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CONVERGENCE THEOREMS BASED ON THE SHRINKING PROJECTION METHOD FOR HEMI-RELATIVELY NONEXPANSIVE MAPPINGS, VARIATIONAL INEQUALITIES AND EQUILIBRIUM PROBLEMS

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Abstract. In this paper, hemi-relatively nonexpansive mappings, variational inequalities and equilibrium problems are considered based on a shrinking projection method. Strong convergence of iterative sequences is obtained in a uniformly convex and uniformly smooth Banach space. As an application, the problem of finding zeros of maximal monotone operators is studied.

1. Introduction

Let E be a Banach space and E^* the dual space of E. Let C be a nonempty closed convex subset of E. Let J be the normalized duality mapping from E into 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing.

It is known that the duality mapping J has the following properties:

(1) If E is smooth, then J is single-valued.
(2) If E is strictly convex, then J is one-to-one.
(3) If E is reflexive, then J is surjective.

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(4) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
(5) If E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E and J is single-valued and also one-to-one; see, [6, 12, 27, 35].

Let $A : C \to E^*$ be an operator. We consider the following variational inequality: Find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

A point $x_0 \in C$ is called a solution of the variational inequality (1.1) if $\langle Ax_0, y - x_0\rangle \geq 0$. The solutions set of the variational inequality (1.1) is denoted by $VI(A, C)$. The variational inequality (1.1) has been intensively considered due to its various applications in operations research, economic equilibrium and engineering design. When A has some monotonicity, many iterative methods for solving the variational inequality (1.1) have been developed; see, [1, 2, 3, 4, 7, 8, 24, 25, 26].

Let C is a nonempty closed and convex subset of a Hilbert space H and $P_C : H \to C$ be the metric projection of H onto C, then P_C is nonexpansive, that is,

$$\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall x, y \in H.$$

This fact actually characterizes Hilbert spaces, however, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Recently, applying the generalized projection operator, Li [16] established the following Mann type iterative scheme for solving variational inequalities without assuming the monotonicity of A in compact subset of Banach spaces.

Theorem 1.1. ([16], Theorem 3.1) *Let E be a uniformly convex and uniformly smooth Banach space and C be a compact convex subset of E. Let $A : C \to E^*$ be a continuous mapping on C such that*

$$\langle Ax - \xi, J^{-1}(Jx - (Ax - \xi))\rangle \geq 0, \quad \forall x \in C,$$

where $\xi \in E^*$. For any $x_0 \in C$, define the Mann type iteration scheme as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \Pi_C(Jx_n - (Ax_n - \xi)), \quad \forall n \geq 1,$$

where the sequence $\{\alpha_n\}$ satisfies the following conditions:

(a) $0 \leq \alpha_n \leq 1$ *for all* $n \in N$;
(b) $\Sigma_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Then the variational inequality $\langle Ax - \xi, y - x\rangle \geq 0$ *for all* $y \in C$ (*when* $\xi = 0$, *the variational inequality* (1.1)) *has a solution* $x^* \in C$ *and there exists a subsequence* $\{n_i\} \subset \{n\}$ *such that*

$$x_{n_i} \to x^* \quad (i \to \infty).$$

In addition, Fan [11] established some existence results of solutions and the convergence of the Mann type iterative scheme for the variational inequality (1.1) in a noncompact subset of a Banach space and proved the following theorem.

Theorem 1.2. ([11], Theorem 3.3) *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a compact convex subset of* E. *Suppose that there exists a positive number* β *such that*

$$\langle Ax, J^{-1}(Jx - \beta Ax)\rangle \geq 0, \quad \forall\, x \in C,$$

and $J - \beta A : C \to E^*$ *is compact. if*

$$\langle Ax, y\rangle \leq 0, \quad \forall\, x \in C,\, y \in VI(A, C),$$

then the variational inequality (1.1) *has a solution* $x^* \in C$ *and the sequence* $\{x_n\}$ *defined by the following iteration scheme:*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \Pi_C(Jx_n - \beta Ax_n), \quad \forall\, n \geq 1,$$

where the sequence $\{\alpha_n\}$ *satisfies that* $0 < a \leq \alpha_n \leq b < 1$ *for all* $n \geq 1$ $(a, b \in (0, 1]$ *with* $a < b$), *converges strongly to* $x^* \in C$.

Motivated by Li [16] and Fan [11], Liu [17] introduced the iterative sequence for approximating a common element of the fixed points set of a relatively weak nonexpansive mapping defined by Kohasaka and Takahashi [15] and the solutions set of the variational inequality in a noncompact subset of Banach spaces without assuming the compactness of the operator $J - \beta A$. More precisely, Liu [17] proved the following theorems.

Theorem 1.3. ([17], Lemma 2.5) *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty, closed convex subset of* E. *Suppose that there exists a positive number* β *such that*

$$\langle Ax, J^{-1}(Jx - \beta Ax)\rangle \geq 0, \quad \forall\, x \in C, \tag{1.2}$$

and

$$\langle Ax, y\rangle \leq 0, \quad \forall\, x \in C,\, y \in VI(A, C), \tag{1.3}$$

then $VI(A, C)$ *is closed and convex.*

Theorem 1.4. ([17], Theorem 3.1) *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Assume that A is a continuous operator of C into E^* satisfying the conditions* (1.2) *and* (1.3) *and $S : C \to C$ is a relatively weak nonexpansive mapping with*

$$F := F(S) \cap VI(A, C) \neq \emptyset.$$

Then the sequence $\{x_n\}$ generated by the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)J\Pi_C(Jz_n - \beta Az_n)), \\ C_0 = \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_0 = C, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle Jx_0 - Jx_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} Jx_0, \quad \forall\, n \geq 1, \end{cases} \tag{1.4}$$

where the sequences $\{\alpha_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

$$0 \leq \delta_n < 1, \quad \limsup_{n\to\infty} \delta < 1, \quad 0 < \alpha_n < 1, \quad \liminf_{n\to\infty} \alpha_n(1-\alpha) > 0,$$

converges strongly to $\Pi_{F(S)\cap VI(A,C)}Jx_0$.

