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A MODIFIED GRADIENT PROJECTION ALGORITHM FOR VARIATIONAL INEQUALITIES AND RELATIVELY NONEXPANSIVE MAPPINGS IN BANACH SPACES

Ying Liu[1] and Yuping Cui[2]

[1]College of Mathematics and Information Science
Hebei University, Baoding, Hebei, China
e-mail: ly_cyh2013@163.com

[2]College of Mathematics and Information Science
Hebei University, Baoding, Hebei, China
e-mail: cuiyuping@cmc.hbu.cn

Abstract. In this paper, we modify the gradient projection algorithm proposed by [K. Nakajo, Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces, Applied Mathematics and Computation, 271 (2015), 251-258] for solving a variational inequality problem involving a monotone and Lipschitz continuous mapping and a fixed point problem of a relatively nonexpansive mapping in Banach spaces. The purpose of this modification is to replace a projection onto a general closed convex set to one projection onto a half-space. The latter projection (onto a halfspace) is easier to compute. We prove the iteration sequence generated by this method is weak convergence.

1. Introduction

We consider the following variational inequality problem which is to find a point $x^* \in C$ such that

$$\langle x - x^*, Ax^* \rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where C is a closed convex subset of a Banach space E, $\langle x, f \rangle$ denotes the duality pairing of E and it's dual E^*, and $A : E \to E^*$ is a some mapping. Let $VI(C, A)$ be the solution set of the variational inequality (1.1).

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Definition 1.1. A mapping $A : C \to E^*$ is called monotone, if

$$\langle x - y, Ax - Ay\rangle \geq 0, \quad \forall x, y \in C.$$

Definition 1.2. A mapping $A : C \to E^*$ is called $L-$Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

Variational inequality theory, as a very effective and powerful tool of the current mathematical technology, has been widely applied to mathematical programming, optimization and control, economics and transportation equilibrium, engineering sciences, etc.

Definition 1.3. ([5]) An operator A of C into E^* is said to be $\alpha-$inverse-strongly-monotone, if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

In order to approximate a solution of the variational inequality (1.1), the inverse-strong-monotonicity of A was often assumed (see, for example, [4]-[7]). Especially, in [7], Liu proved the following theorem.

Theorem 1.4. *Let E be a uniformly smooth, 2-uniformly convex Banach space. Let C be a nonempty, closed convex subset of E. Assume that A is an operator of C into E^* that satisfies:*

(A1) *A is $\alpha-$inverse-strongly-monotone,*
(A2) *$VI(C, A) \neq \emptyset$,*
(A3) *$\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(C, A)$.*

Assume that T is a relatively nonexpansive mapping from C into itself such that $F = F(T) \bigcap VI(C, A) \neq \emptyset$. The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ w_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)J\Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n))), \\ z_n = \Pi_C w_n, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTz_n), \\ C_n = \{v \in C : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \bigcap Q_n} x_0. \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy:

$$0 \leq \alpha_n < 1, \ \limsup_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad 0 \leq \beta_n < 1, \ \limsup_{n\to\infty} \beta_n < 1.$$

If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c_1\alpha$, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where c_1 is the $2-$uniformly convexity constant of E.

We know that if A is $\alpha-$inverse-strongly-monotone, then it is monotone and $\frac{1}{\alpha}-$Lipschitz continuous. But, the converse is not true. One question arises naturally:

(Q1) How to extend Theorem 1.4 to the more general class of monotone and continuous mappings? In addition, we also note that the condition (A3) is not easy to be satisfied. The necessity of the condition needs to be checked. Hence, we have the following question.

(Q2) Can the condition (A3) be removed?

Recently, Nakajo in [9] proposed the following three CQ methods:

Algorithm 1.1

$$\begin{cases} x_1 &= x \in E, \\ y_n &= \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ z_n &= Ty_n, \\ C_n &= \{u \in C : \phi(u, z_n) \leq \phi(u, x_n) - \phi(y_n, x_n) \\ & \qquad -2\lambda_n \langle y_n - u, Ax_n - Ay_n \rangle\}, \\ Q_n &= \{u \in C : \langle x_n - u, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \bigcap Q_n} x. \end{cases}$$

Algorithm 1.2

$$\begin{cases} x_1 &= x \in E, \\ y_n &= \Pi_C J^{-1}(JTx_n - \lambda_n AT(x_n)), \\ C_n &= \{u \in C : \phi(u, y_n) \leq \phi(u, x_n) - \phi(y_n, Tx_n) \\ & \qquad -2\lambda_n \langle y_n - u, ATx_n - Ay_n \rangle\}, \\ Q_n &= \{u \in C : \langle x_n - u, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \bigcap Q_n} x. \end{cases}$$

Algorithm 1.3

$$\begin{cases} x_1 &= x \in E, \\ y_n &= \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ z_n &= J^{-1}(\alpha_n JTx_n + (1-\alpha_n)Jy_n), \\ C_n &= \{u \in C : \phi(u, z_n) \leq \phi(u, x_n) - (1-\alpha_n)\phi(y_n, x_n) \\ & \qquad -2\lambda_n(1-\alpha_n) \langle y_n - u, Ax_n - Ay_n \rangle\}, \\ Q_n &= \{u \in C : \langle x_n - u, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \bigcap Q_n} x. \end{cases}$$

In the Algorithms 1.1–1.3, E is a 2-uniformly convex and uniformly smooth Banach space and A is only supposed to be monotone and Lipschitz continuous. The author proved the sequences $\{x_n\}$ generated by Algorithms 1.1–1.3 strongly converge to $\Pi_D x$, where $D = VI(C, A) \bigcap F(T)$ and T is a relatively nonexpansive mapping.

