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THE OVER-RELAXED $\eta-$PROXIMAL POINT ALGORITHM AND NONLINEAR VARIATIONAL INCLUSION PROBLEMS

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Abstract. The over-relaxed $(\eta)-$proximal point algorithm in the context of solving a class of inclusion problems, based on the notion of maximal $(\eta)-$monotonicity, is developed. Convergence analysis for the over-relaxed $(\eta)-$proximal point algorithm is examined, and finally, some specializations are included. Furthermore, we remark that the Yosida approximation can be generalized in light of maximal $(\eta)-$monotonicity, and then it can be applied to first-order evolution equations/inclusions.

1. Introduction

Let X be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. We consider the inclusion problem: find a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

Rockafellar [18] introduced the proximal point algorithm, and examined the general convergence and rate of convergence analysis, while solving (1) by

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showing when M is maximal monotone, that the sequence $\{x^k\}$ generated for an initial point x^0 by

$$x^{k+1} \approx P_k(x^k), \tag{2}$$

converges weakly to a solution of (1), provided the approximation is made sufficiently accurate as the iteration proceeds, where $P_k = (I + c_k M)^{-1}$ for a sequence $\{c_k\}$ of positive real numbers that is bounded away from zero. It follows from (2) that x^{k+1} is an approximate solution to inclusion problem

$$0 \in M(x) + c_k^{-1}(x - x^k). \tag{3}$$

We recall the relaxed proximal point algorithm introduced in [7].

Algorithm 1.1. Let $M : X \to 2^X$ be a set-valued maximal monotone mapping on X with $0 \in \text{range}(M)$, and let the sequence $\{x^k\}$ be generated by the iterative procedure

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k w^k \, \forall k \geq 0, \tag{4}$$

where w^k is such that

$$\|w^k - (I + c_k M)^{-1}(x^k)\| \leq \epsilon_k \, \forall k \geq 0,$$

and $\{\epsilon_k\}$, $\{\alpha_k\}$ and $\{c_k\} \subseteq [0, \infty)$ are scalar sequences.

Eckstein and Bertsekas [7] applied Algorithm 1.1 to approximate a weak solution to (1).

Theorem 1.1. [7, Theorem 3] *Let $M : X \to 2^X$ be a set-valued maximal monotone mapping on X with $0 \in range(M)$, and let the sequence $\{x^k\}$ be generated by Algorithm 1.1. If the scalar sequences $\{\epsilon_k\}$, $\{\alpha_k\}$ and $\{c_k\}$ satisfy*

$$E_1 = \Sigma_{k=0}^{\infty} \epsilon_k < \infty, \ \triangle_1 = \inf\ \alpha_k > 0, \ \triangle_2 = \sup\ \alpha_k < 2, \ and\, c = \inf\ c_k > 0,$$

then the sequence $\{x^k\}$ converges weakly to a zero of M.

Convergence analysis for *Algorithm* 1.1 is achieved using the notion of the firm nonexpansiveness of the resolvent operator $(I + cM)^{-1}$. As a whole, the maximal monotonicity has played a powerful role to studying convex programming and variational inequalities. Later it turned out that one of the fundamental algorithms applied to solve these problems was the proximal point algorithm. In [7], Eckstein and Bertsekas has shown that much of the theory of the relaxed proximal point algorithm and related algorithms can be carried over to the Douglas-Rachford splitting method and its special cases, for instance, the alternating direction method of multipliers.

Just recently, Verma [26] generalized the relaxed proximal point algorithm, and applied to the approximation solvability of variational inclusion problems of the form (1). Recently, a great deal of research on the solvability of inclusion problems is carried out using resolvent operator techniques, that have applications to other problems such as equilibria problems in economics, optimization and control theory, operations research, and mathematical programming.

In this paper, we first introduce the over-relaxed $(\eta)-$proximal point algorithm based on the notion of $(\eta)-$ monotonicity, and then apply it for approximating a solution to a general class of nonlinear inclusion problems involving $(\eta)-$ monotone mappings in a Hilbert space setting. Second, we examine the convergence analysis of the over-relaxed $(\eta)-$proximal point algorithm for solving a class of nonlinear inclusions. Also, several results on the generalized firm nonexpansiveness and generalized resolvent mapping are given. The results, thus obtained here, are general and in some cases new. For more details, we refer the reader [1–36].

