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A STRONG FRACTIONAL CALCULUS THEORY FOR BANACH SPACE VALUED FUNCTIONS

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Abstract. We develop here a strong left fractional calculus theory for Banach space valued functions of Caputo type. Then we establish many Bochner integral inequalities of various types.

1. Introduction

Here we use extensively the Bochner integral for Banach space valued functions, which is a direct generalization of Lebesgue integral to this case. The reader may read about Bochner integral and its properties from [2], [5], [6], [8], [9], [11], [12] and [13].

Using Bochner integral properties and the great article [13], we develop a left Caputo type strong fractional theory for the first time in the literature, which is the direct analog of the real one, but now dealing with Banach space valued functions.

In the literature there are very few articles about the weak fractional theory of Banach space valued functions with the best one [1].

However we found the weak theory, using Pettis integral and functionals, complicated, less clear, dificult and unnecessary.

With this article we try to simplify matters and put the related theory on its natural grounds and resemble theory on real numbers.

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We define the Riemann-Liouville fractional Bochner integral operator, see Definition 2.2, and we prove the commutative semigroup property, see Theorem 2.7.

We use the general Fundamental theorem of calculus for Bochner integration, see Theorem 2.9 here, from [13].

Based on the last we produce a related general Taylor's formula for Banach valued functions.

We introduce then the left Caputo type fractional derivative in our setting, see Definition 2.13. Then we are able to produce the related left fractional Taylor's formula in Banach space setting, which involves the Hausdorff measure. With this developed machinery we derive left fractional: Ostrowski type inequalities, Landau type inequalities, Poincaré and Sobolev types, we finish with the Opial type and Hilbert-Pachpatte type.

All these fractional inequalities for Banach space valued functions and using always the Hausdorff measure. We cover these inequalities to all possible directions, acting at the introductory basic level, which leaves room for expansions later.

2. Main results

Definition 2.1. Let $U \subseteq \mathbb{R}$ be an interval, and X be a Banach space, we denote by $L_1(U, X)$ the Bochner integrable functions from U into X.

Definition 2.2. Let $n \in \mathbb{R}_+$, and $[a,b] \subset \mathbb{R}$, X a Banach space, and $f \in L_1([a,b], X)$. The Bochner integral operator

$$\left(J_a^n f\right)(x) := \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)\, dt, \tag{1}$$

for $a \leq x \leq b$, is called the Riemann-Liouville fractional Bochner integral operator of order n, where Γ is the gamma function. For $n = 0$, we set $J_a^0 := I$, the identity operator.

Theorem 2.3. *Let $f \in L_1([a,b], X)$ and $n > 0$. Then, the Bochner integral $\left(J_a^n f\right)(x)$ exists for almost every $x \in [a,b]$. Furthermore, $J_a^n f \in L_1([a,b], X)$.*

Proof. We can write (see, [7, p.14])

$$\int_a^x (x-t)^{n-1} f(t)\, dt = \int_{\mathbb{R}} \Phi_1(x-t)\, \Phi_2(t)\, dt = (\Phi_1 * \Phi_2)(x), \tag{2}$$

which is a convolution, where

$$\Phi_1(u) = \begin{cases} u^{n-1}, & 0 < u \leq b-a, \\ 0, & \text{elsewhere}, \end{cases} \tag{3}$$

and

$$\Phi_2(u) = \begin{cases} f(u), & a \le u \le b, \\ 0, & \text{elsewhere.} \end{cases} \tag{4}$$

Notice that $\Phi_1 \in L_1(\mathbb{R})$ and $\Phi_2 \in L_1(\mathbb{R}, X)$, *i.e.*, both functions are Bochner integrable. Hence the convolution $\Phi_1 * \Phi_2$ exists, and it is Bochner integrable, according to [11, Theorem 5.1, p.194]. Equivalently, $\|\Phi_1 * \Phi_2\|$ is Lebesgue integrable (see [6]), hence finite almost everywhere. Thus $(J_a^n f) \in L_1([a,b], X)$ and exists almost everywhere on $[a,b]$. The claim is proved. ☐

Remark 2.4. Let $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ a Banach space. Let also $f : [a,b] \to X$. If f is continuous, *i.e.*, $f \in C([a,b], X)$, then f is strongly measurable, by [9, Corollary 2.3, p.5].

Furthermore $f([a,b]) \subseteq X$ is compact, thus it is closed and bounded, hence f is bounded, i.e. $\|f(t)\| \le M$, $\forall\, t \in [a,b]$, $M > 0$.

Let $x_n, x \in [a,b] : x_n \to x$, as $n \to \infty$, then $f(x_n) \to f(x)$ in $\|\cdot\|$, that is $|\|f(x_n)\| - \|f(x)\|| \le \|f(x_n) - f(x)\| \to 0$, proving $\|f\|$ is continuous, hence bounded, so that $\|f\|_{L_\infty([a,b],X)} := es\sup_{t\in[a,b]} \|f(t)\| < +\infty$, that is $f \in L_\infty([a,b], X)$, and hence $f \in L_1([a,b], X)$. Consequently, f is Bochner integrable ([2, p.426]), given that f is continuous.

For the last we used the fact:

$$\int_{[a,b]} \|f(t)\|\, dt \le \|f\|_{L_\infty([a,b],X)} (b-a) < +\infty,$$

proving that $f \in L_1([a,b], X)$.

Also, clearly, absolute continuity of $f : [a,b] \to X$, implies uniform continuity and continuity of f.

We denote by $AC([a,b], X)$ the space of absolutely continuous functions from $[a,b]$ into X.

We present the following useful result.

Theorem 2.5. *Here $[a,b] \subset \mathbb{R}$, X is a Banach space, $F : [a,b] \to X$. Let $r > 0$ and $F \in L_\infty([a,b], X)$, and the Bochner integral*

$$G(s) := \int_a^s (s-t)^{r-1} F(t)\, dt, \tag{5}$$

all $s \in [a,b]$. Then $G \in AC([a,b], X)$ for $r \ge 1$ and $G \in C([a,b], X)$ for $r \in (0,1)$.

Proof. Denote by $\|F\|_\infty := \|F\|_{L_\infty([a,b],X)} := es\sup_{t\in[a,b]} \|F(t)\|_X < +\infty$. Hence $F \in L_1([a,b], X)$. By [11, Theorem 5.4, p.101], $(s-t)^{r-1} F(t)$ is a strongly measurable function in t, $t \in [a,s]$, $s \in [a,b]$. So that $(s-t)^{r-1} F(t) \in L_1([a,s], X)$, see [6].

(1) Case $r \geq 1$.

We use the definition of absolute continuity. So for every $\varepsilon > 0$ we need $\delta > 0$: whenever (a_i, b_i), $i = 1, \cdots, n$, are disjoint open subintervals of $[a,b]$, then

$$\sum_{i=1}^{n} (b_i - a_i) < \delta \quad \Rightarrow \quad \sum_{i=1}^{n} \| G(b_i) - G(a_i) \| < \varepsilon. \tag{6}$$

If $\|F\|_\infty = 0$, then $G(s) = 0$, for all $s \in [a,b]$, the trivial case and all fulfilled. So we assume $\|F\|_\infty \neq 0$. Hence we have (see, [5])

$$\begin{aligned} & G(b_i) - G(a_i) \\ &= \int_a^{b_i} (b_i - t)^{r-1} F(t)\,dt - \int_a^{a_i} (a_i - t)^{r-1} F(t)\,dt \\ &= \int_a^{a_i} (b_i - t)^{r-1} F(t)\,dt - \int_a^{a_i} (a_i - t)^{r-1} F(t)\,dt \\ &\quad + \int_{a_i}^{b_i} (b_i - t)^{r-1} F(t)\,dt \quad \text{(see [2, Theorem 11.43, p.426])} \\ &= \int_a^{a_i} \left((b_i - t)^{r-1} - (a_i - t)^{r-1} \right) F(t)\,dt + \int_{a_i}^{b_i} (b_i - t)^{r-1} F(t)\,dt. \end{aligned} \tag{7}$$

Call

$$I_i := \int_a^{a_i} \left| (b_i - t)^{r-1} - (a_i - t)^{r-1} \right| dt. \tag{8}$$

Thus

$$\| G(b_i) - G(a_i) \| \leq \left[I_i + \frac{(b_i - a_i)^r}{r} \right] \|F\|_\infty := T_i. \tag{9}$$

If $r = 1$, then $I_i = 0$ and

$$\| G(b_i) - G(a_i) \| \leq \|F\|_\infty (b_i - a_i), \tag{10}$$

for all $i := 1, \cdots, n$.