In the Algorithms 1.1–1.3, the condition (A3) assumed in Theorem 1.4 is removed and the inverse-strong-monotonicity of A is successfully weakened to monotonicity and Lipschitz continuity. Therefore, the work done by Nakajo [9] is very meaningful. However, we should also note that the Algorithms 1.1–1.3 seem to be difficult to use in practice because the computation of the next iterate becomes a subproblem of finding a general minimal distance onto the intersection of two additional closed and convex subsets of a Banach space E. As mentioned in [3], it is not easy to solve a minimal distance onto a general closed and convex set even if in a Hilbert space. This might seriously affect the efficiency of the Algorithms 1.1–1.3. Therefore, the purpose of this paper is to modify the Algorithms 1.1–1.3 to avoid this subproblem produced in Algorithms 1.1–1.3 and answer the questions (Q1) and (Q2).

2. Preliminaries

Throughout this paper, let E be a Banach space, and E^* be the dual space of E. $\langle \cdot, \cdot \rangle$ denotes the duality pairing of E and E^*. Let $\mathbb{N}$ be the set of all positive integers. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$Jx := \{v \in E^* : \langle x, v \rangle = \|v\|^2 = \|x\|^2\}, \quad \forall x \in E.$$

The following properties of the duality mapping J can be found in [2]:

(i) If E is smooth, then J is single-valued.
(ii) If E is strictly convex, then J is one-to-one and strictly monotone.
(iii) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
(iv) If E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one, onto and J^{-1} is also single-valued, one-to-one, surjective and it is the normalized duality mapping from E^* into E.

Let E be a smooth Banach space. Define

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \ \ \forall x, y \in E. \tag{2.1}$$

Remark 2.1. We have from Remark 2.1 in [8] that if E is a strictly convex and smooth Banach space, then for $x, y \in E, \phi(y, x) = 0$ if and only if $x = y$.

Let E be a reflexive, strictly convex, and smooth Banach space. K denotes a nonempty, closed, and convex subset of E. By [1], for each $x \in E$, there exists a unique element $x_0 \in K$ (denoted by $\Pi_K(x)$) such that

$$\phi(x_0, x) = \min_{y \in K} \phi(y, x).$$

The mapping $\Pi_K : E \to K$ defined by $\Pi_K(x) = x_0$ is called the generalized projection operator from E onto K. Moreover, x_0 is called the generalized projection of x.

Lemma 2.2. ([1]) *Let E be a reflexive, strictly convex, and smooth Banach space. Let C be a nonempty, closed, and convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.3. ([1]) *Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and let $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4. ([8]) *Let E be a uniformly convex and smooth Banach space. Let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$ as $n \to \infty$.*

Lemma 2.5. ([5]) *Let S be a nonempty, closed, and convex subset of a uniformly convex, smooth Banach space E. Let $\{x_n\}$ be a sequence in E. Suppose that, for all $u \in S$,*

$$\phi(u, x_{n+1}) \leq \phi(u, x_n)$$

for every $n = 1, 2, \cdots$. Then $\{\Pi_S x_n\}$ is a Cauchy sequence.

Lemma 2.6. ([9]) *Let E be a $2-$uniformly convex and smooth Banach space. Then, there exists a constant $c_1 > 0$, such that, for every $x, y \in E$,*

$$\phi(x, y) \geq c_1 \|x - y\|^2,$$

where c_1 is called the $2-$uniformly convex constant.

Lemma 2.7. ([9]) *Let C be a nonempty, closed convex subset of E and A be a monotone and hemicontinuous operator of C into E^*. Then*

$$VI(C, A) = \{u \in C : \langle v - u, Av \rangle \geq 0, \text{ for all } v \in C\}.$$

It is obvious from Lemma 2.6 that the set $VI(C, A)$ is a closed convex subset of C.

Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space E and T be a mapping from C into itself. A point $x \in C$ is said to be a fixed point of T if $Tx = x$. We denote the set of fixed points of T by $F(T)$. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F(T)}$. Following Matsushita and Takahashi [8], a mapping T of C into itself is said to be *relatively nonexpansive* (see [9]) if the following conditions are satisfied:

(i) $F(T)$ is nonempty;
(ii) $\phi(u, Tx) \leq \phi(u, x), \ \ \forall u \in F(T), \ \ x \in C$;
(iii) $\hat{F(T)} = F(T)$.

Lemma 2.8. ([8]) *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

3. Main results

In this section, we modify the iterative Algorithm 1.1 by replacing the generalized projection onto $C_n \bigcap Q_n$ with a generalized projection onto a half-space C_n and construct the following iterative Algorithm 3.1 for finding a common element of the set of solutions of the variational inequality (1.1) involving a monotone Lipschitz continuous operator A and the set of fixed points of a relatively nonexpansive mapping T.