We note that the solution set for (1) turns out to be the same as of the Yosida inclusion

$$0 \in M_\rho,$$

where $M_\rho = \rho^{-1}(I - (I + \rho M)^{-1})$ is the Yosida approximation of M with parameter $\rho > 0$. It seems in certain ways that it is easier to solve the Yosida inclusion than (1). In other words, M_ρ provides better solvability conditions under right choice for ρ than M itself. On the other hand, M_ρ has also been applied to first-order evolution equations/inclusions in Hilbert space as well as Banach space settings. As in our present situation, resolvent operator $(I + \rho M)^{-1}$ is empowered by $(\eta)-$maximal monotonicity, the Yosida approximation can be generalized in the context of solving first-order evolution equations/inclusions.

The contents for the paper are organized as: Section 1 deals with a historical development of the relaxed proximal point algorithm in conjunction with the $(\eta)-$ maximal monotonicity, and with the approximation solvability of a class of nonlinear inclusion problems using the convergence analysis for the proximal point algorithm, as well as for the relaxed proximal point algorithm. Section 2 introduces and derives some results on unifying $(\eta)-$ maximal monotonicity and generalized firm nonexpansiveness of the generalized resolvent operator. In Section 3, the over-relaxed $(\eta)-$proximal point algorithm is introduced, and then it is applied to approximate the solution to inclusion (1).

2. Maximal η-monotonicity and firm nonexpansiveness

In this section we discus some results based on basic properties of $\eta-$ monotonicity, and then we derive some results involving $\eta-$ monotonicity and the generalized firm nonexpansiveness. Let X denote a real Hilbert space with the norm $\|\cdot\|$ and inner product $< \cdot, \cdot >$. Let $M : X \to 2^X$ be a multivalued mapping on X. We shall denote both the map M and its graph by M, that is, the set $\{(x, y) : y \in M(x)\}$. This is equivalent to stating that a mapping is any subset M of $X \times X$, and $M(x) = \{y : (x, y) \in M\}$. If M is single-valued, we shall still use $M(x)$ to represent the unique y such that $(x, y) \in M$ rather than the singleton set $\{y\}$. This interpretation shall much depend on the context. The domain of a map M is defined (as its projection onto the first argument) by

$$\text{dom}(M) = \{x \in X : \exists\, y \in X : (x, y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

dom(T)=X, shall denote the full domain of M, and the range of M is defined by

$$\text{range}(M) = \{y \in X : \exists\, x \in X : (x, y) \in M\}.$$

The inverse M^{-1} of M is $\{(y, x) : (x, y) \in M\}$. For a real number ρ and a mapping M, let $\rho M = \{x, \rho y) : (x, y) \in M\}$. If L and M are any mappings, we define

$$L + M = \{(x, y + z) : (x, y) \in L, (x, z) \in M\}.$$

Definition 2.1. Let $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be:

(i) Monotone if
$$\langle u^* - v^*, u - v\rangle \geq 0\, \forall\, (u, u^*), (v, v^*) \in \text{graph}(M).$$

(ii) $(r)-$ strongly monotone if there exists a positive constant r such that
$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2\, \forall\, (u, u^*), (v, v^*) \in \text{graph}(M).$$

(iii) Strongly monotone if
$$\langle u^* - v^*, u - v\rangle \geq \|u - v\|^2\, \forall\, (u, u^*), (v, v^*) \in \text{graph}(M).$$

(iv) $(r)-$strongly pseudomonotone if
$$\langle v^*, u - v\rangle \geq 0$$
implies
$$\langle u^*, u - v\rangle \geq r\|u - v\|^2\, \forall\, (u, u^*), (v, v^*) \in \text{graph}(M).$$

(v) Pseudomonotone if
$$\langle v^*, u - v\rangle \geq 0$$
implies
$$\langle u^*, u - v\rangle \geq 0\, \forall\, (u, u^*), (v, v^*) \in \text{graph}(M).$$

(vi) $(m)-$relaxed monotone if there exists a positive constant m such that
$$\langle u^* - v^*, u - v\rangle \geq (-m)\|u - v\|^2\, \forall\, (u, u^*), (v, v^*) \in \text{graph}(M).$$

(vii) $(c)-$ cocoercive if there is a positive constant c such that
$$\langle u^* - v^*, u - v\rangle \geq c\|u^* - v^*\|^2\, \forall\, (u, u^*), (v, v^*) \in \text{graph}(M).$$

Definition 2.2. Let $M : X \to 2^X$ be a mapping on X. The map M is said to be:

(i) Nonexpansive if

$$\|u^* - v^*\| \leq \|u - v\| \,\forall\, (u, u^*), (v, v^*) \in \text{graph}(M).$$

(ii) Firmly nonexpansive if

$$\|u^* - v^*\|^2 \leq \langle u^* - v^*, u - v\rangle \,\forall\, (u, u^*), (v, v^*) \in \text{graph}(M).$$