A mapping $A : D(A) \subset E \to E^*$ is said to be monotone if the following inequality holds:

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall\, x, y \in D(A). \tag{1.5}$$

A is said to be $\lambda-$inverse strongly monotone if there exists a positive real number λ such that

$$\langle x - y, Ax - Ay\rangle \geq \lambda \|Ax - Ay\|^2, \quad \forall\, x, y \in D(A). \tag{1.6}$$

If A is $\lambda-$inverse strongly monotone, then it is *Lipschitz* continuous with constant $\frac{1}{\lambda}$, *i.e.*, $\|Ax - Ay\| \leq \frac{1}{\lambda}\|x - y\|$, $\forall\, x, y \in D(A)$, and hence uniformly continuous.

For finding an element of a nonexpansive mapping and $VI(A, C)$, Takahashi and Toyoda [38] introduced the following iterative scheme in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \mu_n Ax_n), \quad n \geq 1, \tag{1.7}$$

where $x_0 \in C$, P_C is a metric projection of H onto C, A is a $\lambda-$inverse strongly monotone operator. Furthermore they proved a weak convergence theorem.

Theorem 1.5. ([38], Theorem 3.1) *Let C be a closed convex subset of a real Hilbert space H. Let $\lambda > 0$. Let A be an $\lambda-$inverse strongly-monotone mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(A, C) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by (1.7) for all $n \in \mathbb{N} \cup \{0\}$, where $\{\mu_n\} \subset [a, b]$ for some $a,\ b \in (0, 2\lambda)$ and $\{\alpha_n\} \subset [c, d]$ for some $c,\ d \in (0, 1)$. Then, $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(A, C)$, where $z = \lim_{n\to\infty} P_{F(S)\cap VI(A,C)}x_n$.*

Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for f is as follows: Find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall\, y \in C. \tag{1.8}$$

The set of solutions of the problem (1.8) is denoted by $EP(f)$. For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x,\ y,\ z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x, y) = \langle Ax, y - x\rangle, \quad \forall\, x, y \in C.$$

Then f satisfies (A1)-(A4).

Let $f : C \times C \to \mathbb{R}$ be a bifunction and let $B : C \to E^*$ be a monotone mapping. The generalized equilibrium problem (for short, GEP) for f and B is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle B\hat{x}, y - \hat{x}\rangle \geq 0, \quad \forall\ y \in C. \tag{1.9}$$

The set of solutions for the problem (1.9) is denoted by $GEP(f, B)$, *i.e.*,

$$GEP(f, B) := \{\hat{x} \in C : f(\hat{x}, y) + \langle B\hat{x}, y - \hat{x}\rangle \geq 0, \quad \forall\, y \in C\}.$$

If $B = 0$ in (1.9), then $GEP(1.9)$ reduces to the classical equilibrium problem and $GEP(f, 0)$ is denoted by $EP(f)$, *i.e.*,

$$EP(f) := \{\hat{x} \in C : f(\hat{x}, y) \geq 0, \quad \forall\, y \in C\}.$$

Equilibrium problems, which were introduced in [5] in 1994, have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, physics, image reconstruction, ecology, transportation, network, elasticity and optimization. Numerous problems in physics, optimization and economics reduce to finding a solution of the problem (1.5). Some methods have been proposed to solve the equilibrium problem in a Hilbert space; See [5, 20, 21].

In this paper, motivated and inspired by the results mentioned above, we introduce a new hybrid projection algorithm based on the shrinking projection method [31, 37] for two hemi-relatively nonexpansive mappings. Using the new algorithm, we prove some strong convergence theorem which approximate a common element in the fixed points set of two hemi-relatively nonexpansive mappings, the solutions set of a variational inequality and the solutions set of the equilibrium problem in a uniformly convex and uniformly smooth Banach space. Our results extend and improve the recent ones announced by Li [16], Fan [11], Liu [17], Takahashi and Toyoda [38], Kamraksa and Wangkeeree [14] and many others.

2. Preliminaries

A Banach space E is said to be strictly convex if $\frac{x+y}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{\to\infty} \|\frac{x_n+y_n}{2}\| = 1$.

Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U_E$.

It is well known that, if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E and, if E is uniformly smooth if and only if E^* is uniformly convex.

A Banach space E is said to have the Kadec-Klee property if, for a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, $x_n \to x$. It is known that, if E is uniformly convex, then E has the Kadec-Klee property; see, [9, 35, 36] for more details.

Let C be a closed convex subset of E and T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong

$\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Recall that an operator T in Banach space is said to be closed if $x_n \to x$ and $Tx_n \to y$ implies $Tx = y$.

A mapping T from C into itself is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in C.$$

The mapping T is said to be relatively nonexpansive [13, 18, 19] if

$$\widehat{F}(T) = F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall\, x \in C,\, p \in F(T).$$

The asymptotic behavior of a relatively nonexpansive mapping was studied in [13, 18, 19]. A point $p \in C$ is called a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of strong asymptotic fixed points of T is denoted by $\widetilde{F}(T)$.

A mapping T from C into itself is said to be relatively weak nonexpansive if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall\, x \in C,\, p \in F(T).$$

The mapping T is said to be hemi-relatively nonexpansive if

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall\, x \in C,\, p \in F(T).$$

It is obvious that a relatively nonexpansive mapping is a relatively and weakly nonexpansive mapping and, further, a relatively and weakly nonexpansive mapping is a hemi-relatively nonexpansive mapping, but the converses are not true as in the following example.

Example 2.1. ([34]) Let E be any smooth Banach space and $x_0 \neq 0$ be any element of E. We define a mapping $T : E \to E$ as follows: For all $n \geq 1$,

$$T(x) = \begin{cases} (\frac{1}{2} + \frac{1}{2^{n+1}})x_0, & \text{if } x = (\frac{1}{2} + \frac{1}{2^n})x_0, \\ -x, & \text{if } x \neq (\frac{1}{2} + \frac{1}{2^n})x_0. \end{cases}$$

Then T is a hemi-relatively nonexpansive mapping, but it is not relatively nonexpansive mapping.

Next, we give some important examples which are hemi-relatively nonexpansive.