From now on, we adopt the following assumptions.

(B1) $VI(C, A) \bigcap F(T) \neq \emptyset$.
(B2) $A : E \to E^*$ is monotone on C.
(B3) $A : E \to E^*$ is $L-$Lipschitz continuous on C.

Algorithm 3.1

$$\begin{cases} x_0 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ z_n = Ty_n, \\ \text{If} \ \ x_n = y_n = z_n, \ \ \text{then stop.} \\ \text{Otherwise,} \ \ \text{construct} \ \ C_n \ \ \text{as} \\ C_n := \{u \in E : \phi(u, z_n) \leq \phi(u, x_n) - \phi(y_n, x_n) - 2\lambda_n \langle y_n - u, Ax_n - Ay_n\rangle\}, \\ x_{n+1} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) J\Pi_{C_n} x_n), \end{cases}$$

where $0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < \frac{c_1}{2L}$ and $\limsup_{n\to\infty} \alpha_n < 1$.

Remark 3.1. (1) If $x_n = y_n = z_n$, then $x_n \in VI(C,A)\bigcap F(T)$, which implies that the iterative sequence $\{x_n\}$ is finite, and the last term is an element of $VI(C,A)\bigcap F(T)$. Otherwise, $x_n \notin C_n$. In fact, if $x_n \in C_n$, then it follows from Lemma 2.6 and $0 < \inf_{n\in\mathbb{N}} \lambda_n \le \sup_{n\in\mathbb{N}} \lambda_n < \frac{c_1}{2L}$ that

$$\begin{aligned}\phi(x_n, z_n) &\le \phi(x_n, x_n) - \phi(y_n, x_n) - 2\lambda_n\langle y_n - x_n, Ax_n - Ay_n\rangle \\ &\le -c_1\|x_n - y_n\|^2 + 2\lambda_n L\|x_n - y_n\|^2 \le 0.\end{aligned}$$

This implies that $x_n = y_n = z_n$. Hence, if $x_n = y_n = z_n$ doesn't hold, then, $x_n \notin C_n$.
(2) Since $\phi(u, z_n) \le \phi(u, x_n) - \phi(y_n, x_n) - 2\lambda_n\langle y_n - u, Ax_n - Ay_n\rangle\}$ is equivalent to

$$\begin{aligned}&2\langle u, Jz_n - Jx_n + \lambda_n(Ax_n - Ay_n)\rangle \\ &\ge 2\lambda_n\langle y_n, Ax_n - Ay_n\rangle + \|z_n\|^2 - \|x_n\|^2 + \phi(y_n, x_n),\end{aligned}$$

we have C_n is a half-space. The next iterate x_{n+1} is a convex combination of x_n and a generalized projection of x_n onto the half-space C_n, which is not expensive at all from a numerical point of view.

Theorem 3.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space with the 2-uniformly convexity constant c_1. Let the duality mapping J is weakly sequentially continuous and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a relatively nonexpansive mapping. Then, under conditions* (B1)–(B3)*, the sequence $\{x_n\}$ generated by Algorithm* 3.1 *converges weakly to a point $x^* \in VI(C,A)\bigcap F(T)$ and $x^* = \lim_{n\to\infty} \Pi_{VI(C,A)\bigcap F(T)}x_n$.*

Proof. The proof will be split into four steps.

Step 1. Show that Algorithm 3.1 is well-defined.

Suppose $x^* \in VI(C,A)\bigcap F(T)$. By $x^* \in VI(C,A)$, we have $\langle y_n - x^*, Ax^*\rangle \ge 0$. Since A is monotone, we have $\langle y_n - x^*, Ay_n\rangle \ge 0$. It follows from $y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$ and Lemma 2.3 that

$$\begin{aligned}\langle y_n - x^*, Jx_n - Jy_n\rangle &\ge \lambda_n\langle y_n - x^*, Ax_n\rangle \\ &= \lambda_n\langle y_n - x^*, Ax_n - Ay_n\rangle + \lambda_n\langle y_n - x^*, Ay_n\rangle \\ &\ge \lambda_n\langle y_n - x^*, Ax_n - Ay_n\rangle,\end{aligned}$$

which implies that

$$\phi(x^*, y_n) \le \phi(x^*, x_n) - \phi(y_n, x_n) - 2\lambda_n\langle y_n - x^*, Ax_n - Ay_n\rangle$$

for all $n \in \mathbb{N}$. On the other hand, since $x^* \in F(T)$, we have

$$\phi(x^*, z_n) = \phi(x^*, Ty_n) \le \phi(x^*, y_n).$$

So, we obtain that

$$\phi(x^*, z_n) \le \phi(x^*, y_n) \le \phi(x^*, x_n) - \phi(y_n, x_n) - 2\lambda_n \langle y_n - x^*, Ax_n - Ay_n \rangle$$

for all $n \in \mathbb{N}$. This and Remark 3.1 imply that Algorithm 3.1 is well-defined.

Step 2. Show that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ have the same weak accumulation points.