(iii) $(c)-$Firmly nonexpansive if there exists a constant $c > 0$ such that

$$\|u^* - v^*\|^2 \leq c\langle u^* - v^*, u - v\rangle \,\forall\, (u, u^*), (v, v^*) \in \text{graph}(M).$$

Definition 2.3. A map $\eta : X \times X \to X$ is said to be:

(i) Monotone if

$$\langle x - y, \eta(x, y)\rangle \geq 0 \,\forall\, (x, y) \in X.$$

(ii) (t)-strongly monotone if there exists a positive constant t such that

$$\langle x - y, \eta(x, y)\rangle \geq t\|x - y\|^2 \,\forall\, (x, y) \in X.$$

(iii) Strongly monotone if

$$\langle x - y, \eta(x, y)\rangle \geq \|x - y\|^2 \,\forall\, (x, y) \in X.$$

(iii) (τ)-Lipschitz continuous if there exists a positive constant τ such that

$$\|\eta(x, y)\| \leq \tau\|x - y\|.$$

Definition 2.4. A map $M : X \to 2^X$ is said to be maximal (η)- monotone if

(i) M is $(\eta)-$ monotone

(ii) $R(I + cM) = X$ for $c > 0$.

Proposition 2.1. *Let $M : X \to 2^X$ be a maximal $(\eta)-$ monotone mapping. Then $(I + cM)$ is maximal monotone for $c > 0$, where I is the identity mapping.*

Proposition 2.2. *Let $M : X \to 2^X$ be a maximal $(\eta)-$monotone mapping. Then generalized resolvent operator $(I + cM)^{-1}$ is single-valued, where I is the identity mapping.*

Definition 2.5. Let $M : X \to 2^X$ be a maximal $(\eta)-$ monotone mapping. Then the generalized resolvent operator $J_c^{M,\eta} : X \to X$ is defined by

$$J_c^{M,\eta}(u) = (I + cM)^{-1}(u).$$

Definition 2.6. Let $M : X \to 2^X$ be a multivalued mapping on X, and let $\eta : X \times X \to X$ be another mapping. The map M is said to be:

(i) $(\eta)-$ monotone if

$$\langle u^* - v^*, \eta(u,v)\rangle \geq 0 \,\forall\, (u,u^*), (v,v^*) \in \text{graph}(M).$$

(ii) $(r,\eta)-$ strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, \eta(u,v)\rangle \geq r\|u-v\|^2 \,\forall\, (u,u^*), (v,v^*) \in \text{graph}(M).$$

(iii) $(\eta)-$ strongly monotone if

$$\langle u^* - v^*, \eta(u,v)\rangle \geq \|u-v\|^2 \,\forall\, (u,u^*), (v,v^*) \in \text{graph}(M).$$

(iv) $(r,\eta)-$strongly pseudomonotone if

$$\langle v^*, \eta(u,v)\rangle \geq 0$$

implies

$$\langle u^*, \eta(u,v)\rangle \geq r\|u-v\|^2 \,\forall\, (u,u^*), (v,v^*) \in \text{graph}(M).$$

(v) $(\eta)-$ pseudomonotone if

$$\langle v^*, \eta(u,v)\rangle \geq 0$$

implies

$$\langle u^*, \eta(u,v)\rangle \geq 0 \,\forall\, (u,u^*), (v,v^*) \in \text{graph}(M).$$

(vi) $(m,\eta)-$relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, \eta(u,v)\rangle \geq (-m)\|u-v\|^2 \,\forall\, (u,u^*), (v,v^*) \in \text{graph}(M).$$

(vii) $(c,\eta)-$ cocoercive if there is a positive constant c such that

$$\langle u^* - v^*, \eta(u,v)\rangle \geq c\|u^*-v^*\|^2 \,\forall\, (u,u^*), (v,v^*) \in \text{graph}(M).$$

Proposition 2.3. *Let X be a real Hilbert space, let $M : X \to 2^X$ be maximal $(\eta)-$ monotone, and let $\eta : X \times X \to X$ be $(t)-$ strongly monotone. Then the resolvent operator associated with M and defined by*

$$J_\rho^{M,\eta}(u) = (I + \rho M)^{-1}(u) \,\forall\, u \in X,$$

satisfies the following:

$$\langle u - v, \eta(J_\rho^{M,\eta}(u), J_\rho^{M,\eta}(v))\rangle \geq t\|J_\rho^{M,\eta}(u) - J_\rho^{M,\eta}(v)\|^2. \tag{5}$$