Example 2.2. ([23]) Let E be a strictly convex reflexive smooth Banach space. Let A be a maximal monotone operator of E into E^* and J_r be the resolvent for A with $r > 0$. Then $J_r = (J + rA)^{-1}J$ is a hemi-relatively nonexpansive mapping from E onto $D(A)$ with $F(J_r) = A^{-1}0$.

In [4, 12], Alber introduced the functional $V : E^* \times E \to \mathbb{R}$ defined by

$$V(\phi, x) = \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2, \tag{2.2}$$

where $\phi \in E^*$ and $x \in E$. It is easy to see that

$$V(\phi, x) \geq (\|\phi\| - \|x\|)^2 \tag{2.3}$$

and so the functional $V : E^* \times E \to \mathbb{R}^+$ is nonnegative.

In order to prove our results in the next section, we present several definitions and lemmas here.

Definition 2.3. ([13]) If E be a uniformly convex and uniformly smooth Banach space, then the generalized projection $\Pi_C : E^* \to C$ is a mapping that assigns an arbitrary point $\phi \in E^*$ to the minimum point of the functional $V(\phi, x)$, *i.e.*, a solution to the minimization problem

$$V(\phi, \Pi_C(\phi)) = \inf_{y \in C} V(\phi, y). \tag{2.4}$$

Li [16] proved that the generalized projection operator $\Pi_C : E^* \to C$ is continuous if E is a reflexive, strictly convex and smooth Banach space.

Consider the function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = V(Jy, x), \quad \forall\, x, y \in E.$$

The following properties of the operator Π_C and V are useful for our paper (see, for example, [1, 16]).

(B1) $V : E^* \times E \to \mathbb{R}$ is continuous.
(B2) $V(\phi, x) = 0$ if and only if $\phi = Jx$.
(B3) $V(J\Pi_C(\phi), x) \leq V(\phi, x)$ for all $\phi \in E^*$ and $x \in E$.
(B4) The operator Π_C is J fixed at each point $x \in E^*$ and $x \in E$.
(B5) If E is smooth, then, for any given $\phi \in E^*$ and $x \in C$, $x \in \Pi_C(\phi)$ if and only if

$$\langle \phi - Jx, x - y\rangle \geq 0, \quad \forall y \in C.$$

(B6) The operator $\Pi_C : E^* \to c$ is single valued if and only if E is strictly convex.
(B7) If E is smooth, then, for any given point $\phi \in E^*$ and $x \in \Pi_C(\phi)$, the following inequality holds:

$$V(Jx, y) \leq V(\phi, y) - V(\phi, x), \quad \forall y \in C.$$

(B8) $v(\phi, X)$ is convex with respect to ϕ when x is fixed and with respect to x when ϕ is fixed.
(B9) If E is reflexive, then, for any point $\phi \in E^*$, $\Pi_C(\phi)$ is a nonempty closed convex and bounded subset of C.

Using some properties of the generalized projection operator Π_C, Alber [1] proved the following theorem.

Lemma 2.4. ([1]) *Let E be a strictly convex reflexive smooth Banach space. Let A be an arbitrary operator from a Banach space E to E^* and β be an arbitrary fixed positive number. Then $x \in C \subset E$ is a solution of the variational inequality* (1.1) *if and only if x is a solution of the following operator equation in E:*

$$x = \Pi_C(Jx - \beta Ax). \tag{2.5}$$

Lemma 2.5. ([13]) *Let E be a uniformly convex smooth Banach space and $\{y_n\}$, $\{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to\infty} \phi(y_n, z_n) = 0$, then $\lim_{n\to\infty} \|y_n - z_n\| = 0$.*

Lemma 2.6. ([7]) *Let E be a uniformly convex and uniformly smooth Banach space. We have*

$$\|\phi + \Phi\|^2 \leq \|\phi\|^2 + 2\langle \Phi, J(\phi + \Phi)\rangle, \quad \forall\, \phi, \Phi \in E^*. \tag{2.6}$$

From Qin *et al.* [22], the following lemma can be obtained immediately.

Lemma 2.7. *Let E be a uniformly convex Banach space, $s > 0$ be a positive number and $B_s(0)$ be a closed ball of E. Then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\Sigma_{i=1}^N(\alpha_i x_i)\|^2 \leq \Sigma_{i=1}^N(\alpha_i\|x_i\|^2) - \alpha_i\alpha_j g(\|x_i - x_j\|) \tag{2.7}$$

for all $x_1, x_2, \cdots, x_N \in B_s(0) = \{x \in E : \|x\| \leq s\}$, $i \neq j$ for all $i, j \in \{1, 2, \cdots, N\}$ and $\alpha_1, \alpha_2, \cdots, \alpha_N \in [0, 1]$ such that $\Sigma_{i=1}^N \alpha_i = 1$.

Lemma 2.8. ([5]) *Let C be a closed and convex subset of a smooth, strictly convex and reflexive Banach spaces E, f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions $(B1)$-$(B4)$ and let $r > 0$, $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall\, y \in C. \tag{2.8}$$

Lemma 2.9. ([39]) *Let C be a closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach spaces E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (B1)-(B4)*. For all $r > 0$ and $x \in E$, define the mapping*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall\, y \in C \right\}.$$

Then, the following statements hold:

(C1) *T_r is single-valued;*

(C2) *T_r is a firmly nonexpansive-type mapping, that is, for all $x,\ y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(C3) $F(T_r) = \hat{F}(T_r) = EP(f)$;
(C4) $EP(f)$ *is closed and convex.*

Lemma 2.10. ([39]) *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (C1)-(A4)*, and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.9}$$

Remark 2.11. Replacing x with $J^{-1}(Jx - rB(x))$ in (2.8), where B is monotone mapping from C into E^*, then there exists $z \in C$ such that

$$f(z, y) + \langle Bx, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall\, y \in C. \tag{2.10}$$

Lemma 2.12. *Let C be a closed and convex subset of a smooth, strictly convex and reflexive Banach space E, $B : C \to E^*$ a monotone and continuous mapping, f a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions $(A1)$–$(A4)$. For all $r > 0$, the following statements hold.*

(i) *for $x \in E$, there exists $z \in C$ such that*

$$f(z, y) + \langle Bz, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall\, y \in C;$$

(ii) *if E is additionally uniformly smooth and $K_r : E \to C$ is defined as*

$$K_r x = \Big\{ z \in C : f(z, y) + \langle Bz, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall\, y \in C \Big\}, \quad \forall\, x \in E. \tag{2.11}$$

Then the mapping K_r has the following Properties:

(D1) *K_r is single-valued;*
(D2) *K_r is a firmly nonexpansive-type mapping, that is, for all $x,\ y \in E$,*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle; \tag{2.12}$$

(D3) $F(K_r) = \hat{F}(K_r) = EP(f, B)$;
(D4) $EP(f, B)$ *is closed and convex subset of C;*
(D5) $\phi(p, K_r x) + \phi(K_r x, x) \leq \phi(p, x), \quad \forall\, p \in F(K_r)$.