If $r > 1$, then since $\left[(b_i - t)^{r-1} - (a_i - t)^{r-1} \right] \geq 0$ for all $t \in [a, a_i]$, we find

$$\begin{aligned} I_i &= \int_a^{a_i} \left((b_i - t)^{r-1} - (a_i - t)^{r-1} \right) dt \\ &= \frac{(b_i - a)^r - (a_i - a)^r - (b_i - a_i)^r}{r} \\ &= \frac{r(\xi - a)^{r-1}(b_i - a_i) - (b_i - a_i)^r}{r}, \quad \text{for some } \xi \in (a_i, b_i). \end{aligned} \tag{11}$$

Therefore, it holds

$$I_i \leq \frac{r\,(b-a)^{r-1}\,(b_i-a_i)-(b_i-a_i)^r}{r} \tag{12}$$

and

$$\left(I_i+\frac{(b_i-a_i)^r}{r}\right) \leq (b-a)^{r-1}\,(b_i-a_i)\,. \tag{13}$$

That is

$$T_i \leq \|F\|_\infty\,(b-a)^{r-1}\,(b_i-a_i)\,, \tag{14}$$

so that

$$\|G(b_i)-G(a_i)\| \leq \|F\|_\infty\,(b-a)^{r-1}\,(b_i-a_i)\,,\ \text{ for all } i=1,\cdots,n. \tag{15}$$

So in the case of $r=1$, and by choosing $\delta := \frac{\varepsilon}{\|F\|_\infty}$, we get

$$\sum_{i=1}^{n}\|G(b_i)-G(a_i)\| \overset{(10)}{\leq} \|F\|_\infty\left(\sum_{i=1}^{n}(b_i-a_i)\right) \leq \|F\|_\infty\,\delta=\varepsilon, \tag{16}$$

proving for $r=1$ that G is absolutely continuous. In the case of $r>1$, and by choosing $\delta := \frac{\varepsilon}{\|F\|_\infty (b-a)^{r-1}}$, we get

$$\begin{aligned}\sum_{i=1}^{n}\|G(b_i)-G(a_i)\| &\overset{(15)}{\leq} \|F\|_\infty\,(b-a)^r\left(\sum_{i=1}^{n}(b_i-a_i)\right)\\ &\leq \|F\|_\infty\,(b-a)^{r-1}\,\delta=\varepsilon,\end{aligned} \tag{17}$$

proving for $r>1$ that G is absolutely continuous again.

(2) Case of $0<r<1$.

Let $a_{i*}, b_{i*} \in [a,b] : a_{i*} \leq b_{i*}$. Then $(a_{i*}-t)^{r-1} \geq (b_{i*}-t)^{r-1}$, for all $t \in [a, a_{i*}]$. Then

$$\begin{aligned}I_{i*} &= \int_a^{a_{i*}}\left((a_{i*}-t)^{r-1}-(b_{i*}-t)^{r-1}\right)dt\\ &= \frac{(b_{i*}-a_{i*})^r}{r}+\left(\frac{(a_{i*}-a)^r-(b_{i*}-a)^r}{r}\right) \leq \frac{(b_{i*}-a_{i*})^r}{r},\end{aligned} \tag{18}$$

by $(a_{i*}-a)^r-(b_{i*}-a)^r<0$. Therefore

$$I_{i*} \leq \frac{(b_{i*}-a_{i*})^r}{r} \tag{19}$$

and

$$T_{i*} \leq \frac{2\,(b_{i*}-a_{i*})^r}{r}\,\|F\|_\infty\,, \tag{20}$$

proving that

$$\|G(b_{i*}) - G(a_{i*})\| \leq \left(\frac{2\|F\|_{\infty}}{r}\right)(b_{i*} - a_{i*})^{r}, \tag{21}$$

which is proving that G is continuous. The theorem is proved. □

Theorem 2.6. *Let $m, n \in \mathbb{R}_+$ and $f \in L_1([a,b], X)$. Then*

$$J_a^m J_a^n f = J_a^{m+n} f = J_a^n J_a^m f, \tag{22}$$

holds almost everywhere on $[a,b]$.

If $f \in C([a,b], X)$ or $m+n \geq 1$, then identity in (22) is valid everywhere on $[a,b]$.

Proof. If $m = 0$, or $n = 0$, or $m = n = 0$, (22) is trivial. We assume $m, n > 0$. See also [7, pp.14-15], for the case $X = \mathbb{R}$. Here $a \leq x \leq b$. We have

$$\begin{aligned} &(J_a^m J_a^n f)(x) \\ &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x (x-t)^{m-1} \int_a^t (t-\tau)^{n-1} f(\tau)\, d\tau dt \\ &= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \int_a^x \chi_{[a,t]}(\tau)(x-t)^{m-1}(t-\tau)^{n-1} f(\tau)\, d\tau dt =: (\xi), \end{aligned} \tag{23}$$

where $\chi_{[a,t]}$ is the indicator function. Hence by Fubini theorem (see, [11, Theorem 2, p.93]), we obtain that

$$(\xi) = \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \int_a^x \chi_{[\tau,x]}(t)(x-t)^{m-1}(t-\tau)^{n-1} f(\tau)\, d\tau dt \tag{24}$$

$$= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x f(\tau)\left(\int_\tau^x (x-t)^{m-1}(t-\tau)^{n-1} dt\right) d\tau$$

$$= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x f(\tau)\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}(x-\tau)^{m+n-1} d\tau \tag{25}$$

$$= \frac{1}{\Gamma(m+n)} \int_a^x f(\tau)(x-\tau)^{m+n-1} d\tau = \left(J_a^{m+n} f\right)(x), \tag{26}$$

almost everywhere on $[a,b]$.

We have proved that

$$J_a^m J_a^n f = J_a^{m+n} f, \tag{27}$$

almost everywhere on $[a,b]$. Hence (22) is valid.

By Remark 2.4 if f is continuous, $f \in C([a,b], X)$, then $f \in L_\infty([a,b], X)$, and by Theorem 2.5 we get that $J_a^n f \in C([a,b], X)$, and therefore $J_a^m J_a^n f \in C([a,b], X)$, and $J_a^{m+n} f \in C([a,b], X)$ too.

Because these two continuous functions coincide almost everywhere, see (27), the must be equal everywhere on $[a,b]$.

At the end, if $f \in L_1([a,b],X)$ and $m+n \geq 1$, we have by (27) that

$$J_a^m J_a^n f = J_a^{m+n} f = J_a^{m+n-1} J_a^1 f, \tag{28}$$

almost everywhere. Here we have that $\|f\| \in L_1([a,b])$ and $\int_a^x \|f(t)\| \, dt$ is continuous in $x \in [a,b]$.

Let now $x_n \to x$, x_n, $x \in [a,b]$, $n \to \infty$. Then,

i) case $x_n \geq x$: we have

$$\begin{aligned}
&\left\| \left(J_a^1 f\right)(x_n) - \left(J_a^1 f\right)(x) \right\| = \left\| \int_a^{x_n} f(t)\, dt - \int_a^x f(t)\, dt \right\| \\
&= \left\| \int_a^x f(t)\, dt - \int_a^x f(t)\, dt + \int_x^{x_n} f(t)\, dt \right\| = \left\| \int_x^{x_n} f(t)\, dt \right\| \\
&\leq \int_x^{x_n} \|f(t)\|\, dt = \left(\int_a^{x_n} \|f(t)\|\, dt - \int_a^x \|f(t)\|\, dt \right) \to 0.
\end{aligned}$$

ii) case $x_n \leq x$: we have

$$\begin{aligned}
&\left\| \left(J_a^1 f\right)(x_n) - \left(J_a^1 f\right)(x) \right\| = \left\| \int_a^{x_n} f(t)\, dt - \int_a^x f(t)\, dt \right\| \\
&= \left\| \int_a^{x_n} f(t)\, dt - \int_a^{x_n} f(t)\, dt - \int_{x_n}^x f(t)\, dt \right\| = \left\| \int_{x_n}^x f(t)\, dt \right\| \\
&\leq \int_{x_n}^x \|f(t)\|\, dt = \left(\int_a^x \|f(t)\|\, dt - \int_a^{x_n} \|f(t)\|\, dt \right) \to 0.
\end{aligned} \tag{29}$$

We have proved that $J_a^1 f \in C([a,b],X)$. Therefore by Theorem 2.5 we also have that $J_a^{m+n} f = J_a^{m+n-1}\left(J_a^1 f\right)$ is continuous.

Now looking at (28), all three sides of the equality almost everywhere are continuous (by Tonelli theorem, [11, p.100]), hence they must be identical over $[a,b]$.

For the last statement, also we prove directly that $J_a^{m+n} f$ is continuous, we set $\alpha := m+n \geq 1$.

Let $x_n \to x$, as $n \to \infty$, where $x_n, x \in [a,b]$. We notice that

$$\begin{aligned}
\left(J_a^\alpha f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt \\
&= \frac{1}{\Gamma(\alpha)} \int_a^b \chi_{[a,x]}(t)\,(x-t)^{\alpha-1} f(t)\, dt
\end{aligned} \tag{30}$$

and

$$\left(J_a^\alpha f\right)(x_n) = \frac{1}{\Gamma(\alpha)} \int_a^b \chi_{[a,x_n]}(t)\,(x_n-t)^{\alpha-1} f(t)\, dt. \tag{31}$$

Clearly, here the function $\chi_{[a,x_n]}(t)(x_n-t)^{\alpha-1}f(t)$, $\chi_{[a,x]}(t)(x-t)^{\alpha-1}f(t)$ are Bochner integrable, and

$$\chi_{[a,x_n]}(t)(x_n-t)^{\alpha-1}f(t) \quad \to \quad \chi_{[a,x]}(t)(x-t)^{\alpha-1}f(t),$$

pointwise almost everywhere in $\|\cdot\|$, as $n\to\infty$.

