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SOME COMMON AND COINCIDENCE FIXED POINT RESULTS IN CONE METRIC SPACES

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Abstract. In this paper, we establish the existence and uniqueness of common and coincidence fixed points for a family of self mappings satisfying a generalized contractive condition in cone metric spaces. Examples are given to support the results. The presented results generalize many known results in cone metric spaces.

1. Introduction

It is well known that the classical contraction mapping principle of Banach is a fundamental result in fixed point theory. Several authors have obtained various extensions and generalizations of Banachs theorems by considering contractive mappings on different metric spaces, for instance [6]-[8], [14]-[20], [22, 23, 28]. In 2007, Huang and Zhang [10] generalized the concept of metric spaces by replacing the real numbers by ordered Banach space and introduced cone metric spaces. They have proved Banach contraction mapping theorem and some other fixed point theorems of contractive type mappings in cone metric spaces. Subsequently, Abbas and Jungck [1] obtained the common fixed points for noncommuting mappings by dropping the continuity property. In [2], Abbas and Rhoades studied the fixed and periodic point results. Ilić and

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Rakočević [11] extended the concept of quasi-contraction mappings to cone metric spaces and provided a generalized result of [10] to quasi-contraction mappings in complete cone metric spaces. In 2009, Radenović [21] has obtained coincidence point result for two mappings in cone metric spaces which satisfy new contractive conditions. Meanwhile, Semwel and Dimri [26] investigated the couple fixed point results for Suzuki type mappings in cone metric spaces. Recently, many authors studied several variants of contraction conditions and proved some fixed point theorems in a cone metric space when the underlying cone is normal or not normal. The existence of a common fixed point in cone metric space has been considered recently in [4, 6, 9, 12, 13, 24, 25, 29, 30].

Motivated by the above work, in this paper, we obtain a unique common fixed point and coincidence point for a family of self mappings satisfying a generalized contractive condition in cone metric spaces without the normality. Illustrative examples are provided to justify our results. Also, we prove the convergence of the sequence of fixed points. The presented results generalize many known results in cone metric spaces.

2. Preliminaries

In this section, We recall the definition of cone metric spaces and some of their properties.

Definition 2.1. Let E be a real Banach space. A subset P of E is called a cone if the following conditions are satisfied:

(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$.
(iii) $P \cap (-P) = \{0\}$.

Given a cone P of E, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$.

A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P.

Definition 2.2. Let X be a nonempty set and $d : X \times X \to E$ be a mapping such that the following conditions hold:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.3. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$ and $d : X \times X \to E$ such that $d(x, y) = (|x-y|, \delta|x-y|)$, where $\delta \geq 0$ is a constant. Then (X,d) is a cone metric space.

Definition 2.4. Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is N such that for all $m, n > N, d(x_n, x_m) \ll c$;

(ii) a convergent sequence if for every $c \in E$ with $0 \ll c$, there is N such that for all $n > N, d(x_n, x) \ll c$, for some $x \in X$. We denote it by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. The limit of a convergent sequence is unique provided P is a normal cone with normal constant K (see [10]).

Proposition 2.5. ([1]) *Let f and g be weakly compatible self mappings on a non empty set X. If f and g have a unique point of coincidence $v = fu = gu$, then v is the unique common fixed point of f and g.*

3. Main results

Theorem 3.1. *Let (X, d) be a cone metric space, T and f be two self mappings of X such that*

(i) *$T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X;*

(ii)

$$d(Tx, Ty) \leq \alpha \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\right\}, \quad \forall x, y \in X, \tag{3.1}$$

where $0 \leq \alpha < 1$.

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible then T and f have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $T(X) \subset f(X)$, there exists $x_1 \in X$ such that $Tx_0 = fx_1$. Proceeding like this way, for $x_n \in X$, we get $x_{n+1} \in X$ such that $Tx_n = fx_{n+1}$, $n = 0, 1, 2, \cdots$. Then, using (3.1), we obtain

$$\begin{aligned} d(fx_{n+1}, fx_n) &\leq d(Tx_n, Tx_{n-1}) \\ &\leq \alpha \max\Big\{ d(fx_n, fx_{n-1}), d(fx_n, Tx_n), d(fx_{n-1}, Tx_{n-1}), \\ &\qquad \frac{d(fx_n, Tx_{n-1}) + d(fx_{n-1}, Tx_n)}{2} \Big\} \\ &\leq \alpha \max\Big\{ d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_n), \\ &\qquad \frac{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})}{2} \Big\}, \end{aligned}$$

which yields

$$d(fx_{n+1}, fx_n) \leq \alpha \max\Big\{ d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}) \Big\}.$$

Case I: $d(fx_{n+1}, fx_n) \leq \alpha d(fx_{n+1}, fx_n)$ which implies $1 \leq \alpha$, which is a contradiction.