Proof. Define a bifunction $F : C \times C \to \mathbb{R}$ as follows:

$$F(z, y) = f(z, y) + \langle Bz, y - z\rangle, \quad \forall\, z, y \in C.$$

Then it is easy to imply that F satisfies conditions $(A1)$–$(A4)$. Therefore, from Lemma 2.8–2.10, statements (i), (ii) of Lemma 2.12 follow immediately. $\square$

Lemma 2.13. ([17], Lemma 2.6) *If E is a reflexive, strictly convex and smooth Banach space, then $\Pi_C = J^{-1}$.*

Lemma 2.14. ([34], Lemma 2.6) *Let E be a strictly convex and smooth real Banach space, C be a closed convex subset of E and T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

3. Main results

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (A_1)-(A_4). Assume that A_1, A_2 are two continuous operators of C into E^* satisfying the conditions* (1.2) *and* (1.3)*, B is a continuous and monotone operator of C into E^* and $S, T : C \to C$ are two closed hemi-relatively nonexpansive mappings with $F := F(S) \cap F(T) \cap VI(A_1, C) \cap VI(A_2, C) \cap GEP(f, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n^i = \Pi_C(Jx_n - \eta_i A_i x_n), \quad i = 1,\ 2, \\ y_n = \Pi_C(\beta_n^0 Jx_n + \beta_n^1 JTz_n^1 + \beta_n^2 JSz_n^2), \\ u_n \in C \text{ such that } f(u_n, y) + \langle Bu_n, y - u_n\rangle \\ \qquad\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \bigcap_{i=1,2} \phi(z, u_n) \leq \phi(z, y_n) \\ \qquad\qquad \leq (1 - \beta_n^i)\phi(z, x_n) + \beta_n^i \phi(z, z_n^i) \\ \qquad\qquad \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\beta_n^0\}$, $\{\beta_n^1\}$ and $\{\beta_n^2\}$ are the sequences in $[0, 1]$ with the following restrictions:

(a) $\beta_n^0 + \beta_n^1 + \beta_n^2 = 1$;
(b) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;
(c) $\liminf_{n\to\infty} \beta_n^0\beta_n^1 > 0$ *and* $\liminf_{n\to\infty} \beta_n^0\beta_n^2 > 0$.

Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_F Jx_0$, where Π_F is the generalized projection from C onto F.

Proof. We divide the proof into five steps.

Step 1. $\Pi_F Jx_0$ and $\Pi_{C_{n+1}} Jx_0$ are well defined.

From Lemma 2.12 $(D5)$, Lemma 2.14 and Theorem 1.1, one has that $\Pi_F Jx_0$ is well defined. Next, we show that C_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. From the definitions of C_n, it is obvious that C_n is closed for all $n \in \mathbb{N} \cup \{0\}$.

Next, we prove that C_n is convex for all $n \in \mathbb{N} \cup \{0\}$. Since $\phi(z, u_n) \leq \phi(z, y_n)$ is equivalent to $2\langle z, Jy_n - Ju_n\rangle \leq \|y_n\|^2 - \|u_n\|^2$, for $i = 1, 2$, we have

$$\phi(z, y_n) \leq (1 - \beta_n^i)\phi(z, x_n) + \beta_n^i \phi(z, z_n^i)$$

is equivalent to

$$2\langle z, (1 - \beta_n^i)Jx_n + \beta_n^i J z_n^i - Jy_n\rangle \leq (1 - \beta_n^i)\|x_n\|^2 + \beta_n^i \|z_n^i\|^2 - \|y_n\|^2,$$

and

$$(1 - \beta_n^i)\phi(z, x_n) + \beta_n^i \phi(z, z_n^i) \leq \phi(z, x_n)$$

is equivalent to

$$2\langle z, Jx_n - Jz_n^i\rangle \leq \|x_n\|^2 - \|z_n^i\|^2,$$

it follows that C_n is convex for all $n \in \mathbb{N} \cup \{0\}$. Thus, for all $n \in \mathbb{N} \cup \{0\}$, C_n is closed and convex and so $\Pi_{C_{n+1}} Jx_0$ is well defined.

Step 2. $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Observe that $F \subset C_0 = C$ is obvious. Suppose that $F \subset C_n$ for some $n \in \mathbb{N}$. Let $w \in F \subset C_n$. Then, from the definition of ϕ and V, the property $(B3)$ of V, Lemma 2.6, the conditions (1.2) and (1.3), for all $n \in \mathbb{N} \cup \{0\}$, $i = 1,\ 2$, it follows that

$$\begin{aligned}
&\phi(w, \Pi_C(Jx_n - \eta_i A_i x_n)) \\
&= V(J\Pi_C(Jx_n - \eta_i A_i x_n), w) \\
&\leq V(Jx_n - \eta_i A_i x_n, w) \\
&= \|Jx_n - \eta_i A_i x_n\|^2 - 2\langle Jx_n - \eta_i A_i x_n, w\rangle + \|w\|^2 \\
&\leq \|Jx_n\|^2 - 2\eta_i \langle A_i x_n, J^{-1}(Jx_n - \eta_i A_i x_n)\rangle \\
&\quad - 2\langle Jx_n - \eta_i A_i x_n, w\rangle + \|w\|^2 \\
&\leq \|Jx_n\|^2 - 2\langle Jx_n, w\rangle + \|w\|^2 \\
&= \phi(w, x_n).
\end{aligned} \tag{3.2}$$