By the construction of x_{n+1} and $VI(C, A) \bigcap F(T) \subset C_n$, we deduce that

$$\begin{aligned} \phi(x^*, x_{n+1}) &\le \|x^*\|^2 - 2\langle x^*, \alpha_n Jx_n + (1-\alpha_n) J\Pi_{C_n} x_n \rangle + \alpha_n \|x_n\|^2 \\ &\quad + (1-\alpha_n)\|\Pi_{C_n} x_n\|^2 \\ &= \alpha_n \phi(x^*, x_n) + (1-\alpha_n)\phi(x^*, \Pi_{C_n} x_n) \\ &\le \alpha_n \phi(x^*, x_n) + (1-\alpha_n)(\phi(x^*, x_n) - \phi(\Pi_{C_n} x_n, x_n)) \\ &\le \phi(x^*, x_n) - (1-\alpha_n)\phi(\Pi_{C_n} x_n, x_n)) \le \phi(x^*, x_n), \end{aligned} \tag{3.1}$$

which yields that the sequence $\{\phi(x^*, x_n)\}$ is convergent. Hence $\{x_n\}$ is bounded. Since $y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$, we have

$$\phi(x^*, y_n) \le \phi(x^*, J^{-1}(Jx_n - \lambda_n Ax_n)) \le (\|x^*\| + \|Jx_n - \lambda_n Ax_n\|)^2,$$

which implies from the boundness of $\{x_n\}$ that $\{y_n\}$ is bounded. It follows from $\phi(x^*, z_n) \le \phi(x^*, y_n)$ that $\{z_n\}$ is also bounded. Since $\{\phi(x^*, x_n)\}$ converges and $\limsup_{n\to\infty} \alpha_n < 1$, we get from (3.1) that

$$\lim_{n\to\infty} \phi(\Pi_{C_n} x_n, x_n) = 0. \tag{3.2}$$

From Lemma 2.4, we have

$$\lim_{n\to\infty} \|\Pi_{C_n} x_n - x_n\| = 0. \tag{3.3}$$

Since $\Pi_{C_n} x_n \in C_n$, we have

$$\begin{aligned} &\phi(\Pi_{C_n} x_n, z_n) \\ &\le \phi(\Pi_{C_n} x_n, x_n) - \phi(y_n, x_n) - 2\lambda_n \langle y_n - \Pi_{C_n} x_n, Ax_n - Ay_n \rangle \\ &\le \phi(\Pi_{C_n} x_n, x_n) - c_1\|y_n - x_n\|^2 + 2\lambda_n L\|y_n - x_n\|^2 \\ &\quad - 2\lambda_n \langle x_n - \Pi_{C_n} x_n, Ax_n - Ay_n \rangle \\ &\le \phi(\Pi_{C_n} x_n, x_n) + (2\lambda_n L - c_1)\|y_n - x_n\|^2 + 2\lambda_n L\|x_n - \Pi_{C_n} x_n\|\|x_n - y_n\| \\ &\le \phi(\Pi_{C_n} x_n, x_n) + 2\lambda_n L\|x_n - \Pi_{C_n} x_n\|\|x_n - y_n\|, \end{aligned}$$

which implies from (3.2),(3.3) and the boundness of $\{x_n\}$ and $\{y_n\}$ that

$$\lim_{n\to\infty} \phi(\Pi_{C_n} x_n, z_n) = \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.4}$$

It follows from Lemma 2.4 that

$$\lim_{n\to\infty} \|z_n - \Pi_{C_n} x_n\| = 0. \tag{3.5}$$

Since $\|x_n - z_n\| \leq \|x_n - \Pi_{C_n} x_n\| + \|\Pi_{C_n} x_n - z_n\|$, we have from (3.3) and (3.5) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0, \tag{3.6}$$

which leads to $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ have the same weak accumulation points.

Step 3. Show that each weak accumulation point of $\{x_n\}$ is an element of $VI(C, A) \bigcap F(T)$.

Since J is uniformly norm-to-norm continuous on bounded sets, we have from (3.4) that

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.7}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. By (3.4), we have $y_{n_k} \rightharpoonup \hat{x} \in C$. By $y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$, we have

$$\langle y_n - u, Jx_n - Jy_n\rangle \geq \lambda_n \langle y_n - u, Ax_n\rangle$$

for all $u \in C$, which leads to

$$\begin{aligned} &\langle y_n - u, Jx_n - Jy_n\rangle - \lambda_n \langle y_n - u, Ax_n - Ay_n\rangle \\ &\geq \lambda_n \langle y_n - u, Ay_n\rangle \geq \lambda_n \langle y_n - u, Au\rangle. \end{aligned} \tag{3.8}$$

This implies that

$$\|y_{n_k} - u\|(\|Jx_{n_k} - Jy_{n_k}\| + \lambda_{n_k} L\|x_{n_k} - y_{n_k}\|) \geq \lambda_{n_k} \langle y_{n_k} - u, Au\rangle.$$

By (3.4) and (3.7), we obtain that

$$\langle \hat{x} - u, Au\rangle \leq 0, \quad \forall u \in C.$$

By Lemma 2.7, we have $\hat{x} \in VI(C, A)$. Since $\|z_n - y_n\| \leq \|z_n - x_n\| + \|x_n - y_n\|$, from (3.4) and (3.6), we get that $\lim_{n\to\infty} \|z_n - y_n\| = 0$. So, we have $\hat{x} \in F(T)$. This implies that $\hat{x} \in VI(C, A) \bigcap F(T)$.