Case II: $d(fx_{n+1}, fx_n) \leq \alpha d(fx_n, fx_{n-1}) \leq \alpha^2 d(fx_{n-1}, fx_{n-2})$, continuing in this fashion, we get

$$d(fx_{n+1}, fx_n) \leq \alpha^n d(fx_1, fx_0).$$

For $m > n$, we have

$$\begin{aligned} d(fx_m, fx_n) &\leq d(fx_m, fx_{m-1}) + d(fx_{m-1}, fx_{m-2}) + \cdots + d(fx_{n+1}, fx_n) \\ &\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) d(fx_1, fx_0) \\ &\leq \frac{\alpha^n}{1-\alpha} d(fx_1, fx_0). \end{aligned}$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that $\{x \in E : \|x\| < \delta\} + c \subseteq P$. Also, choose a natural number N_0 such that

$$\frac{\alpha^n}{1-\alpha} d(fx_1, fx_0) \in \{x \in E : \|x\| < \delta\}, \ \ \forall n \geq N_0.$$

Then

$$\frac{\alpha^n}{1-\alpha} d(fx_1, fx_0) \ll c, \ \ \forall n \geq N_0,$$

which implies that

$$d(fx_m, fx_n) \leq \frac{\alpha^n}{1-\alpha} d(fx_1, fx_0) \ll c, \ \ \forall m \geq n.$$

This shows that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Because of the completeness of $f(X)$, $fx_n \to v$ and there exists $u \in X$ such that $fu = v$. Further, we have

$$\begin{aligned} d(fx_n, Tu) &= d(Tx_{n-1}, Tu) \\ &\leq \alpha \max \Big\{ d(fx_{n-1}, fu), d(fx_{n-1}, Tx_{n-1}), d(fu, Tu), \\ &\qquad \frac{d(fx_{n-1}, Tu) + d(fu, Tx_{n-1})}{2} \Big\} \\ &\leq \alpha \max \Big\{ d(fx_{n-1}, fu), d(fx_{n-1}, fx_n), d(fu, Tu), \\ &\qquad \frac{d(fx_{n-1}, Tu) + d(fu, fx_n)}{2} \Big\}. \end{aligned}$$

Now, letting $n \to \infty$, we obtain

$$d(v, Tu) \leq \alpha d(v, Tu),$$

which implies that $Tu = v = fu$, since $\alpha < 1$. Therefore, v is a point of coincidence of T and f. Now we claim that v is a unique point of coincidence of T and f. Suppose there exists another point of coincidence v' of T and f, that is, $Tu' = v' = fu'$. Then, from (3.1), we have

$$\begin{aligned} d(fu, fu') &\leq d(Tu, Tu') \\ &\leq \alpha \max \Big\{ d(fu, fu'), d(fu, Tu), d(fu', Tu'), \\ &\qquad \frac{d(fu, Tu') + d(fu', Tu)}{2} \Big\}, \end{aligned}$$

which gives

$$d(fu, fu') \leq \alpha d(fu, fu').$$

Thus, $fu = fu'$ which shows that v is a unique point of coincidence of T and f. Now, by Proposition 2.5, T and f have a unique common fixed point v in X. □

Remark 3.2. Theorem 3.1 generalizes the following:

(i) Theorems 2.1, 2.3, 2.4 of [1].
(ii) Theorem 1, 2 and 3 of [4].
(iii) Theorems 1, 3, 4 of [10].
(iv) Theorems 2.3, 2.6, 2.7, 2.8 of [24].
(v) Theorems 2.1 of [27].

Example 3.3. Let $X = \mathbb{R}, E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$. Now, define

$$d(x, y) = \left(\frac{\|x - y\|}{2},\ \beta\frac{\|x - y\|}{2}\right), \text{ where } \beta > 0.$$

Let $T, f : X \to X$ be defined by

$$Tx = x^2 + 3x + 3 \quad \text{and} \quad fx = 2x^2 + 6x + 5.$$

Note that the condition (3.1) holds for $\frac{1}{2} < \alpha < 1$. Clearly, $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Thus, by Theorem 3.1, T and f have a unique point of coincidence $1 \in X$, as $f(-1) = T(-1) = 1$.

Remark 3.4. As $fT(-1) = 13$ and $Tf(-1) = 7$, T and f are not weakly compatible. So,

$$T(1) \neq f(1) \neq 1.$$

This shows that the weak compatibility for T and f in Theorem 3.1 is an essential condition.

Example 3.5. If we take $Tx = x^2 + 3x + 1$ and $fx = 2x^2 + 6x + 3$ in Example 3.3 then T and f become weakly compatible and all conditions of Theorem 3.1 are satisfied. Hence we obtain a a unique common fixed point $1 = f(1) = T(1)$.

If $f = Id_X$ in Theorem 3.1, then we get the following result of Ćirić [7].

Corollary 3.6. *Let (X, d) be a cone metric space and T be a self mapping of X such that*

$$d(Tx, Ty) \leq \alpha \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \tag{3.2}$$

for all $x, y \in X$, where $0 \leq \alpha < 1$. Then T has a unique fixed point in X.