Since $u_n = K_{r_n} y_n$, applying Lemma 2.12, the properties (B3) and (B8) of the operator V and (3.4), we obtain

$$\begin{aligned}
\phi(w, u_n) = \phi(w, K_{r_n} y_n) \leq \phi(w, y_n) &= V(Jy_n, w) \\
&\leq \beta_n^0 V(Jx_n, w) + \beta_n^1 V(JTz_n^1, w) + \beta_n^2 V(JSz_n^2, w) \\
&\leq \beta_n^0 \phi(w, x_n) + \beta_n^1 \phi(w, z_n^1) + \beta_n^2 \phi(w, z_n^2) \\
&\leq \beta_n^0 \phi(w, x_n) + \beta_n^1 \phi(w, x_n) + \beta_n^2 \phi(w, x_n) \\
&= \phi(w, x_n),
\end{aligned} \tag{3.3}$$

which shows that $w \in C_{n+1}$. This implies that $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Step 3. $\{x_n\}$ is a Cauchy sequence.

Since $x_n = \Pi_{C_n} Jx_0$ and $F \subset C_n$, we have $V(Jx_0, x_n) \leq V(Jx_0, w)$ for all $w \in F$. Therefore, $\{V(Jx_0, x_n)\}$ is bounded. Moreover, from the definition of V, it follows that $\{x_n\}$ is bounded. Since $x_{n+1} = \Pi_{C_{n+1}} Jx_0 \in C_{n+1}$ and $x_n = \Pi_{C_n} Jx_0$, we have $V(Jx_0, x_n) \leq V(Jx_0, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Hence $\{V(Jx_0, x_n)\}$ is nondecreasing and so $\lim_{n\to\infty} V(Jx_0, x_n)$ exists. By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m} Jx_0 \in C_n$ for any positive integer $m \geq n$. From the property (B3), we have

$$V(Jx_n, x_m) \leq V(Jx_0, x_m) - V(Jx_0, x_n)$$

for all $n \in \mathbb{N} \cup \{0\}$ and any positive integer $m \geq n$. This implies that

$$V(Jx_n, x_m) \to 0 \quad \text{as} \quad n, m \to \infty.$$

The definition of ϕ implies that

$$\phi(x_m, x_n) \to 0 \quad \text{as} \quad n, m \to \infty. \tag{3.4}$$

Applying Lemma 2.5, we obtain

$$\|x_m - x_n\| \to 0 \quad \text{as} \quad n, m \to \infty. \tag{3.5}$$

Hence $\{x_n\}$ is a Cauchy sequence. In view of the completeness of a Banach space E and the closeness of C, it follows that

$$\lim_{n\to\infty} x_n = p \tag{3.6}$$

for some $p \in C$.

Step 4. $p \in F$.

First, we show that $p \in F(S) \cap F(T)$. In fact, since $x_{n+1} \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n).$$

Thus, by (3.4) and Lemma 2.5, we have that

$$\|x_{n+1} - u_n\| \to 0 \quad \text{as} \quad n \to \infty,$$

and hence

$$\|x_n - u_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\| \to 0 \quad \text{as} \quad n \to \infty, \tag{3.7}$$

which implies that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} x_n = p. \tag{3.8}$$

On the other hand, since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.9}$$

Since $\{x_n\}$ is bounded, $\{Jx_n\}$, $\{JTx_n\}$ and $\{JSx_n\}$ are also bounded. Since E is a uniformly smooth Banach space, one knows that E^* is a uniformly convex Banach space. Let $r = \sup_{n\geq 0}\{\|Jx_n\|, \|JTx_n\|, \|JSx_n\|\}$. Therefore, from Lemma 2.7, it follows that there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ satisfying $g(0) = 0$ and the inequality (2.7). It follows from the property (B3) of the operator V, (3.2) and the definition of S and T that

$$\begin{aligned}
&\phi(w, y_n)\\
&= V(Jy_n, w)\\
&\leq V(\beta_n^0 Jx_n + \beta_n^1 JTz_n^1 + \beta_n^2 JSz_n^2, w)\\
&= \phi(w, J^{-1}(\beta_n^0 Jx_n + \beta_n^1 JTz_n^1 + \beta_n^2 JSz_n^2))\\
&= \|w\|^2 - 2\beta_n^0\langle w, Jx_n\rangle - 2\beta_n^1\langle w, JTz_n^1\rangle - 2\beta_n^2\langle w, JSz_n^2\rangle\\
&\quad + \|\beta_n^0 Jx_n + \beta_n^1 JTz_n^1 + \beta_n^2 JSz_n^2\|^2\\
&\leq \|w\|^2 - 2\beta_n^0\langle w, Jx_n\rangle - 2\beta_n^1\langle w, JTz_n^1\rangle - 2\beta_n^2\langle w, JSz_n^2\rangle\\
&\quad + \beta_n^0\|Jx_n\|^2 + \beta_n^1\|JTz_n^1\|^2 + \beta_n^2\|JSz_n^2\|^2 - \beta_n^0\beta_n^1 g(\|Jx_n - JTz_n^1\|)\\
&= \beta_n^0\phi(w, x_n) + \beta_n^1\phi(w, Tz_n^1) + \beta_n^2\phi(w, Sz_n^2) - \beta_n^0\beta_n^1 g(\|Jx_n - JTz_n^1\|)\\
&\leq \beta_n^0\phi(w, x_n) + \beta_n^1\phi(w, x_n) + \beta_n^2\phi(w, x_n) - \beta_n^0\beta_n^1 g(\|Jx_n - JTz_n^1\|)\\
&= \phi(w, x_n) - \beta_n^0\beta_n^1 g(\|Jx_n - JTz_n^1\|).
\end{aligned} \tag{3.10}$$