Proof. For any $u, v \in X$, it follows from the definition of the resolvent operator $J_\rho^{M,\eta}$ that

$$\frac{1}{\rho}[u - J_\rho^{M,\eta}(u)] \in M(J_\rho^{M,\eta}(u)),$$

and

$$\frac{1}{\rho}[v - J_\rho^{M,\eta}(v)] \in M(J_\rho^{M,\eta}(v)).$$

Since M is $(\eta)-$ monotone, we have

$$\begin{aligned}&\frac{1}{\rho}\langle u - v - [J_\rho^{M,\eta}(u) - J_\rho^{M,\eta}(v)],\\ &\eta(J_\rho^{M,\eta}(u), J_\rho^{M,\eta}(v))\rangle \geq 0.\end{aligned} \tag{6}$$

In light of (6), we have

$$\begin{aligned}&\langle u - v, \eta(J_\rho^{M,\eta}(u), J_\rho^{M,\eta}(v))\rangle\\ \geq\ &\langle J_\rho^{M,\eta}(u) - J_\rho^{M,\eta}(v), \eta(J_\rho^{M,\eta}(u), J_\rho^{M,\eta}(v))\rangle\\ \geq\ &t\|J_\rho^{M,\eta}(u) - J_\rho^{M,\eta}(v)\|^2.\end{aligned}$$

□

Proposition 2.4. *Let X be a real Hilbert space, let $M : X \to 2^X$ be maximal $(\eta)-$ monotone, and let $\eta : X \times X \to X$ be $(t)-$ strongly monotone and $(\tau)-$ Lipschitz continuous. Then the resolvent operator associated with M and defined by*

$$J_\rho^{M,\eta}(u) = (I + \rho M)^{-1}(u)\,\forall\, u \in X,$$

satisfies the following:

(i) *For $\tau < 1$, we have*

$$\langle u - v, \eta(J_\rho^{M,\eta}(u), J_\rho^{M,\eta}(v))\rangle \leq \langle u - v, J_\rho^{M,\eta}(u) - J_\rho^{M,\eta}(v)\rangle\,\forall\, u, v \in X. \tag{7}$$

(ii) *For $J_k^* = I - J_\rho^{M,\eta}$, we have (for $t > \frac{1}{2}$)*

$$\langle u - v, J_k^*(u) - J_k^*(v)\rangle \geq \frac{t-1}{2t-1}\|u - v\|^2 + \frac{t}{2t-1}\|J_k^*(u) - J_k^*(v)\|^2. \tag{8}$$

Proof. The proof of (i) follows from the $(\tau)-$Lipschitz continuity of η for $\tau < 1$. To prove (ii), we apply (i) to Proposition 2.3, and we get

$$\langle u - v, J_\rho^{M,\eta}(u) - J_\rho^{M,\eta}(v)\rangle \geq t\|J_\rho^{M,\eta}(u) - J_\rho^{M,\eta}(v)\|^2. \tag{9}$$

It further follows that

$$\begin{aligned}&\langle u - v, u - v - (J_k^*(u) - J_k^*(v))\rangle\\ \geq\ &t[\|J_k^*(u) - J_k^*(v)\|^2 + \|u - v\|^2 - 2\langle J_k^*(u) - J_k^*(v), u - v\rangle].\end{aligned}$$

□

3. The over-relaxed $(\eta)-$proximal point algorithm

This section deals with the over-relaxed $(\eta)-$ proximal point algorithm and its application to approximation solvability of the inclusion problem (1) based on the maximal (η)-monotonicity. Furthermore, some results connecting the $(\eta)-$ monotonicity and corresponding resolvent operator are established, that generalize the results on the firm nonexpansiveness [7], while the auxiliary results on maximal $(\eta)-$ monotonicity and general maximal monotonicity are obtained.

Theorem 3.1. *Let X be a real Hilbert space, and let $M : X \to 2^X$ be maximal $(\eta)-$ monotone. Then the following statements are mutually equivalent:*

(i) *An element $u \in X$ is a solution to* (1).

(ii) *For an $u \in X$, we have*

$$u = J_c^{M,\eta}(u)\, for\, c > 0, .$$

where

$$J_c^{M,\eta}(u) = (I + cM)^{-1}(u).$$

Proof. It follows from the definition of the generalized resolvent operator corresponding to M. □

Note that Theorem 3.1 generalizes [7, Lemma 2] to the case of a maximal $(\eta)-$ monotone mapping.

Next, we present a generalization to the relaxed Proximal point algorithm [26] based on the $(\eta)-$monotonicity.

