvement.

Proof of Theorem 2.1 *and Theorem* 2.2. Simply notice that the iterates $\{x_k\}$ lie in D_1 which is a more precise location that $\bar{U}(\bar{x}, \rho)$ used in [18], since $D_1 \subseteq \bar{U}(\bar{x}, \rho^*)$. Then, follow the proofs in [18] using f_1, Lemma 2.9, Lemma 2.11 instead of f, Proposition 12, Proposition 17, respectively. □

3. Special cases and a numerical example

We present a robust semi-local convergence result in Newton's method for solving nonlinear inclusion problem using Lipschitz-like condition. In particular, Theorem 2.2 and Theorem 2.3, reduce respectively to:

Theorem 3.1. *Let $\mathbb{X}$, $\mathbb{Y}$ be Banach spaced with $\mathbb{X}$ being reflexive, and $D \subseteq \mathbb{X}$ be an open set. Let also $F : D \to \mathbb{Y}$ be a continuously differentiable operator and $C \subset \mathbb{Y}$ be nonempty closed convex cone. Let $\bar{x} \in D$, $L_0 > 0$, $L_1 > 0$ and $R > 0$. Suppose that F satisfies Robinson's condition at $\bar{x}$,*

$$\|T_{\bar{x}}^{-1}[F'(x) - F'(\bar{x})]\| \leq L_0\|x - \bar{x}\| \text{ for each } x \in U(\bar{x}, R), \tag{3.1}$$

$$\|T_{\bar{x}}^{-1}[F'(x) - F'(\bar{x})]\| \leq L_1\|x - y\| \text{ for each } x,\ y \in D_1, \tag{3.2}$$

$$\|T_{\bar{x}}^{-1}(-F(\bar{x}))\| \leq \eta,$$

and

$$h_1 = 2L_1\eta \leq 1. \tag{3.3}$$

Define $f_1 : [0, +\infty) \to \mathbb{R}$ by $f_1(t) := \frac{L_1}{2}t^2 - t + n$ and $t_* := \frac{1-\sqrt{1-h_1}}{L_1}$. Then, the sequences generated by Newtons method for solving the inclusion $F(x) \in C$ and the equation$f_1(t) = 0$, with starting points $x_0 = \bar{x}$ and t_0, respectively,

$$x_{k+1} \in x_k + argmin\{\|d\| : F(x_k) + F'(x_k)d \in C\},$$

$$t_{k+1} = t_k - \frac{f_1(t_k)}{f_1'(t_k)},$$

are well defined, $\{x_k\}$ is contained in $B(\bar{x}, t^*)$, $\{t_k\}$ is strictly increasing, is contained in $[0, t_*)$ and converges to t_*. Moreover, the following estimates hold

$$\|x_{k+1} - x_k\| \leq \frac{L_1\|x_k - x_{k-1}\|^2}{2(1 - L_0\|x_k - \bar{x}\|)}. \tag{3.4}$$

Furthermore, $\{x_k\}$ converges to $x_* \in B[\bar{x}, t_*, \rho)$ such that $F(x_*) \in C$.

Proof. It is easy to see that function f_1 is a majorant function for F at $\bar{x}$ and function $f_0(t) = \frac{L_0}{2}t^2 - t + n$ is a center majorant function for F at $\bar{x}$. The rest follows from the proof of Theorem 2.2. □

Theorem 3.2. *Under the hypotheses of Theorem 3.1, further suppose that*

$$0 \leq \rho < B := \frac{(1 - 2L_1 n)}{4L_1}.$$

Define $g : [0, +\infty) \to \mathbb{R}$ by

$$g(t) = \frac{-1}{L_0\rho - 1}\left[\frac{L_1}{2}(t + \rho)^2 - (t + \rho) + n + 2\rho\right].$$

Denote by $t_{*,\rho}$ the smallest zero of g. Then, the sequences generated by Newton's method for solving the inclusion $F(x) \in C$ and the equation $g(t) = 0$ with starting point $x_0 = \bar{x}$ and for any $\hat{x} \in B(\bar{x}, \rho)$ and $t_0 = 0$, respectively

$$x_{k+1} = x_k + argmin\{\|d\| : F(x_k) + F'(x_k)d \in C\},$$
$$t_{k+1} = t_k - \frac{g(t_k)}{g'(t_k)}$$

are well defined, $\{x_k\}$ is contained in $B(\bar{x}, t_{*,\rho})$, converges to $x_* \in B[\bar{x}, t_{*,\rho})$ with $F(x_*) \in C$, $\{t_k\}$is strictly increasing, is contained in $[0, t_{*,\rho})$ and converged to $t_{*,\rho}$.