Furthermore, we notice that

$$\begin{aligned}\left\|\chi_{[a,x_n]}(t)(x_n-t)^{\alpha-1}f(t)\right\| &\leq |x_n-t|^{\alpha-1}\|f(t)\| \\ &\leq (b-a)^{\alpha-1}\|f(t)\|\in L_1([a,b]),\end{aligned} \tag{32}$$

for all $n\in\mathbb{N}$. Thus, by Dominated convergence theorem for Bochner integrals, see [5], we get that

$$\begin{aligned}&\lim_{n\to\infty}\int_a^b \chi_{[a,x_n]}(t)(x_n-t)^{\alpha-1}f(t)\,dt \\ &= \int_a^b \chi_{[a,x]}(t)(x-t)^{\alpha-1}f(t)\,dt \\ &= \int_a^x (x-t)^{\alpha-1}f(t)\,dt, \quad \text{in norm } \|\cdot\|.\end{aligned} \tag{33}$$

Hence $(J_a^\alpha f)(x_n)\to(J_a^\alpha f)(x)$, as $n\to\infty$, in $\|\cdot\|$. That is proving continuity of $J_a^{m+n}f$, when $f\in L_1([a,b],X)$ and $m+n\geq 1$. The theorem is completely proved in detail. ☐

The algebraic version of previous Theorem follows:

Theorem 2.7. *The Bochner integral operators*

$$\{J_a^n : L_1([a,b],X)\to L_1([a,b],X);\, n\in\mathbb{R}_+\}$$

make a commutative semigroup with respect to composition. The identity operator $J_a^0=I$ is the neutral element of this semigroup.

Definition 2.8. ([13]) A definition of the Hausdorff measure h_α goes as follows: if (T,d) is a metric space, $A\subseteq T$ and $\delta>0$, let $\Lambda(A,\delta)$ be the set of all arbitrary collections $(C)_i$ of subsets of T, such that $A\subseteq\cup_i C_i$ and $diam(C_i)\leq\delta$ ($diam$ =diameter) for every i. Now, for every $\alpha>0$ define

$$h_\alpha^\delta(A):=\inf\left\{\sum(diamC_i)^\alpha \mid (C_i)_i\in\Lambda(A,\delta)\right\}. \tag{34}$$

Then there exists $\lim\limits_{\delta\to0}h_\alpha^\delta(A)=\sup\limits_{\delta>0}h_\alpha^\delta(A)$, and $h_\alpha(A):=\lim\limits_{\delta\to0}h_\alpha^\delta(A)$ gives an outer measure on the power set $\mathcal{P}(T)$, which is countably additive on the σ-field of all Borel subsets of T. If $T=\mathbb{R}^n$, then the Hausdorff measure h_n, restricted to the σ-field of the Borel subsets of $\mathbb{R}^n$, equals the Lebesgue measure on $\mathbb{R}^n$ up to a constant multiple. In particular, $h_1(C) = \mu(C)$ for every Borel set $C \subseteq \mathbb{R}$, where μ is the Lebesgue measure.

We will use the following spectacular result.

Theorem 2.9. ([13], Fundamental Theorem of Calculus for Bochner integration) *Suppose that for the given function $f : [a, b] \to X$, there exists $F : [a, b] \to X$, which is continuous, the derivative $F'(t)$ exists and $F'(t) = f(t)$ outside a μ-null Borel set $B \subseteq [a, b]$ such that $h_1(F(B)) = 0$. Then f is μ-measurable (i.e., strongly measurable), and if we assume the Bochner integrability of f,*

$$F(b) - F(a) = \int_a^b f(t)\, dt. \tag{35}$$

Notice here that the derivatives of a function $f : [a, b] \to X$, where X is a Banach space, are defined exactly as the numerical ones, see for definitions and properties, [12, pp.83-86 and p.93], that is they are strong derivatives.

We will use the last theorem to give a general Taylor's formula for Banach space valued functions with a Bochner integral remainder.

Theorem 2.10. *Let $n \in \mathbb{N}$ and $f \in C^{n-1}([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and X is a Banach space. Set*

$$F(x) := \sum_{i=0}^{n-1} \frac{(b-x)^i}{i!} f^{(i)}(x), \quad x \in [a, b]. \tag{36}$$

Assume that $f^{(n)}$ exists outside a μ-null Borel set $B \subseteq [a, b]$ such that

$$h_1(F(B)) = 0. \tag{37}$$

We further assume the Bochner integrability of $f^{(n)}$. Then

$$f(b) = f(a) + \sum_{i=1}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) + \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x)\, dx. \tag{38}$$

Proof. We get that $F \in C([a, b], X)$. Notice that $F(b) = f(b)$, and

$$F(a) = \sum_{i=0}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a).$$

Clearly F' exists outside B. Infact it holds

$$F'(x) = \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x), \quad \forall x \in [a, b] - B. \tag{39}$$

Also F' is Bochner integrable. By Theorem 2.9 now we get that

$$F(b) - F(a) = \int_a^b F'(t)\, dt. \tag{40}$$

That is, we have

$$f(b) - \left(\sum_{i=0}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) \right) = \int_a^b \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x)\, dx, \tag{41}$$

proving the claim. □

We mention the following classical result of a Taylor's formula:

Theorem 2.11. ([12, pp.93-94]) *Let $n \in \mathbb{N}$ and $f \in C^n([a,b], X)$, $[a,b] \subset \mathbb{R}$, X a Banach space. Then*

$$f(b) = f(a) + \sum_{i=1}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) + \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x)\, dx. \tag{42}$$

The remainder here is the Riemann X-valued integral (defined similar to numerical one) given by

$$Q_{n-1} = \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x)\, dx, \tag{43}$$

with the property:

$$\|Q_{n-1}\| \leq \max_{a \leq x \leq b} \left\| f^{(n)}(x) \right\| \frac{(b-a)^n}{(n-1)!}. \tag{44}$$

Important Note 2.12. *By* [8]*, a Riemann integrable X-valued function is not necessarily a Bochner integrable one.*

Definition 2.13. Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\nu > 0$; $n := \lceil \nu \rceil \in \mathbb{N}$, $\lceil \cdot \rceil$ is the ceiling of the number, $f : [a,b] \to X$. We assume that $f^{(n)} \in L_1([a,b], X)$. We call the Caputo-Bochner left fractional derivative of order ν:

$$\left(D_{*a}^{\nu} f\right)(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \quad \forall\, x \in [a,b]. \tag{45}$$

If $\nu \in \mathbb{N}$, we set $D_{*a}^{\nu} f := f^{(\nu)}$ the ordinary X-valued derivative, and also set $D_{*a}^{0} f := f$.

By Theorem 2.3, $\left(D_{*a}^{\nu} f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\nu} f \in L_1([a,b], X)$.

We notice that

$$D_{*a}^{\nu} f = J_a^{n-\nu} f^{(n)}. \tag{46}$$

If $\left\| f^{(n)} \right\|_{L_\infty([a,b],X)} < \infty$, then by Theorem 2.5 $D_{*a}^{\nu} f \in C([a,b], X)$.

Remark 2.14. (to Definition 2.13) We notice that (by Theorem 2.6)

$$\begin{aligned} \left(J_a^{\nu} D_{*a}^{\nu} f\right)(x) &= \left(J_a^{\nu} J_a^{n-\nu} f^{(n)}\right)(x) \\ &= \left(J_a^{\nu+n-\nu} f^{(n)}\right)(x) = \left(J_a^{n} f^{(n)}\right)(x), \end{aligned} \tag{47}$$

almost everywhere.

We also notice that

$$\left(J_a^{n} f^{(n)}\right)(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f^{(n)}(t)\, dt, \tag{48}$$

and given that $f^{(n)} \in L_1([a,b], X)$, we get again that $\left(J_a^n f^{(n)}\right) \in L_1([a,b], X)$, and it exists a.e. on $[a,b]$, by Theorem 2.3. We have proved that

$$\left(J_a^{\nu} D_{*a}^{\nu} f\right)(x) = \left(J_a^{n} f^{(n)}\right)(x), \tag{49}$$

for almost all $x \in [a,b]$.

We present the following left fractional Taylor's formula.

Theorem 2.15. *Let $n \in \mathbb{N}$ and $f \in C^{n-1}([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space, and let $\nu \geq 0 : n = \lceil \nu \rceil$. Set*

$$F_x(t) := \sum_{i=0}^{n-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [a,x], \tag{50}$$

where $x \in [a,b]$. Assume that $f^{(n)}$ exists outside a μ-null Borel set $B_x \subseteq [a,x]$, such that

$$h_1(F_x(B_x)) = 0, \tag{51}$$

where $x \in [a,b]$. We also assume that $f^{(n)} \in L_1([a,b], X)$. Then

$$f(x) = \sum_{i=0}^{n-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \left(D_{*a}^{\nu} f\right)(z)\, dz, \tag{52}$$

for $x \in [a,b]$.

Proof. We use Theorem 2.9 and (35). Clearly it holds

$$\left(f(\cdot) - \sum_{i=0}^{n-1} \frac{(\cdot - a)^i}{i!} f^{(i)}(a)\right) \in C([a,b], X),$$

that is (by (38)) $\left(J_a^n f^{(n)}\right) \in C([a,b], X)$. Hence (49) holds as equality over $[a,b]$ (by Tonelli's theorem), therefore $\left(J_a^{\nu} D_{*a}^{\nu} f\right) \in C([a,b], X)$. Thus (52) is valid. □

More generally we get

Theorem 2.16. *Let $n \in \mathbb{N}$ and $f \in C^{n-1}([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space, and let $\nu \geq 0 : n = \lceil \nu \rceil$. Set*

$$F_x(t) := \sum_{i=0}^{n-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [a,x], \tag{53}$$

where $x \in [a,b]$.