Algorithm 3.1. Let $M : X \to 2^X$ be a set-valued maximal $(\eta)-$ monotone mapping on X with $0 \in \mathrm{range}(M)$, and let the sequence $\{x^k\}$ be generated by the iterative procedure

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \,\forall\, k \geq 0, \tag{10}$$

and y^k satisfies

$$\|y^k - J_{c_k}^{M,\eta}(x^k)\| \leq \delta_k \|y^k - x^k\|,$$

where $J_{c_k}^{M,\eta} = (I + c_k M)^{-1}$, $\delta_k \to 0$ and

$$y^{k+1} = (1 - \alpha_k)x^k + \alpha_k J_{c_k}^{M,\eta}(x^k) \,\forall\, k \geq 0.$$

Here

$$\{\delta_k\}, \{\alpha_k\}, \{c_k\} \subseteq [0, \infty)$$

are scalar sequences such that $\inf \alpha_k > 0$, $\sup \alpha_k < 2$, and $\sum_{k=0}^{\infty} \delta_k < \infty$.

Algorithm 3.2. Let $M : X \to 2^X$ be a set-valued maximal $(\eta)-$monotone mapping on X with $0 \in \text{range}(M)$, and let the sequence $\{x^k\}$ be generated by the iterative procedure

$$x^{k+1} = (1 - \alpha_k - \beta_k)x^k + \alpha_k y^k \, \forall \, k \geq 0, \tag{11}$$

and y^k satisfies

$$\|y^k - J_{\rho_k}^M(x^k)\| \leq \delta_k \|y^k - x^k\|,$$

where $J_{\rho_k}^M = (I + \rho_k M)^{-1}$, and

$$\{\delta_k\}, \{\alpha_k\}, \{\beta_k\}\{c_k\} \subseteq [0, \infty)$$

are scalar sequences such that $\inf \alpha_k > 0$, $\sup \alpha_k < 2$, and $\sum_{k=0}^{\infty} \delta_k < \infty$.

For $\delta_k = (1 + \frac{1}{1+k})$ in Algorithm 3.1, for $k > 0$, we have

Algorithm 3.3. Let $M : X \to 2^X$ be a set-valued maximal $(\eta)-$ monotone mapping on X with $0 \in \text{range}(M)$, and let the sequence $\{x^k\}$ be generated by the iterative procedure

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \, \forall \, k \geq 0, \tag{12}$$

and y^k satisfies

$$\|y^k - J_{c_k}^{M,\eta}(x^k)\| \leq (1 + \frac{1}{1+k})\|y^k - x^k\|,$$

where $J_{c_k}^{M,\eta} = (I + c_k M)^{-1}$, and

$$y^{k+1} = (1 - \alpha_k)x^k + \alpha_k J_{c_k}^{M,\eta}(x^k) \, \forall \, k \geq 0.$$

Here

$$\{\alpha_k\}, \{c_k\} \subseteq [0, \infty)$$

are scalar sequences such that inf $\alpha_k > 0$, and sup $\alpha_k < 2$, and $\sum_{k=1}^{\infty} \frac{1}{k} < \infty$.

Theorem 3.3. *Let X be a real Hilbert space. Let $M : X \to 2^X$ be maximal $(\eta)-$monotone and x^* be a zero of M. Let $\eta : X \times X \to X$ be $(t)-$ strongly monotone and $(\tau)-$Lipschitz continuous. Let the sequence $\{x^k\}$ be generated by Algorithm 3.1. Suppose that the sequence $\{x^k\}$ is bounded in the sense that there exists at least one solution to $0 \in M(x)$. Then (for $\tau < 1$ and $t > \frac{1}{2}$)*

$$(2t - 1)\|J_{c_k}^{M,\eta}(x^k) - x^*\|^2 \leq \|x^k - x^*\|^2 - \|J_k^*(x^k)\|^2, \tag{13}$$

where

$$J_k^* = I - J_{c_k}^{M,\eta}.$$

In addition, assume that M^{-1} is $(a)-$Lipschitz continuous at 0, that is, there exists a unique solution z^* to $0 \in M(z)$ (equivalently, $M^{-1}(0) = \{z^*\}$) and for constants $a \geq 0$ and $b > 0$, we have

$$\|z - z^*\| \leq a\|w\| \text{ whenever } z \in M^{-1}(w) \text{ and } \|w\| \leq b. \tag{14}$$

Then the sequence $\{x^k\}$ *converges linearly to a unique solution* x^* *with rate*

$$\sqrt{1-\alpha^*(2t-(2t-1)\alpha^*)\frac{c^{*2}}{a^2(2t-1)+c^{*2}}}<1 \quad for \quad t>\frac{1}{2},$$

where $\alpha^* = \limsup_{k\to\infty} \alpha_k$, *and sequences* $\{\alpha_k\}$, *and* $\{c_k\}$ *satisfy* $\alpha_k \geq 1$, $c_k \nearrow c^* \leq \infty$, *and* $\inf_{k\geq 0}\ \alpha_k > 0$ *and* $\sup_{k\geq 0}\ \alpha_k < \frac{2t}{2t-1}$.