Step 4. Show that the entire sequence $\{x_n\}$ weakly converges to $\hat{x}$ and $\hat{x} = \lim_{n\to\infty} \Pi_{VI(C,A)}(x_n)$.

Put $u_n = \Pi_{VI(C,A) \bigcap F(T)}(x_n)$. It follows from (3.1) and Lemma 2.5 that $\{u_n\}$ is a Cauchy sequence. Since $VI(C, A) \bigcap F(T)$ is closed, we have that $\{u_n\}$ converges strongly to $z \in VI(C, A) \bigcap F(T)$. By the uniform smoothness of E, we also have that $\lim_{n\to\infty} \|Ju_n - Jz\| = 0$. Now, we prove that $z = \hat{x}$. In fact, it follows from Lemma 2.3, $u_n = \Pi_{VI(C,A) \bigcap F(T)}(x_n)$ and $\hat{x} \in VI(C, A) \bigcap F(T)$ that $\langle \hat{x} - u_n, Ju_n - Jx_n\rangle \geq 0$. By the weakly sequential continuity of J, we infer that $\langle \hat{x} - z, Jz - J\hat{x}\rangle \geq 0$. Hence we have from the monotonicity of J that $\langle \hat{x} - z, Jz - J\hat{x}\rangle = 0$. Since E is strictly convex, we have that $z = \hat{x}$. Therefore, the sequence $\{x_n\}$ converges weakly to $\hat{x} = \lim_{k\to\infty} \Pi_{VI(C,A)}(x_n)$. □

Remark 3.3. Algorithm 3.1 replaces the second general projection onto the closed and convex set $C_n \bigcap Q_n$ in Algorithm 1.1 with the one onto the half-space C_n.

We consider the following Algorithm 3.2 which is different from Algorithm 3.1 and prove the new weak convergence theorem.

Algorithm 3.2.

$$\begin{cases} x_0 \in E, \\ y_n = \Pi_C J^{-1}(JTx_n - \lambda_n AT(x_n)), \\ \text{If } \ x_n = y_n = Tx_n, \ \text{ then stop.} \\ \text{Otherwise, } \ \text{construct } \ C_n \ \text{ as} \\ C_n := \{u \in E : \phi(u, y_n) \leq \phi(u, x_n) - \phi(y_n, Tx_n) - 2\lambda_n \langle y_n - u, ATx_n - Ay_n \rangle\}, \\ x_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) J\Pi_{C_n} x_n), \end{cases}$$

where $0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < \frac{c_1}{L}$ and $\limsup_{n\to\infty} \alpha_n < 1$.

Remark 3.4. (1) If $x_n = y_n = Tx_n$, then $x_n \in VI(C, A) \bigcap F(T)$, which implies that the iterative sequence $\{x_n\}$ is finite, and the last term is an element of $VI(C, A) \bigcap F(T)$. Otherwise, $x_n \notin C_n$. In fact, if $x_n \in C_n$, then it follows from Lemma 2.6 that

$$\begin{aligned} & c_1 \|x_n - y_n\|^2 \\ & \leq \phi(x_n, y_n) \leq \phi(x_n, x_n) - \phi(y_n, Tx_n) - 2\lambda_n \langle y_n - x_n, ATx_n - Ay_n \rangle \\ & \leq -c_1 \|y_n - Tx_n\|^2 + 2\lambda_n L \|x_n - y_n\| \|y_n - Tx_n\| \\ & \leq -c_1 \|y_n - Tx_n\|^2 + \lambda_n L \|x_n - y_n\|^2 + \lambda_n L \|y_n - Tx_n\|^2, \end{aligned}$$

which implies that

$$(c_1 - \lambda_n L)\|x_n - y_n\|^2 \leq 0 \quad \text{ and } \quad (c_1 - \lambda_n L)\|Tx_n - y_n\|^2 \leq 0.$$

Since $0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < \frac{c_1}{L}$, we have $x_n = y_n = Tx_n$ Therefore, if $x_n = y_n = Tx_n$ doesn't hold, then $x_n \notin C_n$.
(2) Since $\phi(u, y_n) \leq \phi(u, x_n) - \phi(y_n, Tx_n) - 2\lambda_n \langle y_n - u, ATx_n - Ay_n \rangle\}$ is equivalent to

$$\begin{aligned} & 2\langle u, Jx_n - Jy_n - \lambda_n (ATx_n - Ay_n) \rangle \\ & \geq \|x_n\|^2 - 2\|y_n\|^2 + 2\langle y_n, JTx_n \rangle + \|Tx_n\|^2 - 2\lambda_n \langle y_n, ATx_n - Ay_n \rangle, \end{aligned}$$

we have C_n is a half-space. The next iterate x_{n+1} is a convex combination of x_n and a generalized projection of x_n onto the half-space C_n, which is not expensive at all from a numerical point of view.

Theorem 3.5. *Assume that E, C, J are the same as Theorem 3.2. Let $T : E \to E$ be a relatively nonexpansive mapping. Then, under conditions (B1)–(B3), the sequence $\{x_n\}$ generated by Algorithm 3.2 converges weakly to a point $x^* \in VI(C,A)\bigcap F(T)$ and $x^* = \lim_{n\to\infty} \Pi_{VI(C,A)\bigcap F(T)} x_n$.*

Proof. The proof will be split into four steps.