On the other hand, from (3.3), we get that

$$\phi(w, u_n) = \phi(w, K_{r_n} y_n) \leq \phi(w, y_n). \tag{3.11}$$

Substituting (3.10) into (3.11), we obtain that

$$\phi(w, u_n) \leq \phi(w, x_n) - \beta_n^0\beta_n^1 g(\|Jx_n - JTz_n^1\|).$$

The above inequality implies that

$$\beta_n^0\beta_n^1 g(\|Jx_n - JTz_n^1\|) \leq \phi(w, x_n) - \phi(w, u_n), \tag{3.12}$$

and we have

$$\begin{aligned}
\phi(w, x_n) - \phi(w, u_n) &= 2\langle Ju_n - Jx_n, w\rangle + \|x_n\|^2 - \|u_n\|^2\\
&\leq 2\langle Ju_n - Jx_n, p\rangle + (\|x_n\| - \|u_n\|)(\|x_n\| + \|u_n\|)\\
&\leq 2\|Ju_n - Jx_n\|\|w\| + \|x_n - u_n\|(\|x_n\| + \|u_n\|).
\end{aligned}$$

It follows from (3.7) and (3.8) that

$$\lim_{n\to\infty}(\phi(w, x_n) - \phi(w, u_n)) = 0. \tag{3.13}$$

In view of $\liminf_{n\to\infty}\beta_n^0\beta_n^1 > 0$, the inequality (3.12) implies that

$$g(\|Jx_n - JTz_n^1\|) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, from the property of g, we get that

$$\|Jx_n - JTz_n^1\| \to 0 \quad \text{as} \quad n \to \infty.$$

Furthermore, since J^{-1} is uniformly norm to norm continuous on bounded sets, we see that

$$\|x_n - Tz_n^1\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.14}$$

On the other hand, By the construction of C_n, we know that

$$\phi(z, u_n) \le (1-\beta_n^1)\phi(z, x_n) + \beta_n^1\phi(z, z_n^1) \le \phi(z, x_n).$$

From $x_{n+1} = \Pi_{C_{n+1}} Jx_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \le (1-\beta_n^1)\phi(x_{n+1}, x_n) + \beta_n^1\phi(x_{n+1}, z_n^1) \le \phi(x_{n+1}, x_n).$$

It follows from (3.8) that

$$\phi(x_{n+1}, z_n^1) \to 0 \quad \text{as} \quad n \to \infty.$$

Applying Lemma 2.5, one has

$$\|x_{n+1} - z_n^1\| \to 0 \quad \text{as} \quad n \to \infty,$$

and by (3.5), we obtain that

$$\|x_n - z_n^1\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n^1\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.15}$$

Thus, from (3.14) and (3.15), we obtain that

$$\|z_n^1 - Tz_n^1\| \le \|z_n^1 - x_n\| + \|x_n - Tz_n^1\| \to 0 \quad \text{as} \quad n \to \infty.$$

Similarly, one can obtain that

$$\|x_n - z_n^2\| \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\|z_n^2 - Sz_n^2\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, from the closedness of S, T, we obtain that $p \in F(S) \cap F(T)$.

Secondly, we show that $p \in GEP(f, B)$, from $u_n = K_{r_n} x_n$ and the construction of C_n, one has

$$\begin{aligned} \phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \\ &\le \phi(w, y_n) - \phi(w, K_{r_n} y_n) \\ &\le \phi(w, x_n) - \phi(w, K_{r_n} y_n) \\ &\le \phi(w, x_n) - \phi(w, u_n). \end{aligned}$$

And by (3.13) it follows that

$$\phi(u_n, y_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Applying Lemma 2.5, we obtain

$$\|u_n - y_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Since J is a uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0.$$

From the assumption that $r_n \geq a$, one has

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0.$$

Observing that $u_n = K_{r_n} y_n$, one obtains

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy\rangle \geq 0, \quad \forall\, y \in C,$$

where $F(u_n, y) = f(u_n, y) + \langle Bu_n, y - u_n\rangle$. From (A2), one gets

$$\begin{aligned} \|y_n - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\ &\leq -F(u_n, y) \\ &\leq F(y, u_n), \quad \forall\, y \in C. \end{aligned}$$

Taking $n \to \infty$ in above inequality, it follows from (A4) and (3.8) that

$$F(y, p) \leq 0, \quad \forall\, y \in C.$$

For all $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Note that $y,\ p \in C$, one obtains $y_t \in C$, which yields that $F(y_t, p) \leq 0$. It follows from $(A1)$ that

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, p) \leq tF(y_t, y),$$

that is

$$F(y_t, y) \geq 0.$$

Let $t \downarrow 0$. From $(A3)$, we obtain $F(p, y) \geq 0$ for all $y \in C$, which imply that $p \in GEP(f, B)$.

Finally, we show that $p \in VI(A_1, C) \cap VI(A_2, C)$. In fact, by (3.15), we have

$$\|\Pi_C(Jx_n - \eta_1 A_1 x_n) - x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Since $\lim_{n\to\infty} x_n = p$, we obtain

$$\lim_{n\to\infty} z_n^1 = p.$$

Similarly, one can also have

$$\lim_{n\to\infty} z_n^2 = p.$$

By the continuity of the operator J, A_1, Π_C, we have

$$\lim_{n\to\infty} \|\Pi_C(Jx_n - \eta_1 A_1 x_n) - \Pi_C(Jp - \eta_1 A_1 p)\| = 0$$

Note that

$$\begin{aligned}\|\Pi_C(Jx_n-\eta_1A_1x_n)-p)\| &\leq \|\Pi_C(Jx_n-\eta_1A_1x_n)-x_n\|+\|x_n-p\| \\ &\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Hence it follows from the uniqueness of the limit that $p=\Pi_C(Jp-\eta_1A_1p)$. From Lemma 2.4, we have $p\in VI(A_1,C)$. By the same way, we can also know that $p\in VI(A_2,C)$ and so $p\in VI(A_1,C)\cap VI(A_2,C)$. Therefore, we have $p\in F$.

Step 5. $p=\Pi_F Jx_0$.