Proof. The proof follows from the proof of Theorem 2.2 using the special choices of the functions f_0 and f_1. □

Remark 3.3. Let $f(t) = \frac{L}{2}t^2 - t + n$. Then, the corresponding Kantorovich condition to (3.3) given in [18] is $h_k = 2L\eta \le 1$.

However, we have that

$$L_1 \le L, \tag{3.5}$$

so

$$h_k \le 1 \quad \Rightarrow \quad h \le 1, \tag{3.6}$$

but not necessarily vice versa unless, if $L_1 = L$. It follows from the proof of Theorem 3.1 that the following sequence $\{s_n\}$ defined by

$$s_0 = 0, \;\; s_1 = n, \;\; s_2 = s_1 + \frac{L_0(s_1 - s_0)^2}{2(1 - L_0 s_1)}, \tag{3.7}$$

$s_{n+2} = s_{n+1} + \frac{L_1(s_{n+1} - s_n)^2}{2(1 - L_0 s_{n+1})}$ is also majorizing for the sequence $\{x_n\}$ which certainly converges, if (3.3) holds. Moreover, we have that

$$t_0 = 0, \;\; t_1 = n, \;\; t_{n+1} = t_n + \frac{L_1(t_n - t_{n-1})^2}{2(1 - L_0 t_n)}, \tag{3.8}$$

and

$$u_0 = 0, \; u_1 = n, \; u_{n+1} = u_n + \frac{L(u_n - u_{n-1})^2}{2(1 - L u_n)}. \tag{3.9}$$

Sequence $\{u_n\}$ was used in [18]. Then, we have for $L_0 < L_1 < L$ that

$$s_n < t_n < u_n, \; n = 2, 3, \cdots \tag{3.10}$$

$$s_{n+1} - s_n < t_{n+1} - t_n < u_{n+1} - u_n, \; n = 1, 2, \cdots \tag{3.11}$$

and

$$s_* = \lim_{n\to\infty} s_n \le t_* = \lim_{n\to\infty} t_n \le u_* = \lim_{n\to\infty} u_n. \tag{3.12}$$

Furthermore, we have that sequence $\{s_k\}$ converges provided that

$$h_1 = \bar{L}\eta \leq 1, \tag{3.13}$$

where $\bar{L} = \frac{1}{4}\left(4L_0 + \sqrt{L_1 L_0} + \sqrt{L_1 L_0 + 8L_0^2}\right)$. It follows from (3.3)and (3.12) that

$$h_k \leq 1 \;\; \Rightarrow \;\; h \leq 1 \;\; \Rightarrow \;\; h_1 \leq 1. \tag{3.14}$$

Estimates (3.10)-(3.12) and (3.13) justify the advantages of our new approach over the one in [18]. Notice also that these advantages are obtained under the same computational cost, since in practice the computation of the function f requires the computation of f_0 or f_1 as special cases. Next, we present as academic example to show that $L_0 < L_1 < L$, so that the aforementioned advantages will hold.

Example 3.4. Let $X = Y = \mathbb{R}$, $\bar{x} = 1$, $D = U(1, 1-q)$, $q \in [0, 1/2)$ and define function F on D by

$$F(x) = x^3 - q.$$

Then, we get that $\eta = (1-q)/3$, $L_0 = 3-q$, $L = 2(2-q)$ and $L_1 = 2(1+1/L_0)$, So $L_0 < L_1 < L$ holds for each $q \in [0, 1/2)$. However, the old Kantorovich convergence condition is not satisfied, since $h_k > 1$ for each $q \in [0, 1/2)$. However, conditions (3.3) and (3.13) hold respectively for $q \in [0.4620, 0.5]$ and $q \in [0.2757, 0.5]$. Hence, our results can apply.

Hence, there is no guarantee that sequence $\{x_k\}$ converges according to Theorem 18 in [18]. We leave the details to the motivated readers. Our results can also improve along the same lines to the corresponding ones given in [21] concerning Smale's alpha theory and Wang's theory [22].

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