Step 1. Show that Algorithm 3.2 is well-defined.

Suppose $x^* \in VI(C,A)\bigcap F(T)$. As in the proof of Theorem 3.2, we obtain from $y_n = \Pi_C J^{-1}(JTx_n - \lambda_n ATx_n)$ that

$$\langle y_n - x^*, JTx_n - Jy_n\rangle \geq \lambda_n \langle y_n - x^*, ATx_n\rangle.$$

By $x^* \in VI(C,A)$, we have $\langle y_n - x^*, Ax^*\rangle \geq 0$. Since A is monotone, we have $\langle y_n - x^*, Ay_n\rangle \geq 0$. Hence

$$\langle y_n - x^*, JTx_n - Jy_n\rangle \geq \lambda_n \langle y_n - x^*, ATx_n - Ay_n\rangle,$$

which implies that

$$\phi(x^*, y_n) \leq \phi(x^*, Tx_n) - \phi(y_n, Tx_n) - 2\lambda_n \langle y_n - x^*, ATx_n - Ay_n\rangle$$

for all $n \in \mathbb{N}$. Since $x^* \in F(T)$, we have $\phi(x^*, Tx_n) \leq \phi(x^*, x_n)$. So, we get

$$\phi(x^*, y_n) \leq \phi(x^*, x_n) - \phi(y_n, Tx_n) - 2\lambda_n \langle y_n - x^*, ATx_n - Ay_n\rangle$$

for all $n \in \mathbb{N}$. This and Remark 3.4 imply that Algorithm 3.2 is well-defined.

Step 2. Show that $\{x_n\}$, $\{y_n\}$ and $\{Tx_n\}$ have the same weak accumulation points. As in the proof of Theorem 3.2, we can obtain that $\{x_n\}, \{y_n\}$ and $\{Tx_n\}$ are bounded and

$$\lim_{n\to\infty} \phi(\Pi_{C_n} x_n, x_n) = 0. \tag{3.9}$$

By Lemma 2.4, we have

$$\lim_{n\to\infty} \|\Pi_{C_n} x_n - x_n\| = 0. \tag{3.10}$$

Since $\Pi_{C_n} x_n \in C_n$, we have

$$\begin{aligned}
&c_1\|\Pi_{C_n} x_n - y_n\|^2 \leq \phi(\Pi_{C_n} x_n, y_n)\\
&\leq \phi(\Pi_{C_n} x_n, x_n) - \phi(y_n, Tx_n) - 2\lambda_n \langle y_n - \Pi_{C_n} x_n, ATx_n - Ay_n\rangle\\
&\leq \phi(\Pi_{C_n} x_n, x_n) - c_1\|y_n - Tx_n\|^2 + 2\lambda_n L \|y_n - \Pi_{C_n} x_n\| \|Tx_n - y_n\|\\
&\leq \phi(\Pi_{C_n} x_n, x_n) - c_1\|y_n - Tx_n\|^2 + \lambda_n L \|y_n - \Pi_{C_n} x_n\|^2 + \lambda_n L \|Tx_n - y_n\|^2,
\end{aligned}$$

which implies that

$$(c_1 - \lambda_n L)\|Tx_n - y_n\|^2 + (c_1 - \lambda_n L)\|\Pi_{C_n} x_n - y_n\|^2 \leq \phi(\Pi_{C_n} x_n, x_n).$$

It follows from (3.9) that

$$\lim_{n\to\infty} \|y_n - Tx_n\| = \lim_{n\to\infty} \|y_n - \Pi_{C_n} x_n\| = 0. \tag{3.11}$$

Since $\|x_n - y_n\| \leq \|x_n - \Pi_{C_n} x_n\| + \|\Pi_{C_n} x_n - y_n\|$, we have from (3.9) and (3.10) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.12}$$

On the other hand, since $\|x_n - Tx_n\| \leq \|x_n - y_n\| + \|y_n - Tx_n\|$, it follows from (3.11) and (3.12) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0, \tag{3.13}$$

which leads to $\{x_n\}$, $\{y_n\}$ and $\{Tx_n\}$ have the same weak accumulation points.

Step 3. Show that each weak accumulation point of $\{x_n\}$ is an element of $VI(C, A) \bigcap F(T)$. Since J is uniformly norm-to-norm continuous on bounded sets, we have from (3.11) that

$$\lim_{n\to\infty} \|JTx_n - Jy_n\| = 0. \tag{3.14}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. By (3.12), we have $y_{n_k} \rightharpoonup \hat{x} \in C$. By $y_n = \Pi_C J^{-1}(JTx_n - \lambda_n ATx_n)$, we have

$$\langle y_n - u, JTx_n - Jy_n \rangle \geq \lambda_n \langle y_n - u, ATx_n \rangle$$

for all $u \in C$, which leads to

$$\begin{aligned} &\langle y_n - u, JTx_n - Jy_n \rangle - \lambda_n \langle y_n - u, ATx_n - Ay_n \rangle \\ &\geq \lambda_n \langle y_n - u, Ay_n \rangle \geq \lambda_n \langle y_n - u, Au \rangle. \end{aligned} \tag{3.15}$$

This implies that

$$\|y_{n_k} - u\|(\|JTx_{n_k} - Jy_{n_k}\| + \lambda_{n_k} L \|Tx_{n_k} - y_{n_k}\|) \geq \lambda_{n_k} \langle y_{n_k} - u, Au \rangle.$$

By (3.11) and (3.14), we obtain that

$$\langle \hat{x} - u, Au \rangle \leq 0, \quad \forall u \in C.$$

By Lemma 2.7, we have $\hat{x} \in VI(C, A)$. From (3.13), we have $\hat{x} \in F(T)$. This implies that $\hat{x} \in VI(C, A) \bigcap F(T)$.