Assume that $f^{(n)}$ exists outside a μ-null Borel set $B_x \subseteq [a,x]$, such that

$$h_1(F_x(B_x)) = 0, \ \ \forall\, x \in [a,b]. \tag{54}$$

We also assume that $f^{(n)} \in L_1([a,b], X)$. Then

$$f(x) = \sum_{i=0}^{n-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} (D_{*a}^{\nu} f)(z)\, dz, \tag{55}$$

for all $x \in [a,b]$.

Proof. By Theorem 2.15. □

Remark 2.17. (to Theorem 2.16) By (55), we notice that

$$(J_a^{\nu} D_{*a}^{\nu} f)(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} (D_{*a}^{\nu} f)(z)\, dz \in C([a,b], X)$$

as a function of $x \in [a,b]$.

We have also

Corollary 2.18. (to Theorem 2.16) *All as in Theorem 2.16. Additionally we assume that $f^{(i)}(a) = 0$, $i = 0, 1, \cdots, n-1$. Then*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} (D_{*a}^{\nu} f)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{56}$$

Next we present an Ostrowski type inequality at left fractional level for Banach valued functions.

Theorem 2.19. *Let $\nu \geq 0$, $n = \lceil \nu \rceil$. Here all as in Theorem 2.16. Assume that $f^{(i)}(a) = 0$, $i = 1, \cdots, n-1$, and that $D_{*a}^{\nu} f \in L_{\infty}([a,b], X)$. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\| D_{*a}^{\nu} f \|_{L_{\infty}([a,b],X)}}{\Gamma(\nu+2)} (b-a)^{\nu}. \tag{57}$$

Proof. By Theorem 2.16 when $f^{(i)}(a) = 0$, $i = 1, \cdots, n-1$, we get that

$$f(x) - f(a) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} (D_{*a}^{\nu} f)(z)\, dz, \ \ \forall\, x \in [a,b]. \tag{58}$$

Thus

$$\begin{aligned}
&\|f(x)-f(a)\| \\
&=\frac{1}{\Gamma(\nu)}\left\|\int_a^x (x-z)^{\nu-1}\left(D_{*a}^{\nu}f\right)(z)\,dz\right\| \quad \text{(by [9, Theorem 2.5, p.7])} \\
&\leq \frac{1}{\Gamma(\nu)}\int_a^x (x-z)^{\nu-1}\left\|D_{*a}^{\nu}f(z)\right\|dz \\
&\leq \frac{\left\|D_{*a}^{\nu}f\right\|_{L_\infty([a,b],X)}}{\Gamma(\nu+1)}(x-a)^{\nu}.
\end{aligned} \tag{59}$$

We got that

$$\|f(x)-f(a)\| \leq \frac{\left\|D_{*a}^{\nu}f\right\|_{L_\infty([a,b],X)}}{\Gamma(\nu+1)}(x-a)^{\nu}, \quad \forall\, x\in[a,b]. \tag{60}$$

Therefore,

$$\begin{aligned}
&\left\|\frac{1}{b-a}\int_a^b f(x)\,dx-f(a)\right\| = \left\|\frac{1}{b-a}\int_a^b (f(x)-f(a))\,dx\right\| \\
&\leq \frac{1}{b-a}\int_a^b \|f(x)-f(a)\|\,dx \leq \frac{\left\|D_{*a}^{\nu}f\right\|_{L_\infty([a,b],X)}}{(b-a)\,\Gamma(\nu+1)}\int_a^b (x-a)^{\nu}\,dx \\
&= \frac{\left\|D_{*a}^{\nu}f\right\|_{L_\infty([a,b],X)}}{(b-a)\,\Gamma(\nu+1)}\frac{(b-a)^{\nu+1}}{(\nu+1)} = \frac{\left\|D_{*a}^{\nu}f\right\|_{L_\infty([a,b],X)}}{\Gamma(\nu+2)}(b-a)^{\nu}.
\end{aligned} \tag{61}$$

This proves (57). □

We give next the optimality of (57).

Theorem 2.20. *Inequality* (57) *is sharp; namely it is attained by*

$$f(x)=(x-a)^{\nu}\,\overrightarrow{i}, \quad \nu>0,\ \nu\notin\mathbb{N},\ x\in[a,b], \tag{62}$$

$\overrightarrow{i}\in X$ *such that* $\left\|\overrightarrow{i}\right\|=1$.

Proof. (see also [3, pp.621-622]) We observe that

$$\begin{aligned}
f'(x)&=\nu(x-a)^{\nu-1}\,\overrightarrow{i},\quad f''(x)=\nu(\nu-1)(x-a)^{\nu-2}\,\overrightarrow{i},\cdots, \\
f^{(n-1)}(x)&=\nu(\nu-1)(\nu-2)\cdots(\nu-n+2)(x-a)^{\nu-n+1}\,\overrightarrow{i}, \\
f^{(n)}(x)&=\nu(\nu-1)(\nu-2)\cdots(\nu-n+1)(x-a)^{\nu-n}\,\overrightarrow{i}.
\end{aligned} \tag{63}$$

Clearly here $f^{(n)}$ is continuous on $(a,b]$, and $f^{(n)}\in L_1([a,b],X)$.

All assumption of Theorem 2.19 are easily fulfilled.

As in [3, p.622], we get that (see also (45))

$$D_{*a}^{\nu}f(x)=\Gamma(\nu+1)\,\overrightarrow{i}, \quad \forall\, x\in[a,b].$$

Also we see that

$$f^{(k)}(a) = 0, \quad k = 0, 1, \cdots, n-1, \quad \text{and} \quad D_{*a}^{\nu} f \in L_{\infty}([a,b], X). \tag{64}$$

So f fulfills all the assumptions of Theorem 2.19.

Next we find:

$$\begin{aligned} \text{Left hand side of (57)} &= \frac{1}{b-a} \left\| \int_a^b f(x)\, dx \right\| \\ &= \frac{1}{b-a} \int_a^b (x-a)^{\nu}\, dx = \frac{(b-a)^{\nu}}{(\nu+1)}. \end{aligned} \tag{65}$$

And we have

$$\text{Right hand side of (57)} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} (b-a)^{\nu} = \frac{(b-a)^{\nu}}{(\nu+1)}, \tag{66}$$

proving both sides of (57) equal. This completes the proof of the claim. □

When $0 < \nu \leq 1$, Definition 2.13 becomes

Definition 2.21. Let $[A,B] \subset \mathbb{R}$, X be a Banach space, $0 < \nu \leq 1$, $f : [A,B] \to X$. We assume that $f' \in L_1([A,B], X)$.

We define the Caputo-Bochner left fractional derivative of order ν:

$$\left(D_{*A}^{\nu} f\right)(x) := \frac{1}{\Gamma(1-\nu)} \int_A^x (x-t)^{-\nu} f'(t)\, dt, \quad \forall\, x \in [A,B]. \tag{67}$$

We set $D_{*A}^{1} f = f'$, the ordinary X-valued derivative.

One may set

$$D_{*A}^{\nu} f(x) := 0, \ \text{ where } 0 \in X, \text{ if } x < A. \tag{68}$$

Remark 2.22. Let $[A,B] \subset \mathbb{R}$, X be a Banach space, $0 < \nu \leq 1$, $f : [A,B] \to X$. We assume that $f^{(2)} \in L_1([A,B], X)$. Then $\lceil \nu + 1 \rceil = 2$, and

$$\begin{aligned} D_{*A}^{\nu} f'(x) &= \frac{1}{\Gamma(1-\nu)} \int_A^x (x-t)^{-\nu} f''(t)\, dt \\ &= \frac{1}{\Gamma(2-(\nu+1))} \int_A^x (x-t)^{2-(\nu+1)-1} f''(t)\, dt \\ &= D_{*A}^{\nu+1} f(x). \end{aligned} \tag{69}$$

That is

$$D_{*A}^{\nu} f' = D_{*A}^{\nu+1} f. \tag{70}$$

We apply Theorem 2.19 when $0 < \nu \leq 1$.

Theorem 2.23. *Let $f \in C([A,B],X)$, where $[A,B] \subset \mathbb{R}$ and X is a Banach space, $0<\nu\leq 1$. Assume that f' exists outside a μ-null Borel set $B_x \subseteq [A,x]$, such that*

$$h_1(f(B_x)) = 0,\ \ \forall\, x \in [A,B]. \tag{71}$$

We also assume that $f' \in L_1([A,B],X)$, and $D_{*A}^{\nu} f \in L_\infty([A,B],X)$. Then

$$\left\| \frac{1}{B-A}\int_A^B f(x)\,dx - f(A)\right\| \leq \frac{\|D_{*A}^{\nu} f\|_{L_\infty([A,B],X)}(B-A)^{\nu}}{\Gamma(\nu+2)}. \tag{72}$$

We present the following left Caputo-Bochner fractional Landau inequality for $\|\cdot\|_\infty$.