Proof. Suppose that x^* is a zero of M. For all $k \geq 0$, we set

$$J_k^* = I - J_{c_k}^{M,\eta}.$$

Therefore, $J_k^*(x^*) = 0$. Then, in light of Theorem 3.1, any solution to (1) is a fixed point of $J_{c_k}^{M,\eta}$, and hence a zero of J_k^*. First, we express

$$\begin{aligned} y^{k+1} &= (1-\alpha_k)x^k + \alpha_k J_{c_k}^{M,\eta}(x^k) \\ &= (I-\alpha_k J_k^*)(x^k). \end{aligned}$$

Now we begin verifying the boundedness of the sequence $\{x^k\}$ leading to $x^k - J_{c_k}^{M,\eta}(x^k) \to 0$.

Next, we estimate using Proposition 2.4 ($for\, t > \frac{1}{2}$)

$$\begin{aligned} \|y^{k+1}-x^*\|^2 &= \|(1-\alpha_k)x^k + \alpha_k J_{c_k}^{M,\eta}(x^k) - x^*\|^2 \\ &= \|x^k - x^* - \alpha_k J_k^*(x^k)\|^2 \\ &\leq \|x^k-x^*\|^2 - 2\alpha_k\langle x^k-x^*, J_k^*(x^k)-J_k^*(x^*)\rangle + \alpha_k^2\|J_k^*(x^k)\|^2 \\ &\leq \|x^k-x^*\|^2 - \frac{2(t-1)\alpha_k}{2t-1}\|x^k-x^*\|^2 - \frac{2t}{2t-1}\alpha_k\|J_k^*(x^k)\|^2 \\ &\quad +\alpha_k^2\|J_k^*(x^k)\|^2 \\ &= (1-\frac{2(t-1)\alpha_k}{2t-1})\|x^k-x^*\|^2 - \alpha_k(\frac{2t}{2t-1}-\alpha_k)\|J_k^*(x^k)\|^2. \end{aligned}$$

Since under the assumptions $\alpha_k(\frac{2t}{2t-1}-\alpha_k) > 0$, it follows that

$$\|y^{k+1}-x^*\| \leq \|x^k-x^*\|.$$

Therefore,

$$\begin{aligned} &\|x^{k+1}-y^{k+1}\| \\ =\ &\|(1-\alpha_k)x^k + \alpha_k y^k - [(1-\alpha_k)x^k + \alpha_k J_{c_k}^{M,\eta}(x^k)]\| \\ =\ &\|\alpha_k(y^k - J_\rho^{M,\eta}(x^k))\| \\ \leq\ &\alpha_k\delta_k\|y^k-x^k\|. \end{aligned}$$

Now we find the estimate leading to the boundedness of the sequence $\{x^k\}$.

$$\begin{aligned}\|x^{k+1}-x^*\| &\leq \|y^{k+1}-x^*\|+\|x^{k+1}-y^{k+1}\| \\ &\leq \|x^k-x^*\|+\alpha_k\delta_k\|y^k-x^k\| \\ &\leq \|x^0-x^*\|+\sum_{j=0}^{k}\alpha_j\delta_j\|y^j-x^j\|. \end{aligned} \tag{15}$$

Thus, the sequence $\{x^k\}$ is bounded.

We further examine the estimate

$$\begin{aligned} & \|x^{k+1}-x^*\|^2 \\ = & \|y^{k+1}-x^*+x^{k+1}-y^{k+1}\|^2 \\ = & \|y^{k+1}-x^*\|^2+2\langle y^{k+1}-x^*, x^{k+1}-y^{k+1}\rangle+\|x^{k+1}-y^{k+1}\|^2 \\ \leq & \|y^{k+1}-x^*\|^2+2\|y^{k+1}-x^*\|\|x^{k+1}-y^{k+1}\|+\|x^{k+1}-y^{k+1}\|^2 \\ \leq & \|x^k-x^*\|^2-\alpha_k(\frac{2t}{2t-1}-\alpha_k)\|J_k^*(x^k)\|^2-\frac{2(t-1)}{2t-1}\|x^k-x^*\|^2 \\ & +2(\|x^{k+1}-x^*\|+\|x^{k+1}-y^{k+1}\|)\|x^{k+1}-y^{k+1}\|+\|x^{k+1}-y^{k+1}\|^2 \\ \leq & \|x^k-x^*\|^2-\alpha_k(\frac{2t}{2t-1}-\alpha_k)\|J_k^*(x^k)\|^2-\frac{2(t-1)}{2t-1}\|x^k-x^*\|^2 \\ & +2(\|x^{k+1}-x^*\|+\alpha_k\delta_k\|y^k-x^k\|)\alpha_k\delta_k\|y^k-x^k\| \\ & +\alpha_k^2\delta_k^2\|y^k-x^k\|^2, \end{aligned} \tag{16}$$