Step 4. Show that the entire sequence $\{x_n\}$ weakly converges to $\hat{x}$ and $\hat{x} = \lim_{n\to\infty} \Pi_{VI(C,A)}(x_n)$. The proof is the same as Theorem 3.2. Hence, we omit it. □

Remark 3.6. Algorithm 3.2 replaces the second general projection onto the closed and convex set $C_n \bigcap Q_n$ in Algorithm 1.2 with the one onto the half-space C_n.

We consider the following Algorithm 3.3 which is different from Algorithms 3.1 and 3.2.

Algorithm 3.3.

$$\begin{cases} x_0 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ z_n = J^{-1}(\beta_n JTx_n + (1-\beta_n)Jy_n), \\ \text{If } \ x_n = y_n = z_n, \ \text{ then stop.} \\ \text{Otherwise, } \ \text{construct } \ C_n \ \text{ as} \\ C_n := \{u \in E : \phi(u, z_n) \le \phi(u, x_n) - (1-\beta_n)\phi(y_n, x_n) \\ \qquad\qquad\qquad -2(1-\beta_n)\lambda_n\langle y_n - u, Ax_n - Ay_n\rangle\}, \\ x_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J\Pi_{C_n}x_n), \end{cases}$$

where $a \le \beta_n \le 1$ for some $a \in (0,1)$, $0 < \inf\limits_{n\in\mathbb{N}} \lambda_n \le \sup\limits_{n\in\mathbb{N}} \lambda_n < \frac{c_1}{2L}$ and $\limsup\limits_{n\to\infty} \alpha_n < 1$.

Remark 3.7. (1) If $x_n = y_n = z_n$, then $x_n \in VI(C,A)\bigcap F(T)$, which implies that the iterative sequence $\{x_n\}$ is finite, and the last term is an element of $VI(C,A)\bigcap F(T)$. Otherwise, $x_n \notin C_n$. In fact, if $x_n \in C_n$, then it follows from Lemma 2.6 that

$$\begin{aligned} &\phi(x_n, z_n) \\ &\le \phi(x_n, x_n) - (1-\beta_n)\phi(y_n, x_n) - 2(1-\beta_n)\lambda_n\langle y_n - x_n, Ax_n - Ay_n\rangle \\ &\le -c_1(1-\beta_n)\|y_n - x_n\|^2 + 2(1-\beta_n)\lambda_n L\|x_n - y_n\|^2 \\ &= (1-\beta_n)(2\lambda_n L - c_1)\|y_n - x_n\|^2 \le 0, \end{aligned}$$

which implies that $x_n = y_n = z_n$. Therefore, if $x_n = y_n = z_n$ doesn't hold, then $x_n \notin C_n$.
(2) Since $\phi(u, z_n) \le \phi(u, x_n) - (1-\beta_n)\phi(y_n, x_n) - 2(1-\beta_n)\lambda_n\langle y_n - u, Ax_n - Ay_n\rangle\}$ is equivalent to

$$\begin{aligned} &2\langle u, Jx_n - Jz_n - \lambda_n(1-\beta_n)(Ax_n - Ay_n)\rangle \\ &\ge \|x_n\|^2 - \|z_n\|^2 - (1-\beta_n)\phi(y_n, x_n) - 2(1-\beta_n)\lambda_n\langle y_n, Ax_n - Ay_n\rangle, \end{aligned}$$

we have C_n is a half-space. The next iterate x_{n+1} is a convex combination of x_n and a generalized projection of x_n onto the half-space C_n, which is not expensive at all from a numerical point of view.

Theorem 3.8. *Assume that E, C, J, T are the same as Theorem* 3.5. *Then, under conditions* (B1)–(B3)*, the sequence $\{x_n\}$ generated by Algorithm* 3.3 *converges weakly to a point $x^* \in VI(C,A)\bigcap F(T)$ and*

$$x^* = \lim_{n\to\infty} \Pi_{VI(C,A)\bigcap F(T)} x_n.$$

Proof. Suppose $x^* \in VI(C,A)\bigcap F(T)$. We have

$$\phi(x^*, z_n) \le \beta_n\phi(x^*, Tx_n) + (1-\beta_n)\phi(x^*, y_n), \ \ \forall n \in \mathbb{N}.$$

As in the proof of Theorem 3.2, we obtain that

$$\phi(x^*, y_n) \le \phi(x^*, x_n) - \phi(y_n, x_n) - 2\lambda_n\langle y_n - x^*, Ax_n - Ay_n\rangle$$

for all $n \in \mathbb{N}$. Since $x^* \in F(T)$, we have $\phi(x^*, Tx_n) \le \phi(x^*, x_n)$. So, we get

$$\begin{aligned}\phi(x^*, z_n) \le \phi(x^*, x_n) &- (1-\beta_n)\phi(y_n, x_n)\\ &- 2(1-\beta_n)\lambda_n\langle y_n - x^*, Ax_n - Ay_n\rangle\end{aligned}$$

for all $n \in \mathbb{N}$. That is, $VI(C, A)\bigcap F(T) \subset C_n$. This and Remark 3.7 imply that Algorithm 3.3 is well-defined.