Theorem 2.24. *Let $f \in C^1([A_0,+\infty),X)$, where $A_0 \in \mathbb{R}$ is fixed, $0<\nu\leq 1$, X is a Banach space. For any $A,B \in [A_0,+\infty) : A \leq B$, we assume that f fulfills: assume that f'' exists outside a μ-null Borel set $B_x \subseteq [A,x]$ such that*

$$h_1\left(f'(B_x)\right) = 0,\ \ \forall\, x \in [A,B]. \tag{73}$$

We also assume that $f'' \in L_1([A,B],X)$ and $D_{*A}^{\nu+1} f \in L_\infty([A,B],X)$. We further assume that

$$\left\|D_{*A}^{\nu+1} f\right\|_{L_\infty([A,+\infty),X)} \leq \left\|D_{*A_0}^{\nu+1} f\right\|_{L_\infty([A_0,+\infty),X)} < \infty,\ \ \forall\, A \geq A_0, \tag{74}$$

(*the last left inequality is obvious when $\nu=1$*), *and*

$$\|f\|_{\infty,[A_0,+\infty)} := \sup_{t\in[A_0,+\infty)} \|f(t)\| < \infty. \tag{75}$$

Then

$$\begin{aligned} &\left\|f'\right\|_{\infty,[A_0,+\infty)} := \sup_{t\in[A_0,+\infty)} \left\|f'(t)\right\| \\ &\leq (\nu+1)\left(\frac{2}{\nu}\right)^{\left(\frac{\nu}{\nu+1}\right)} (\Gamma(\nu+2))^{-\frac{1}{(\nu+1)}} \\ &\quad \times \left(\|f\|_{\infty,[A_0,+\infty)}\right)^{\frac{\nu}{(\nu+1)}} \left(\left\|D_{*A_0}^{\nu+1} f\right\|_{L_\infty([A_0,+\infty),X)}\right)^{\frac{1}{(\nu+1)}}. \end{aligned} \tag{76}$$

Proof. We have that (by Theorem 2.23)

$$\begin{aligned} &\left\| \frac{1}{B-A}\int_A^B f'(x)\,dx - f'(A)\right\| \leq \frac{\|D_{*A}^{\nu} f'\|_{L_\infty([A,B],X)}(B-A)^{\nu}}{\Gamma(\nu+2)} \\ &= \frac{\left\|D_{*A}^{\nu+1} f\right\|_{L_\infty([A,B],X)}}{\Gamma(\nu+2)}(B-A)^{\nu},\ \ \forall\, A,B \in [A_0,+\infty),\ A \leq B. \end{aligned} \tag{77}$$

Subsequently by Theorem 2.9 we derive

$$\left\| \frac{f(B) - f(A)}{B - A} - f'(A) \right\| \leq \frac{\left\| D_{*A}^{\nu+1} f \right\|_{L_\infty([A,B],X)}}{\Gamma(\nu+2)} (B - A)^\nu, \tag{78}$$

for all $A, B \in [A_0, +\infty)$, $A \leq B$. Hence it holds

$$\left\| f'(A) \right\| - \frac{1}{B - A} \left\| f(B) - f(A) \right\| \leq \frac{\left\| D_{*A}^{\nu+1} f \right\|_{L_\infty([A,B],X)}}{\Gamma(\nu+2)} (B - A)^\nu \tag{79}$$

and

$$\left\| f'(A) \right\| \leq \frac{\left\| f(B) - f(A) \right\|}{B - A} + \frac{\left\| D_{*A}^{\nu+1} f \right\|_{L_\infty([A,B],X)} (B - A)^\nu}{\Gamma(\nu+2)}, \tag{80}$$

for all $A, B \in [A_0, +\infty)$, $A \leq B$. Therefore we obtain

$$\left\| f'(A) \right\| \leq \frac{2 \left\| f \right\|_{\infty,[A_0,+\infty)}}{B - A} + \frac{\left\| D_{*A_0}^{\nu+1} f \right\|_{L_\infty([A_0,+\infty),X)} (B - A)^\nu}{\Gamma(\nu+2)}, \tag{81}$$

for all $A, B \in [A_0, +\infty)$, $A \leq B$.

The right hand side of (81) depends only on $B - A$. Consequently, it holds

$$\left\| f' \right\|_{\infty,[A_0,+\infty)} \leq \frac{2 \left\| f \right\|_{\infty,[A_0,+\infty)}}{B - A} + \frac{\left\| D_{*A_0}^{\nu+1} f \right\|_{L_\infty([A_0,+\infty),X)} (B - A)^\nu}{\Gamma(\nu+2)}. \tag{82}$$

We may call $t = B - A > 0$. Thus by (82),

$$\left\| f' \right\|_{\infty,[A_0,+\infty)} \leq \frac{2 \left\| f \right\|_{\infty,[A_0,+\infty)}}{t} + \frac{\left\| D_{*A_0}^{\nu+1} f \right\|_{L_\infty([A_0,+\infty),X)}}{\Gamma(\nu+2)} t^\nu, \quad \forall\, t > 0. \tag{83}$$

Set

$$\mu := 2 \left\| f \right\|_{\infty,[A_0,+\infty)},$$
$$\theta := \frac{\left\| D_{*A_0}^{\nu+1} f \right\|_{L_\infty([A_0,+\infty),X)}}{\Gamma(\nu+2)}, \tag{84}$$

both are greater than 0.

We consider the function

$$y(t) = \frac{\mu}{t} + \theta t^\nu, \quad 0 < \nu \leq 1, \quad t > 0. \tag{85}$$

As in [4, pp.81-82], y has a global minimum at

$$t_0 = \left(\frac{\mu}{\nu\theta} \right)^{\frac{1}{(\nu+1)}}, \tag{86}$$

which is

$$y(t_0) = (\theta \mu^\nu)^{\frac{1}{(\nu+1)}} (\nu + 1) \nu^{-\left(\frac{\nu}{\nu+1}\right)}. \tag{87}$$

Consequently it is

$$y(t_0) = \left(\frac{\left\| D^{\nu+1}_{*A_0} f \right\|_{L_\infty([A_0,+\infty),X)}}{\Gamma(\nu+2)} \right)^{\frac{1}{(\nu+1)}} \times \left(2 \|f\|_{\infty,[A_0,+\infty)} \right)^{\left(\frac{\nu}{\nu+1}\right)} (\nu+1) \nu^{-\left(\frac{\nu}{\nu+1}\right)}. \tag{88}$$

We have proved that

$$\left\| f' \right\|_{\infty,[A_0,+\infty)} \le (\nu+1) \left(\frac{2}{\nu}\right)^{\left(\frac{\nu}{\nu+1}\right)} (\Gamma(\nu+2))^{-\frac{1}{(\nu+1)}} \times \left(\left\| D^{\nu+1}_{*A_0} f \right\|_{L_\infty([A_0,+\infty),X)} \right)^{\frac{1}{(\nu+1)}} \left(\|f\|_{\infty,[A_0,+\infty)} \right)^{\frac{\nu}{(\nu+1)}}, \tag{89}$$

establishing the claim. □

Corollary 2.25. *All as in Theorem 2.24 for $A_0 = 0$. Then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_+} \le (\nu+1) \left(\frac{2}{\nu}\right)^{\left(\frac{\nu}{\nu+1}\right)} (\Gamma(\nu+2))^{-\frac{1}{(\nu+1)}} \times \left(\|f\|_{\infty,\mathbb{R}_+} \right)^{\left(\frac{\nu}{\nu+1}\right)} \left(\left\| D^{\nu+1}_{*0} f \right\|_{L_\infty(\mathbb{R}_+,X)} \right)^{\frac{1}{(\nu+1)}}. \tag{90}$$

When $\nu = 1$, we get

Corollary 2.26. *Let $f \in C^1([A_0,+\infty),X)$, where $A_0 \in \mathbb{R}$ is fixed, X is a Banach space. For any $A, B \in [A_0,+\infty) : A \le B$, we assume that f fulfills: assume that f'' exists outside a μ-null Borel set $B_x \subseteq [A,x]$ such that $h_1(f'(B_x)) = 0$ for all $x \in [A,B]$.*

We also assume that $f'' \in L_\infty([A_0,+\infty),X)$, and $\|f\|_{\infty,[A_0,+\infty)} < \infty$. Then

$$\left\| f' \right\|_{\infty,[A_0,+\infty)} \le 2 \left(\|f\|_{\infty,[A_0,+\infty)} \right)^{\frac{1}{2}} \left(\left\| f'' \right\|_{L_\infty([A_0,+\infty),X)} \right)^{\frac{1}{2}}. \tag{91}$$

Corollary 2.27. (to Corollary 2.26 for $A_0 = 0$) *It holds*

$$\left\| f' \right\|_{\infty,\mathbb{R}_+} \le 2 \left(\|f\|_{\infty,\mathbb{R}_+} \right)^{\frac{1}{2}} \left(\left\| f'' \right\|_{\infty,\mathbb{R}_+} \right)^{\frac{1}{2}}. \tag{92}$$

When $X = \mathbb{R}$, the last inequality is the Landau inequality [10], with 2 being the best constant.