Theorem 3.7. *Let (X, d) be a cone metric space and $\{T_\alpha\}$ be a family of self mappings of X such that*

$$\begin{aligned} &d(T_\alpha x, T_\alpha y) \\ &\leq \alpha \max\left\{d(x, y), d(x, T_\alpha x), d(y, T_\alpha y), \frac{d(x, T_\alpha y) + d(y, T_\alpha x)}{2}\right\} \end{aligned} \tag{3.3}$$

for all $\alpha \in \Lambda$, $x, y \in X$ and $x \neq y$, where $0 \leq \alpha < 1$. Then $\{T_\alpha\}$ have a unique common fixed point in X.

Proof. The proof follows from Theorem 3.1. □

Theorem 3.8. *Let (X, d) be a cone metric space, T and f be two self mappings of X such that*

(i) $T(X) \subseteq f(X)$ *and* $f(X)$ *is a complete subspace of* X.
(ii)

$$d(Tx, Ty) \leq \alpha \max\left\{d(fx, fy), \frac{d(fx, Tx) + d(fy, Ty)}{2}, \frac{d(fx, Ty) + d(fy, Tx)}{2}\right\}, \quad \forall x, y \in X,$$

where $0 \leq \alpha < 1$.

Then T *and* f *have a unique point of coincidence in* X. *Moreover, if* T *and* f *are weakly compatible then* T *and* f *have a unique common fixed point in* X.

Proof. The proof is immediate from Theorem 3.1. □

Corollary 3.9. *Let* (X, d) *be a cone metric space and* $\{T_\alpha\}$ *be a family of self mappings of* X *such that*

$$d(T_\alpha x, T_\alpha y) \leq \alpha \max\left\{d(x, y), \frac{d(x, T_\alpha x) + d(y, T_\alpha y)}{2}, \frac{d(x, T_\alpha y) + d(y, T_\alpha x)}{2}\right\} \tag{3.4}$$

for all $\alpha \in \Lambda$, $x, y \in X$ *and* $x \neq y$, *where* $0 \leq \alpha < 1$. *Then* $\{T_\alpha\}$ *have a unique common fixed point in* X.

Theorem 3.10. *Let* (X, d) *be a cone metric space,* T *be a self mapping of* X *and* $\{T_n\}$ *be a sequence of self mappings of* X *satisfying*

$$d(T_n x, T_n y) \leq \alpha \max\left\{d(x, y), d(x, T_n x), d(y, T_n y), \frac{d(x, T_n y) + d(y, T_n x)}{2}\right\} \tag{3.5}$$

for all $x, y \in X, x \neq y$, *where* $0 \leq \alpha < 1$ *and the sequence* $\{T_n\}$ *converges pointwise to* T. *If* $\{T_n\}$ *have fixed points* v_n, $n = 1, 2, 3, \cdots$, *then* T *has a unique fixed point* v *and the sequence* $\{v_n\}$ *converges to* v *as* $n \to \infty$.

Proof. For $x, y \in X$, using the continuity of metric and $\{T_n\}$ converging pointwise to T, we get the following

$$d(Tx, Ty) \leq \alpha \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Therefore, from Corollary 3.6, T has a unique fixed point v in X. Further, since $T_n v_n = v_n$ and $Tv = v$, using (3.5) we have

$$\begin{aligned} d(v_n, v) &= d(T_n v_n, Tv) \\ &\leq d(T_n v_n, T_n v) + d(T_n v, Tv) \\ &\leq \alpha \max\left\{ d(v_n, v), d(v_n, T_n v), d(v, T_n v), \frac{d(v_n, T_n v) + d(v, T_n v_n)}{2} \right\} \\ &\quad + d(T_n v, Tv). \end{aligned}$$

Using the fact that $T_n v \to Tv$ as $n \to \infty$, we obtain

$$d(v_n, v) \leq \alpha d(v_n, v),$$

which implies that $v_n \to v$ as $n \to \infty$, since $\alpha < 1$. □

Theorem 3.11. *Let (X, d) be a cone metric space, $\{T_n\}$ be a sequence of self mappings of X with fixed points v_n and T be a self mapping of X such that T satisfies* (3.2) *for all $x, y \in X$ with a fixed point $v \in X$. If T_n converges to T uniformly as $n \to \infty$, then v_n converges to v as $n \to \infty$.*

Proof. Note that $T_n v_n = v_n$ and $Tv = v$. Then, using (3.2), we have

$$\begin{aligned} d(v_n, v) &= d(T_n v_n, Tv) \\ &\leq d(T_n v_n, Tv_n) + d(Tv_n, Tv) \\ &\leq d(T_n v_n, Tv_n) \\ &\quad + \alpha \max\left\{ d(v_n, v), d(v_n, Tv_n), d(v, Tv), \frac{d(v_n, Tv) + d(v, Tv_n)}{2} \right\} \\ &= d(T_n v_n, Tv_n) \\ &\quad + \alpha \max\left\{ d(v_n, v), d(T_n v_n, Tv_n), d(v, Tv), \frac{d(v_n, Tv) + d(T_n v_n, Tv_n)}{2} \right\}, \end{aligned}$$

which yields $d(v_n, v) \leq \alpha d(v_n, v)$. This completes the proof. □

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