Since $p\in F$, from the property (B3) of the operator Π_C, we have

$$V(J\Pi_F Jx_0,p)+V(Jx_0,\Pi_F Jx_0)\leq V(Jx_0,p). \tag{3.16}$$

On the other hand, since $x_{n+1}=\Pi_{C_{n+1}}Jx_0$ and $F\subset C_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$, it follows from the property (B7) of the operator Π_C that

$$V(Jx_{x+1},\Pi_F Jx_0)+V(Jx_0,x_{n+1})\leq V(Jx_0,\Pi_F Jx_0). \tag{3.17}$$

Furthermore, by the continuity of the operator V, we get

$$\lim_{n\to\infty}V(Jx_0,x_{n+1})=V(Jx_0,p). \tag{3.18}$$

Combining (3.16), (3.17) with (3.18), we obtain

$$V(Jx_0,p)=V(Jx_0,\Pi_F Jx_0).$$

Therefore, it follows from the uniqueness of $\Pi_F Jx_0$ that $p=\Pi_F Jx_0$. This completes the proof. □

Remark 3.2. Theorem 3.1 improves Theorem 3.1 of Liu [17] and Theorem 3.1 of Kamraksa *et al.* [14] in the following senses.

(1) The hemi-relatively nonexpansive mapping is more general than the relatively weak nonexpansive one in Liu [17] and Kamraksa *et al.* [14].
(2) The iteration algorithms of Theorem 3.1 is modified Mann iteration which is different from the modified Mann iteration given in Liu [17] and Kamraksa *et al.* [14]; And, in contrast to Theorem 3.1 of Kamraksa *et al.* [14], our algorithm in Theorem 3.1 contacts with generalized equilibrium problem which is more general than equilibrium problem.

Remark 3.3. See Remark 3.1 of Liu [17], Theorem 3.1 also does the corresponding promotions about Liu [17] and Fan [11].

When $S=T=I$ in (3.1), we can obtain the new modified Mann iteration for the variational inequality (1.1), the generalized equilibrium problem (1.9) as follows.

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (A_1)-(A_4). Assume that A_1, A_2 are two continuous operators of C into E^* satisfying the conditions* (1.2) *and* (1.3)*, B is a continuous and monotone operator of C into E^* with $F := VI(A_1, C) \cap VI(A_2, C) \cap GEP(f, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n^i = \Pi_C(Jx_n - \eta_i A_i x_n), \quad i = 1,\ 2, \\ y_n = \Pi_C(\beta_n^0 Jx_n + \beta_n^1 Jz_n^1 + \beta_n^2 Jz_n^2), \\ u_n \in C \text{ such that } f(u_n, y) + \langle Bu_n, y - u_n\rangle \\ \qquad\qquad\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall\, y \in C, \\ C_{n+1} = \{z \in C_n : \bigcap_{i=1,2} \phi(z, u_n) \leq \phi(z, y_n) \\ \qquad\qquad\qquad \leq (1 - \beta_n^i)\phi(z, x_n) + \beta_n^i \phi(z, z_n^i) \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall\, n \geq 1, \end{cases}$$

where $\{\beta_n^0\}$, $\{\beta_n^1\}$ and $\{\beta_n^2\}$ are the sequences in $[0, 1]$ with the following restrictions:

(a) $\beta_n^0 + \beta_n^1 + \beta_n^2 = 1$;
(b) $\{r_n\} \subset [a, \infty)$ *for some $a > 0$;*
(c) $\liminf_{n\to\infty} \beta_n^0 \beta_n^1 > 0$ *and* $\liminf_{n\to\infty} \beta_n^0 \beta_n^2 > 0$.

Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_F Jx_0$, where Π_F is the generalized projection from C onto F.

If $\beta_n^2 = 0$ in (3.1), then the iteration scheme (3.1) reduces to the new modified Mann iteration for one closed hemi-relatively nonexpansive mapping, the variational inequality (1.1) and the generalized equilibrium problem (1.9) as follows.

Corollary 3.5. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (A_1)-(A_4). Assume that A is a continuous operator of C into E^* satisfying the conditions* (1.2) *and* (1.3)*, B is a continuous and monotone operator of C into E^* and $T : C \to C$ is a closed hemi-relatively nonexpansive mapping with*

$$F := F(T) \cap VI(A, C) \cap GEP(f, B) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(Jx_n - \eta Ax_n), \\ y_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ u_n \in C \text{ such that } f(u_n, y) + \langle Bu_n, y - u_n\rangle \\ \qquad\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, y_n) \\ \qquad\qquad \leq \alpha_n\phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall\, n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ with the following restrictions:

(a) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(b) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_F Jx_0$, where Π_F is the generalized projection from C onto F.

If the mapping A is a $\lambda-$inverse strongly monotone mapping in Corollary 3.5, then the following result can be also obtained by Corollary 2.5 and Theorem 3.1.

Corollary 3.6. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (A_1)-(A_4). Assume that A is a $\lambda-$inverse strongly monotone mapping of C into E^* satisfying the conditions (1.2) and (1.3), B is a continuous and monotone operator of C into E^* and $T : C \to C$ is a closed hemi-relatively nonexpansive mapping with $F := F(T) \cap VI(A, C) \cap GEP(f, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(Jx_n - \eta Ax_n), \\ y_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ u_n \in C \text{ such that } f(u_n, y) + \langle Bu_n, y - u_n\rangle \\ \qquad\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall\, y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, y_n) \\ \qquad\qquad \leq \alpha_n\phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall\, n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *with the following restrictions:*

(a) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$*;*

(b) $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

Proof. Since A is $\lambda-$inverse strongly monotone, by (1.6), we have

$$\|Ax - Ay\| \leq \frac{1}{\lambda}\|x - y\|,$$

for all $x,\ y \in C$, then it is *Lipschitz* continuous with constant $\frac{1}{\lambda}$. By Corollary 3.5, we can directly obtain that the sequence $\{x_n\}$ converges strongly to a point $\Pi_F Jx_0$. □

Remark 3.7. Corollary 3.6 improves Theorem 3.1 of Takahashi and Toyoda [38] in the following senses:

(1) The hemi-relatively nonexpansive mapping is more general than a non-expansive one in Takahashi and Toyoda [38].

(2) Our modified Mann iteration obtains strong convergence result about a $\lambda-$inverse strongly monotone mapping and a closed hemi-relatively nonexpansive mapping and generalized equilibrium problem (1.9) in a uniformly convex and uniformly smooth Banach space.