We continue with another Ostrowski type fractional inequality:

Theorem 2.28. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Here all as in Theorem 2.16. Assume that $f^{(k)}(a) = 0$, $k = 1, \cdots, n-1$, and $D_{*a}^{\nu} f \in L_q([a,b], X)$, where X is a Banach space. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\|D_{*a}^{\nu} f\|_{L_q([a,b],X)}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (b-a)^{\nu - \frac{1}{q}}. \quad (93)$$

Proof. We have that

$$f(x) - f(a) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \left(D_{*a}^{\nu} f\right)(z)\, dz, \quad \forall\, x \in [a,b]. \quad (94)$$

Thus

$$\|f(x) - f(a)\| = \frac{1}{\Gamma(\nu)} \left\| \int_a^x (x-z)^{\nu-1} \left(D_{*a}^{\nu} f\right)(z)\, dz \right\| \leq \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \|D_{*a}^{\nu} f(z)\|\, dz \quad (95)$$

$$\leq \frac{1}{\Gamma(\nu)} \left(\int_a^x (x-z)^{p(\nu-1)}\, dz \right)^{\frac{1}{p}} \left(\int_a^x \|D_{*a}^{\nu} f(z)\|^q\, dz \right)^{\frac{1}{q}} \leq \frac{1}{\Gamma(\nu)} \frac{(x-a)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \|D_{*a}^{\nu} f\|_{L_q([a,b],X)}. \quad (96)$$

That is we have

$$\|f(x) - f(a)\| \leq \frac{\|D_{*a}^{\nu} f\|_{L_q([a,b],X)}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} (x-a)^{\nu - \frac{1}{q}}, \quad \forall\, x \in [a,b]. \quad (97)$$

Consequently we get

$$\begin{aligned} &\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \\ &\leq \frac{1}{b-a} \int_a^b \|f(x) - f(a)\|\, dx \\ &\leq \frac{\|D_{*a}^{\nu} f\|_{L_q([a,b],X)}}{(b-a)\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \int_a^b (x-a)^{\nu-\frac{1}{q}}\, dx \\ &= \frac{\|D_{*a}^{\nu} f\|_{L_q([a,b],X)}}{(b-a)\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\nu-\frac{1}{q}+1}}{\nu - \frac{1}{q} + 1} \\ &= \frac{\|D_{*a}^{\nu} f\|_{L_q([a,b],X)}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (b-a)^{\nu-\frac{1}{q}}, \end{aligned} \quad (98)$$

proving the claim. □

Corollary 2.29. (to Theorem 2.28, case of $p = q = 2$) *Let* $\nu > \frac{1}{2}$, $n = \lceil \nu \rceil$. *Here all as in Theorem* 2.16. *Assume that* $f^{(k)}(a) = 0$, $k = 1, \cdots, n-1$, *and* $D_{*a}^{\nu} f \in L_2([a,b], X)$. *Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\| D_{*a}^{\nu} f \|_{L_2([a,b],X)}}{\Gamma(\nu) \left(\sqrt{2\nu - 1}\right) \left(\nu + \frac{1}{2}\right)} (b-a)^{\nu - \frac{1}{2}}. \tag{99}$$

It follows the L_1 case of Ostrowski inequality:

Theorem 2.30. *Let* $\nu \geq 1$, $n = \lceil \nu \rceil$, *and all as in Theorem* 2.16. *Assume that* $f^{(k)}(a) = 0$, $k = 1, \cdots, n-1$, *and* $D_{*a}^{\nu} f \in L_1([a,b], X)$. *Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\| D_{*a}^{\nu} f \|_{L_1([a,b],X)}}{\Gamma(\nu + 1)} (b-a)^{\nu - 1}. \tag{100}$$

Proof. As before we get

$$\begin{aligned} \| f(x) - f(a) \| &\leq \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} \| D_{*a}^{\nu} f(t) \| \, dt \\ &\leq \frac{1}{\Gamma(\nu)} (x-a)^{\nu-1} \int_a^x \| D_{*a}^{\nu} f(t) \| \, dt \\ &\leq \frac{(x-a)^{\nu-1}}{\Gamma(\nu)} \int_a^b \| D_{*a}^{\nu} f(t) \| \, dt \\ &= \frac{(x-a)^{\nu-1}}{\Gamma(\nu)} \| D_{*a}^{\nu} f \|_{L_1([a,b],X)}. \end{aligned} \tag{101}$$

That is, we have

$$\| f(x) - f(a) \| \leq \frac{(x-a)^{\nu-1}}{\Gamma(\nu)} \| D_{*a}^{\nu} f \|_{L_1([a,b],X)}, \quad \forall\, x \in [a,b]. \tag{102}$$

Therefore, we get

$$\begin{aligned} \left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| &\leq \frac{1}{b-a} \int_a^b \| f(x) - f(a) \| \, dx \\ &\leq \frac{\| D_{*a}^{\nu} f \|_{L_1([a,b],X)}}{\Gamma(\nu)(b-a)} \left(\int_a^b (x-a)^{\nu-1} dx \right) \\ &= \frac{\| D_{*a}^{\nu} f \|_{L_1([a,b],X)}}{\Gamma(\nu+1)} (b-a)^{\nu-1}, \end{aligned} \tag{103}$$

proving the claim. ☐

We apply Theorem 2.28 when $0 < \nu \leq 1$.

Theorem 2.31. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $1 \geq \nu > \frac{1}{q}$. Let $f \in C([A, B], X)$, where $[A, B] \subset \mathbb{R}$ and X is a Banach space. Assume that f' exists outside a μ-null Borel set $B_x \subseteq [A, x]$ such that*

$$h_1(f(B_x)) = 0, \quad \forall\, x \in [A, B].$$

We also assume that $f' \in L_1([A, B], X)$ and $D^{\nu}_{*A} f \in L_q([A, B], X)$. Then

$$\left\| \frac{1}{B-A} \int_A^B f(x)\, dx - f(A) \right\| \leq \frac{\| D^{\nu}_{*A} f \|_{L_q([A,B],X)}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B-A)^{\nu - \frac{1}{q}}. \quad (104)$$

We present the following left Caputo-Bochner fractional Landau inequality for L_q norm.

Theorem 2.32. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\frac{1}{q} < \nu \leq 1$. Let $f \in C^1([A_0, +\infty), X)$, where $A_0 \in \mathbb{R}$ is fixed, X is a Banach space. For any $A, B \in [A_0, +\infty) : A \leq B$, we suppose that f fulfills: assume that f'' exists outside a μ-null Borel set $B_x \subseteq [A, x]$, such that*

$$h_1\left(f'(B_x)\right) = 0, \quad \forall\, x \in [A, B]. \quad (105)$$

We also assume that $f'' \in L_1([A, B], X)$ and $D^{\nu+1}_{*A} f \in L_q([A, B], X)$. We further assume that

$$\left\| D^{\nu+1}_{*A} f \right\|_{L_q([A,+\infty),X)} \leq \left\| D^{\nu+1}_{*A_0} f \right\|_{L_q([A_0,+\infty),X)} < \infty, \quad \forall\, A \geq A_0, \quad (106)$$

(the last left inequality is obvious when $\nu = 1$), and

$$\|f\|_{\infty,[A_0,+\infty)} < \infty. \quad (107)$$

Then

$$\begin{aligned} &\|f'\|_{\infty,[A_0,+\infty)} \\ &\leq \left(\frac{2\left(\nu + \frac{1}{p}\right)}{\nu - \frac{1}{q}} \right)^{\left(\frac{\nu - \frac{1}{q}}{\nu + \frac{1}{p}}\right)} \frac{1}{(\Gamma(\nu))^{\frac{1}{\left(\nu + \frac{1}{p}\right)}} (p(\nu-1)+1)^{\frac{1}{(p\nu+1)}}} \\ &\quad \times \left(\|f\|_{\infty,[A_0,+\infty)} \right)^{\left(\frac{\nu - \frac{1}{q}}{\nu + \frac{1}{p}}\right)} \left(\left\| D^{\nu+1}_{*A_0} f \right\|_{L_q([A_0,+\infty),X)} \right)^{\frac{1}{\left(\nu + \frac{1}{p}\right)}}. \end{aligned} \quad (108)$$

Proof. We have that (by Theorem 2.31)

$$\begin{aligned} &\left\| \frac{1}{B-A} \int_A^B f'(x)\, dx - f'(A) \right\| \\ &\leq \frac{\left\| D^{\nu+1}_{*A} f \right\|_{L_q([A,B],X)}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B-A)^{\nu - \frac{1}{q}}, \end{aligned} \quad (109)$$

$\forall\ A, B \in [A_0, +\infty),\ A \leq B$.