Using the same proof as Theorem 3.5, we can obtain $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded, and

$$\begin{aligned}&\lim_{n\to\infty}\phi(\Pi_{C_n}x_n, x_n) = \lim_{n\to\infty}\|\Pi_{C_n}x_n - x_n\| = \lim_{n\to\infty}\|x_n - y_n\|\\ &= \lim_{n\to\infty}\phi(\Pi_{C_n}x_n, z_n) = \lim_{n\to\infty}\|\Pi_{C_n}x_n - z_n\| = 0.\end{aligned} \tag{3.16}$$

Since $\|x_n - z_n\| \le \|x_n - \Pi_{C_n}x_n\| + \|\Pi_{C_n}x_n - z_n\|$, from (3.16), we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.17}$$

Since J is uniformly norm-to-norm continuous on each bounded set, we have

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = \lim_{n\to\infty}\|Jx_n - Jz_n\| = 0. \tag{3.18}$$

Since $z_n = J^{-1}(\beta_n JTx_n + (1-\beta_n)Jy_n)$, it follows from (3.18) and $0 < a \le \beta_n$ that $\lim_{n\to\infty}\|JTx_n - Jx_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on each bounded set, we have $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. As in the proof of Theorem 3.2, each weak accumulation point of $\{x_n\}$ is an element of $VI(C, A)\bigcap F(T)$ and $x_n \rightharpoonup \lim_{n\to\infty}\Pi_{VI(C,A)}(x_n)$. □

Remark 3.9. Algorithm 3.3 replaces the second general projection onto the closed and convex set $C_n\bigcap Q_n$ in Algorithm 1.3 with the one onto the half-space C_n.

Remark 3.10. Theorem 3.5 holds under more mild condition for $\{\lambda_n\}$ than Theorems 3.2 and 3.8.

Remark 3.11. Theorems 3.2, 3.5 and 3.8 improve Theorem 1.4 in the following senses.

(1) The inverse-strong-monotonicity of A is relaxed to monotonicity and Lipschitz continuity.
(2) The assumption (A3) is removed.

Remark 3.12. Remarks 3.3, 3.6, 3.9 and 3.11 show our results answer the questions (Q1) and (Q2) and don't involve this subproblem of finding a general minimal distance onto the intersection of two additional closed and convex subsets of a Banach space E in Algorithms 1.1–1.3.

4. Deduced results

When T is the identity mapping, we get the following weak convergence to an element of $VI(C, A)$ from Theorem 3.5.

Theorem 4.1. *Let C be a nonempty closed convex subset of a $2-$uniformly convex and uniformly smooth Banach space E with the $2-$uniformly convex constant c_1 and the duality mapping J is weakly sequentially continuous. Let $A : E \to E^*$ be monotone and $L-$Lipschitz continuous on C such that $VI(C, A) \neq \emptyset$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, +\infty)$ which satisfies $0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < \frac{c_1}{L}$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ \textit{If } \ x_n = y_n, \ \textit{ then stop.} \\ \textit{Otherwise}, \ \textit{ construct } \ C_n \ \textit{ as} \\ C_n := \{u \in E : \phi(u, y_n) \leq \phi(u, x_n) \ ! - \phi(y_n, x_n) - 2\lambda_n\langle y_n - u, Ax_n - Ay_n\rangle\}, \\ x_{n+1} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)J\Pi_{C_n} x_n). \end{cases}$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to $x^* = \lim_{n\to\infty} \Pi_{VI(C,A)} x_n$.

When $C = E$, we have $VI(E, A) = A^{-1}0 = \{z \in E : Az = 0\}$. So, we have the following theorem by Theorem 4.1.

Theorem 4.2. *Assume E, J are the same as Theorem 4.1. Let $A : E \to E^*$ be monotone and $L-$Lipschitz continuous such that $A^{-1}0 \neq \emptyset$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, +\infty)$ which satisfies $0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < \frac{c_1}{L}$, where c_1 is the $2-$uniformly convex constant. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E, \\ y_n = J^{-1}(Jx_n - \lambda_n A(x_n)), \\ \textit{If } \ x_n = y_n, \ \textit{ then stop.} \\ \textit{Otherwise}, \ \textit{ construct } \ C_n \ \textit{ as} \\ C_n := \{u \in E : \phi(u, y_n) \leq \phi(u, x_n) - \phi(y_n, x_n) - 2\lambda_n\langle y_n - u, Ax_n - Ay_n\rangle\}, \\ x_{n+1} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)J\Pi_{C_n} x_n). \end{cases}$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to $x^* = \lim_{n\to\infty} \Pi_{A^{-1}(0)} x_n$.

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