where $\alpha_k(\frac{2t}{2t-1}-\alpha_k)>0$.

Since $\{\delta_k\}$ is summable, so is $\{\delta_k^2\}$. As $k\to\infty$, we have that

$$\sum_{j=0}^{k}\|J_j^*(x^j)\|^2<\infty \Rightarrow \lim_{k\to\infty} J_k^*(x^k)=0,$$

that is, $x^k-J_{c_k}^{M,\eta}(x^k)\to 0$.

Now we turn our attention (using the above argument) to linear convergence of the sequence $\{x^k\}$. Since $\lim_{k\to\infty} J_k^*(x^k)=0$, it implies for k large that

$$c_k^{-1}J_k^*(x^k)\in M(J_{c_k}^{M,\eta}(x^k)).$$

Therefore, in light of (14), by taking $w=c_k^{-1}J_k^*(x^k)$ and $z=J_{c_k}^{M,\eta}(x^k)$, we have

$$\|J_{c_k}^{M,\eta}(x^k)-x^*\|\leq a\|c_k^{-1}J_k^*(x^k)\|\,\forall\, k\geq k'.$$

Applying (13), we arrive at

$$\|J_{c_k}^{M,\eta}(x^k)-x^*\|^2\leq \frac{a^2}{c_k^2+(2t-1)a^2}\|x^k-x^*\|^2 \quad for \quad t>\frac{1}{2}, \tag{17}$$

where $J_{c_k}^{M,\eta}(x^*) = x^*$. Since $y^{k+1} := (1-\alpha_k)x^k + \alpha_k J_{c_k}^{M,\eta}(x^k)$, we estimate, using (17) and ($\alpha_k \geq 1$) that

$$
\begin{aligned}
& \|y^{k+1} - x^*\|^2 \\
= \ & \|(1-\alpha_k)x^k + \alpha_k J_{c_k}^{M,\eta}(x^k) - x^*\|^2 \\
= \ & \|\alpha_k(J_{c_k}^{M,\eta}(x^k) - x^*) + (1-\alpha_k)(x^k - x^*)\|^2 \\
= \ & \alpha_k^2 \|J_{c_k}^{M,\eta}(x^k) - x^*\|^2 + (1-\alpha_k)^2\|x^k - x^*\|^2 + \\
& 2\alpha_k(1-\alpha_k)\langle J_{c_k}^{M,\eta}(x^k) - x^*, x^k - x^*\rangle \\
\leq \ & \alpha_k^2 \|J_{c_k}^{M,\eta}(x^k) - x^*\|^2 + (1-\alpha_k)^2\|x^k - x^*\|^2 + \\
& 2\alpha_k(1-\alpha_k)\langle \eta(J_{c_k}^{M,\eta}(x^k), x^*), x^k - x^*\rangle \\
\leq \ & \alpha_k^2 \|J_{c_k}^{M,\eta}(x^k) - x^*\|^2 + (1-\alpha_k)^2\|x^k - x^*\|^2 + \\
& 2\alpha_k(1-\alpha_k)t\|J_{c_k}^{M,\eta}(x^k) - x^*\|^2 \\
= \ & [2\alpha_k(1-\alpha_k)t + \alpha_k^2]\|J_{c_k}^{M,\eta}(x^k) - x^*\|^2 + (1-\alpha_k)^2\|x^k - x^*\|^2 \\
= \ & \alpha_k[2t - (2t-1)\alpha_k)]\|J_{c_k}^{M,\eta}(x^k) - x^*\|^2 + (1-\alpha_k)^2\|x^k - x^*\|^2 \\
\leq \ & \alpha_k[2t - (2t-1)\alpha_k)]\frac{a^2}{c_k^2 + (2t-1)a^2}\|x^k - x^*\|^2 + (1-\alpha_k)^2\|x^k - x^*\|^2 \\
= \ & \alpha_k[2t - (2t-1)\alpha_k)]\frac{a^2}{c_k^2 + (2t-1)a^2} + (1-\alpha_k)^2)\|x^k - x^*\|^2,
\end{aligned}
$$

where $\alpha_k[2t - (2t-1)\alpha_k)] > 0$.
Hence, we have

$$\|y^{k+1} - x^*\| \leq \theta_k \|x^k - x^*\|,$$

where

$$\theta_k = \sqrt{\alpha_k[2t - (2t-1)\alpha_k][\frac{a^2}{c_k^2 + (2t-1)a^2}] + (1-\alpha_k)^2} < 1, \tag{18}$$