Subsequently by Theorem 2.9 we derive

$$\left\| \frac{f(B) - f(A)}{B - A} - f'(A) \right\| \leq \frac{\left\| D_{*A}^{\nu+1} f \right\|_{L_q([A,B],X)}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B - A)^{\nu - \frac{1}{q}}, \quad (110)$$

$\forall\ A, B \in [A_0, +\infty),\ A \leq B$. Hence it holds

$$\begin{aligned} &\left\| f'(A) \right\| - \frac{1}{B - A} \left\| f(B) - f(A) \right\| \\ &\leq \frac{\left\| D_{*A}^{\nu+1} f \right\|_{L_q([A,B],X)}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B - A)^{\nu - \frac{1}{q}}, \end{aligned} \quad (111)$$

$\forall\ A, B \in [A_0, +\infty),\ A \leq B$, and

$$\begin{aligned} \left\| f'(A) \right\| \leq &\frac{\left\| f(B) - f(A) \right\|}{B - A} \\ &+ \frac{\left\| D_{*A}^{\nu+1} f \right\|_{L_q([A,B],X)}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B - A)^{\nu - \frac{1}{q}}, \end{aligned} \quad (112)$$

$\forall\ A, B \in [A_0, +\infty),\ A \leq B$. Therefore, we obtain

$$\begin{aligned} \left\| f'(A) \right\| \leq &\frac{2 \left\| f \right\|_{\infty,[A_0,+\infty)}}{B - A} \\ &+ \frac{\left\| D_{*A_0}^{\nu+1} f \right\|_{L_q([A_0,+\infty),X)}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B - A)^{\nu - \frac{1}{q}}, \end{aligned} \quad (113)$$

$\forall\ A, B \in [A_0, +\infty),\ A \leq B$. The R.H.S. (113) depends only on $B - A$. Therefore

$$\begin{aligned} \left\| f' \right\|_{\infty,[A_0,+\infty)} \leq &\frac{2 \left\| f \right\|_{\infty,[A_0,+\infty)}}{B - A} \\ &+ \frac{\left\| D_{*A_0}^{\nu+1} f \right\|_{L_q([A_0,+\infty),X)}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (B - A)^{\nu - \frac{1}{q}}. \end{aligned} \quad (114)$$

We may call $t = B - A > 0$. Thus

$$\left\|f'\right\|_{\infty,[A_0,+\infty)} \leq \frac{2\left\|f\right\|_{\infty,[A_0,+\infty)}}{t} + \frac{\left\|D_{*A_0}^{\nu+1}f\right\|_{L_q([A_0,+\infty),X)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)} t^{\nu-\frac{1}{q}}, \quad \forall\, t \in (0,\infty). \tag{115}$$

Notice that $0 < \nu - \frac{1}{q} < 1$. Call

$$\widetilde{\mu} := 2\left\|f\right\|_{\infty,[A_0,+\infty)}, \tag{116}$$

$$\widetilde{\theta} := \frac{\left\|D_{*A_0}^{\nu+1}f\right\|_{L_q([A_0,+\infty),X)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}, \tag{117}$$

both are positive, and

$$\widetilde{\nu} := \nu - \frac{1}{q} \in (0,1). \tag{118}$$

We consider the function

$$\widetilde{y}(t) = \frac{\widetilde{\mu}}{t} + \widetilde{\theta} t^{\widetilde{\nu}}, \quad t \in (0,\infty). \tag{119}$$

The only critical number here is

$$\widetilde{t_0} = \left(\frac{\widetilde{\mu}}{\widetilde{\nu}\widetilde{\theta}}\right)^{\frac{1}{\widetilde{\nu}+1}} \tag{120}$$

and $\widetilde{y}$ has a global minimum at $\widetilde{t_0}$, which is

$$\widetilde{y}\left(\widetilde{t_0}\right) = \left(\widetilde{\theta}\widetilde{\mu}^{\widetilde{\nu}}\right)^{\frac{1}{\widetilde{\nu}+1}}\left(\widetilde{\nu}+1\right)\widetilde{\nu}^{-\left(\frac{\widetilde{\nu}}{\widetilde{\nu}+1}\right)}. \tag{121}$$

Consequently, we derive

$$\widetilde{y}\left(\widetilde{t_0}\right) = \left(\frac{\left\|D_{*A_0}^{\nu+1}f\right\|_{L_q([A_0,+\infty),X)}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}} \times \left(2\left\|f\right\|_{\infty,[A_0,+\infty)}\right)^{\left(\frac{\nu-\frac{1}{q}}{\nu+\frac{1}{p}}\right)}\left(\nu+\frac{1}{p}\right)\left(\nu-\frac{1}{q}\right)^{-\left(\frac{\nu-\frac{1}{q}}{\nu+\frac{1}{p}}\right)}. \tag{122}$$

We have proved that

$$\|f'\|_{\infty,[A_0,+\infty)} \leq \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-\frac{1}{q}}\right)^{\left(\frac{\nu-\frac{1}{q}}{\nu+\frac{1}{p}}\right)} \frac{1}{(\Gamma(\nu))^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}(p(\nu-1)+1)^{\frac{1}{(p\nu+1)}}} \times \left(\|f\|_{\infty,[A_0,+\infty)}\right)^{\left(\frac{\nu-\frac{1}{q}}{\nu+\frac{1}{p}}\right)} \left(\left\|D_{*A_0}^{\nu+1}f\right\|_{L_q([A_0,+\infty),X)}\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}, \tag{123}$$

establishing the claim. ☐

Corollary 2.33. *All as in Theorem* 2.32 *for* $A_0=0$*. Then*

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-\frac{1}{q}}\right)^{\left(\frac{\nu-\frac{1}{q}}{\nu+\frac{1}{p}}\right)} \frac{1}{(\Gamma(\nu))^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}(p(\nu-1)+1)^{\frac{1}{(p\nu+1)}}} \times \left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\left(\frac{\nu-\frac{1}{q}}{\nu+\frac{1}{p}}\right)} \left(\left\|D_{*0}^{\nu+1}f\right\|_{L_q(\mathbb{R}_+,X)}\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}}. \tag{124}$$

Corollary 2.34. *All as in Theorem* 2.32 *for* $A_0=0$*, and* $p=q=2$*. Here* $\frac{1}{2}<\nu\leq 1$*. Then*

$$\|f'\|_{\infty,\mathbb{R}_+} \leq \left(\frac{2\left(\nu+\frac{1}{2}\right)}{\nu-\frac{1}{2}}\right)^{\left(\frac{\nu-\frac{1}{2}}{\nu+\frac{1}{2}}\right)} \frac{1}{(\Gamma(\nu))^{\frac{1}{\left(\nu+\frac{1}{2}\right)}}(2\nu-1)^{\frac{1}{(2\nu+1)}}} \times \left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\left(\frac{\nu-\frac{1}{2}}{\nu+\frac{1}{2}}\right)} \left(\left\|D_{*0}^{\nu+1}f\right\|_{L_2(\mathbb{R}_+,X)}\right)^{\frac{1}{\left(\nu+\frac{1}{2}\right)}}. \tag{125}$$

Case of $\nu=1$ follows:

Corollary 2.35. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*. Let* $f\in C^1([A_0,+\infty),X)$*, where* $A_0\in\mathbb{R}$ *is fixed,* X *is a Banach space. For any* $A,B\in[A_0,+\infty):A\leq B$*, we assume that* f *fulfills: assume that* f'' *exists outside a* μ*-null Borel set* $B_x\subseteq[A,x]$*, such that*

$$h_1\left(f'(B_x)\right)=0,\ \ \forall\ x\in[A,B]. \tag{126}$$

We further assume that

$$f''\in L_q([A_0,+\infty),X), \tag{127}$$

and

$$\|f\|_{\infty,[A_0,+\infty)} < \infty. \tag{128}$$

Then

$$\begin{aligned} \left\|f'\right\|_{\infty,[A_0,+\infty)} &\leq \left(\frac{2\left(1+\frac{1}{p}\right)}{1-\frac{1}{q}}\right)^{\left(\frac{1-\frac{1}{q}}{1+\frac{1}{p}}\right)} \\ &\times \left(\|f\|_{\infty,[A_0,+\infty)}\right)^{\left(\frac{1-\frac{1}{q}}{1+\frac{1}{p}}\right)} \left(\left\|f''\right\|_{L_q([A_0,+\infty),X)}\right)^{\frac{1}{\left(1+\frac{1}{p}\right)}}. \end{aligned} \tag{129}$$

Corollary 2.36. (to Corollary 2.35) *Assume $A_0 = 0$. Then*

$$\left\|f'\right\|_{\infty,\mathbb{R}_+} \leq \left(\frac{2\left(1+\frac{1}{p}\right)}{1-\frac{1}{q}}\right)^{\left(\frac{1-\frac{1}{q}}{1+\frac{1}{p}}\right)} \|f\|_{\infty,\mathbb{R}_+}^{\left(\frac{1-\frac{1}{q}}{1+\frac{1}{p}}\right)} \left(\left\|f''\right\|_{L_q(\mathbb{R}_+,X)}\right)^{\frac{1}{\left(1+\frac{1}{p}\right)}}. \tag{130}$$

We finish with

Corollary 2.37. (to Corollaries 2.35, 2.36) *Assume $A_0 = 0$ and $p = q = 2$. Then*

$$\left\|f'\right\|_{\infty,\mathbb{R}_+} \leq \sqrt[3]{6}\left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\frac{1}{3}} \left(\left\|f''\right\|_{L_2(\mathbb{R}_+,X)}\right)^{\frac{2}{3}}. \tag{131}$$

We continue with a Poincaré like fractional inequality:

Theorem 2.38. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Here all as in Theorem 2.16. Assume that $f^{(k)}(a) = 0$, $k = 0, 1, \cdots, n-1$, and $D_{*a}^{\nu} f \in L_q([a,b], X)$, where X is a Banach space. Then*

$$\|f\|_{L_q([a,b],X)} \leq \frac{(b-a)^{\nu}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}(q\nu)^{\frac{1}{q}}} \left\|D_{*a}^{\nu} f\right\|_{L_q([a,b],X)}. \tag{132}$$

Proof. We have that

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \left(D_{*a}^{\nu} f\right)(z)\, dz, \quad \forall\, x \in [a,b].$$