4. Applications to maximal monotone operators

In this section, we apply the our main results to proving some strong convergence theorem concerning maximal monotone operators in a Banach space E.

Let B be a multi-valued operator from E to E^* with domain $D(B) = \{z \in E : Bz \neq \emptyset\}$ and range $R(B) = \{z \in E : z \in D(B)\}$. An operator B is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$$

for all $x_1, x_2 \in D(B)$ and $y_1 \in Bx_1$, $y_2 \in Bx_2$. A monotone operator B is said to be maximal if it's graph $G(B) = \{(x,y) : y \in Bx\}$ is not properly contained in the graph of any other monotone operator.

It is well known that, if B is a maximal monotone operator, then $B^{-1}0$ is closed and convex.

The following result is also well known.

Lemma 4.1. ([30]) *Let* E *be a reflexive, strictly convex and smooth Banach space and* B *be a monotone operator from* E *to* E^**. Then* B *is maximal if and only if* $R(J + rB) = E^*$ *for all* $r > 0$*.*

Let E be a reflexive, strictly convex and smooth Banach space and B be a maximal monotone operator from E to E^*. Using Lemma 4.1 and the strict convexity of E, it follows that, for all $r > 0$ and $x \in E$, there exists a unique $x_r \in D(B)$ such that

$$Jx \in Jx_r + rBx_r.$$

If $J_r x = x_r$, then we can define a single valued mapping $J_r : E \to D(B)$ by $J_r = (J + rB)^{-1}J$ and such a J_r is called the resolvent of B. We know that $B^{-1}0 = F(J_r)$ for all $r > 0$ (see [21, 35] for more details).

The following lemma plays an important role in our next theorem.

Lemma 4.2. ([33]) *Let E be a uniformly convex and uniformly smooth Banach space, B be a maximal monotone operator from E to E^* and J_r be a resolvent of B. Then J_r is closed hemi-relatively nonexpansive mapping.*

We consider the problem of strong convergence concerning maximal monotone operators in a Banach space. Such a problem has been also studied in [28, 29, 32]. Using Theorem 3.1, we obtain the following result.

Theorem 4.3. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (A_1)-(A_4). Assume that A_1, A_2 are two continuous operators of C into E^* satisfying the conditions (1.2) and (1.3), B is a continuous and monotone operator of C into $E^*$and B_1, B_2 are two maximal monotone operators from E to E^*, $J_r^{B_1}$ and $J_r^{B_2}$ are two resolvents of B_1 and B_2 with*

$$F := B_1^{-1}0 \cap B_2^{-1}0 \cap VI(A_1, C) \cap VI(A_2, C) \cap GEP(f) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n^i = \Pi_C(Jx_n - \eta_i A_i x_n), \quad i = 1,\ 2, \\ y_n = \Pi_C(\beta_n^0 Jx_n + \beta_n^1 JJ_r^{B_1} z_n^1 + \beta_n^2 JJ_r^{B_2} z_n^2), \\ u_n \in C \text{ such that } f(u_n, y) + \langle Bu_n, y - u_n\rangle \\ \qquad\qquad\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \bigcap_{i=1,2} \phi(z, u_n) \leq \phi(z, y_n) \\ \qquad\qquad\qquad \leq (1 - \beta_n^i)\phi(z, x_n) + \beta_n^i \phi(z, z_n^i) \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where $\{\beta_n^0\}$, $\{\beta_n^1\}$ and $\{\beta_n^2\}$ are the sequences in $[0, 1]$ with the following restrictions:

(a) $\beta_n^0 + \beta_n^1 + \beta_n^2 = 1$;
(b) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;
(c) $\liminf_{n\to\infty} \beta_n^0\beta_n^1 > 0$ *and* $\liminf_{n\to\infty} \beta_n^0\beta_n^2 > 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

Proof. From Lemma 4.2, we know that $J_r^{B_1}$ and $J_r^{B_1}$ are two closed hemi-relatively nonexpansive mappings. Furthermore, applying Theorem 3.1, we can obtain that the sequence $\{x_n\}$ converges strongly to a point $\Pi_F Jx_0$. □

If $\beta_n^2 = 0$ in (4.1), then the iteration scheme (4.1) is reduced to the new modified Mann iteration for zero of maximal monotone operator B_1, the variational inequality (1.1) and the generalized equilibrium problem (1.9) as follows.

Corollary 4.4. *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty closed convex subset of* E*. Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying the conditions* (A_1)-(A_4)*. Assume that* A *is a continuous operator of* C *into* E^* *satisfying the conditions* (1.2) *and* (1.3)*,* B *is a continuous and monotone operator of* C *into* E^* *and* B_1 *is a maximal monotone operator from* E *to* E^**,* $J_r^{B_1}$ *is a resolvent of* B_1 *with*

$$F := B_1^{-1}0 \cap VI(A, C) \cap GEP(f, B) \neq \emptyset.$$

Let $\{x_n\}$ *be a sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(Jx_n - \eta Ax_n), \\ y_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JJ_r^{B_1}z_n), \\ u_n \in C \text{ such that } f(u_n, y) + \langle Bu_n, y - u_n\rangle \\ \qquad\qquad\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, y_n) \\ \qquad\qquad\qquad \leq \alpha_n\phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall\, n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *with the following restrictions:*

(a) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(b) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

Considering $B = 0$ in Corollary 4.4, we can directly obtain the following corollary by applying Corollary 4.4.

Corollary 4.5. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A_1)-(A_4)*. Assume that A is a continuous operator of C into E^* satisfying the conditions* (1.2) *and* (1.3)*, B is a maximal monotone operator from E to E^*, J_r^B is a resolvents of B with*

$$F := B^{-1}0 \cap VI(A, C) \cap EP(f) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(Jx_n - \eta Ax_n), \\ y_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JJ_r^B z_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall\, y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, y_n) \\ \qquad\qquad \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall\, n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ *with the following restrictions:*

(a) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;

(b) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_F Jx_0$, where Π_F is the generalized projection from C onto F.

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