$\alpha_k[2t - (2t-1)\alpha_k)] > 0$ and $\alpha_k \geq 1$. Since Algorithm 3.1 ensures

$$\|y^k - J_{c_k}^{M,\eta}(x^k)\| \leq \delta_k \|y^k - x^k\|,$$

and

$$\alpha_k(y^k - x^k) = x^{k+1} - x^k,$$

we have

$$\begin{aligned}
\|x^{k+1} - x^*\| &= \|y^{k+1} - x^* + x^{k+1} - y^{k+1}\| \\
&\leq \|y^{k+1} - x^*\| + \|x^{k+1} - y^{k+1}\| \\
&\leq \|y^{k+1} - x^*\| + \alpha_k \delta_k \|y^k - x^k\| \\
&= \|y^{k+1} - x^*\| + \delta_k \|x^{k+1} - x^k\| \\
&\leq \|y^{k+1} - x^*\| + \delta_k \|x^{k+1} - x^*\| + \delta_k \|x^k - x^*\|.
\end{aligned}$$

It follows that

$$\|x^{k+1} - x^*\| \leq \frac{\theta_k + \delta_k}{1 - \delta_k} \|x^k - x^*\|,$$

where

$$\begin{aligned}
\limsup \frac{\theta_k + \delta_k}{1 - \delta_k} &= \limsup \theta_k \\
&= \sqrt{1 - \alpha^*[2t - (2t-1)\alpha^*][\frac{c^{*2}}{c^{*2} + (2t-1)a^2}]} < 1.
\end{aligned}$$

□

Theorem 3.4. *Let X be a real Hilbert space, and let $M : X \to 2^X$ be maximal $(\eta)-$ monotone. Let $\eta : X \times X \to X$ be $(t)-$ strongly monotone and $(\tau)-$ Lipschitz continuous. For an arbitrarily chosen initial point x^0, let the sequence $\{x^k\}$ be bounded (in the sense that there exists at least one solution to $0 \in M(x)$) and generated by Algorithm 3.2 as*

$$x^{k+1} = (1 - \alpha_k - \beta_k)x^k + \alpha_k y^k \, for \, k \geq 0$$

with

$$\|y^k - J^{M,\eta}_{c_k}(x^k)\| \leq \epsilon_k,$$

where $J^{M,\eta}_{c_k} = (I + \rho_k M)^{-1}$, and sequences

$$\{c_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty)$$

satisfy $E_1 = \Sigma_{k=0}^{\infty} \epsilon_k < \infty$, $\triangle_1 = \inf \alpha_k > 0$, $\triangle_2 = \sup \alpha_k < 2$, and $c_k^{-1} = \inf \rho_k > 0$. Then the sequence $\{x^k\}$ converges weakly to a solution of (1).

Proof. The proof is similar to that the first part of Theorem 3.3 and then applying the generalized Representation Lemma. □

Theorem 3.5. *Let X be a real Hilbert space, and let $M : X \to 2^X$ be maximal $(\eta)-$-monotone with $0 \in range(M)$. Let $\eta : X \times X \to X$ be $(t)-$ strongly monotone and $(\tau)-$ Lipschitz continuous. Let the sequence $\{x^k\}$ be bounded (in the sense that there exists at least one solution to $0 \in M(x)$) and generated by Algorithm 3.3. Let $0 \in M(x)$ have a solution x^*. Suppose that M^{-1} is*

$(a)-$Lipschitz continuous at 0 for $a \geq 0$. Then the sequence $\{x^k\}$ converges linearly to a unique solution x^* with rate

$$\sqrt{1-\alpha^*(2t-(2t-1)\alpha^*)\frac{c^{*2}}{a^2(2t-1)+c^{*2}}} < 1,$$

where $\alpha^* = \limsup_{k\to\infty} \alpha_k$, and sequences $\{\alpha_k\}$, and $\{c_k\}$ satisfy $\alpha_k \geq 1$, $c_k \nearrow c^* \leq \infty$, and $t > \frac{1}{2}$.

Proof. The proof is similar to that of Theorem 3.3. □

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