Thus

$$\begin{aligned} \|f(x)\| &= \frac{1}{\Gamma(\nu)} \left\| \int_a^x (x-z)^{\nu-1} \left(D_{*a}^{\nu} f\right)(z)\, dz \right\| \\ &\leq \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \left\|D_{*a}^{\nu} f(z)\right\| dz \end{aligned}$$

$$\leq \left(\frac{1}{\Gamma(\nu)}\int_a^x (x-z)^{p(\nu-1)}\,dz\right)^{\frac{1}{p}}\left(\int_a^x \left\|D_{*a}^{\nu}f(z)\right\|^q dz\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\nu)}\frac{(x-a)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b],X)}. \quad (133)$$

We have proved that

$$\|f(x)\| \leq \frac{(x-a)^{\nu-\frac{1}{q}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b],X)}, \quad \forall\, x\in[a,b]. \quad (134)$$

Then

$$\|f(x)\|^q \leq \frac{(x-a)^{q\nu-1}}{(\Gamma(\nu))^q(p(\nu-1)+1)^{\frac{q}{p}}}\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b],X)}^q, \quad (135)$$

and

$$\int_a^b \|f(x)\|^q\,dx \leq \frac{(b-a)^{q\nu}\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b],X)}^q}{(\Gamma(\nu))^q(p(\nu-1)+1)^{\frac{q}{p}}\,q\nu}. \quad (136)$$

This last results into

$$\left(\int_a^b \|f(x)\|^q\,dx\right)^{\frac{1}{q}} \leq \frac{(b-a)^{\nu}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}(q\nu)^{\frac{1}{q}}}\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b],X)}$$

proving the claim. □

Next comes a Sobolev like fractional inequality.

Theorem 2.39. *All as in the last Theorem* 2.38*. Let* $r>0$*. Then*

$$\|f\|_{L_r([a,b],X)} \leq \frac{(b-a)^{\nu-\frac{1}{q}+\frac{1}{r}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}\left(r\left(\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}}\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b],X)}. \quad (137)$$

Proof. As in the last theorem's proof we get that

$$\|f(x)\| \leq \frac{(x-a)^{\nu-\frac{1}{q}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b],X)}, \quad \forall\, x\in[a,b]. \quad (138)$$

Since $r>0$, we get

$$\|f(x)\|^r \leq \frac{(x-a)^{r\left(\nu-\frac{1}{q}\right)}}{(\Gamma(\nu))^r(p(\nu-1)+1)^{\frac{r}{p}}}\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b],X)}^r, \quad \forall\, x\in[a,b]. \quad (139)$$

Hence it holds

$$\int_a^b \|f(x)\|^r dx \leq \frac{(b-a)^{r\left(\nu-\frac{1}{q}\right)+1}}{(\Gamma(\nu))^r (p(\nu-1)+1)^{\frac{r}{p}} \left(r\left(\nu-\frac{1}{q}\right)+1\right)} \|D_{*a}^{\nu} f\|_{L_q([a,b],X)}^r . \tag{140}$$

That is

$$\left(\int_a^b \|f(x)\|^r dx\right)^{\frac{1}{r}} \leq \frac{(b-a)^{\nu-\frac{1}{q}+\frac{1}{r}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(r\left(\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \|D_{*a}^{\nu} f\|_{L_q([a,b],X)} , \tag{141}$$

proving the claim. □

We give the following Opial type fractional inequality:

Theorem 2.40. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n := \lceil \nu \rceil$. Let $[a, b] \subset \mathbb{R}$, X a Banach space, and $f \in C^{n-1}([a, b], X)$. Set*

$$F_x(t) := \sum_{i=0}^{n-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [a, x], \text{ where } x \in [a, b]. \tag{142}$$

Assume that $f^{(n)}$ exists outside a μ-null Borel set $B_x \subseteq [a, x]$, such that

$$h_1(F_x(B_x)) = 0, \quad \forall\, x \in [a, b]. \tag{143}$$

We also assume that $f^{(n)} \in L_\infty([a, b], X)$. Assume also that $f^{(k)}(a) = 0$, $k = 0, 1, \cdots, n-1$. Then

$$\int_a^x \|f(w)\| \|(D_{*a}^{\nu} f)(w)\| dw \leq \frac{(x-a)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}} \Gamma(\nu) ((p(\nu-1)+1)(p(\nu-1)+2))^{\frac{1}{p}}} \left(\int_a^x \|(D_{*a}^{\nu} f)(z)\|^q dz\right)^{\frac{2}{q}}, \tag{144}$$

for all $x \in [a, b]$.

Proof. Clearly here $D_{*a}^{\nu} f \in C([a, b], X)$. We have that

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz, \quad \forall\, x \in [a, b]. \tag{145}$$

Let $a \leq w \leq x$, then we have

$$f(w) = \frac{1}{\Gamma(\nu)} \int_a^w (w-z)^{\nu-1} (D_{*a}^{\nu} f)(z) dz, \tag{146}$$

and

$$\begin{aligned}\|f(w)\| &\leq \frac{1}{\Gamma(\nu)}\int_a^w (w-z)^{\nu-1}\left\|\left(D_{*a}^{\nu}f\right)(z)\right\|dz\\ &\leq \frac{1}{\Gamma(\nu)}\left(\int_a^w (w-z)^{p(\nu-1)}dz\right)^{\frac{1}{p}}\left(\int_a^w \left\|\left(D_{*a}^{\nu}f\right)(z)\right\|^q dz\right)^{\frac{1}{q}}\\ &= \frac{1}{\Gamma(\nu)}\frac{(w-a)^{\frac{(p(\nu-1)+1)}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left(\int_a^w \left\|\left(D_{*a}^{\nu}f\right)(z)\right\|^q dz\right)^{\frac{1}{q}}\\ &= \frac{(w-a)^{\nu-\frac{1}{q}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}(z(w))^{\frac{1}{q}},\end{aligned} \tag{147}$$

where

$$z(w) := \int_a^w \left\|\left(D_{*a}^{\nu}f\right)(z)\right\|^q dz, \tag{148}$$

all $a \leq w \leq x$, $z(a) = 0$. Thus

$$z'(w) = \left\|\left(D_{*a}^{\nu}f\right)(w)\right\|^q, \tag{149}$$

and

$$\left\|\left(D_{*a}^{\nu}f\right)(w)\right\| = \left(z'(w)\right)^{\frac{1}{q}}. \tag{150}$$

Therefore, we obtain

$$\|f(w)\|\left\|\left(D_{*a}^{\nu}f\right)(w)\right\| \leq \frac{(w-a)^{\nu-\frac{1}{q}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}\left(z(w)z'(w)\right)^{\frac{1}{q}}. \tag{151}$$

Integrating the last inequality we get

$$\begin{aligned}&\int_a^x \|f(w)\|\left\|\left(D_{*a}^{\nu}f\right)(w)\right\|dw\\ &\leq \frac{1}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}\int_a^x (w-a)^{\nu-\frac{1}{q}}\left(z(w)z'(w)\right)^{\frac{1}{q}}dw\\ &\leq \frac{1}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\int_a^x (w-a)^{p\left(\nu-\frac{1}{q}\right)}dw\right)^{\frac{1}{p}}\left(\int_a^x \left(z(w)z'(w)\right)dw\right)^{\frac{1}{q}}\\ &= \frac{1}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(x-a)^{\frac{\left(p\left(\nu-\frac{1}{q}\right)+1\right)}{p}}}{\left(p\left(\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{p}}}\left(\int_a^x z(w)\,dz(w)\right)^{\frac{1}{q}}\\ &= \frac{1}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(x-a)^{\nu-1+\frac{2}{p}}}{(p(\nu-1)+2)^{\frac{1}{p}}}\left(\frac{z^2(x)}{2}\right)^{\frac{1}{q}}\end{aligned} \tag{152}$$

$$= \frac{(x-a)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\nu)\left(p\left(\nu-1\right)+1\right)^{\frac{1}{p}}\left(p\left(\nu-1\right)+2\right)^{\frac{1}{p}}}\left(\int_a^x \left\|\left(D_{*a}^{\nu}f\right)(z)\right\|^q dz\right)^{\frac{2}{q}}, \quad (153)$$

proving the claim. □

We finish article with a Hilbert-Pachpatte left fractional inequality:

Theorem 2.41. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu_1 > \frac{1}{q}$, $\nu_2 > \frac{1}{p}$, $n_i := \lceil \nu_i \rceil$, $i = 1, 2$. Here $[a_i, b_i] \subset \mathbb{R}$, $i = 1, 2$; X is a Banach space. Let $f_i \in C^{n_i-1}([a_i, b_i], X)$, $i = 1, 2$. Set*

$$F_{x_i}(t_i) := \sum_{j_i=0}^{n_i-1} \frac{(x_i - t_i)^{j_i}}{j_i!} f_i^{(j_i)}(t_i), \quad (154)$$

$\forall\ t_i \in [a_i, x_i]$, where $x_i \in [a_i, b_i]$; $i = 1, 2$. Assume that $f_i^{(n_i)}$ exists outside a μ-null Borel set $B_{x_i} \subseteq [a_i, x_i]$, such that

$$h_1\left(F_{x_i}\left(B_{x_i}\right)\right) = 0, \ \ \forall\ x_i \in [a_i, b_i];\ i = 1, 2. \quad (155)$$

We also assume that $f_i^{(n_i)} \in L_1([a_i, b_i], X)$, and

$$f_i^{(k_i)}(a_i) = 0, \ \ k_i = 0, 1, \cdots, n_i - 1;\ i = 1, 2, \quad (156)$$

and

$$\left(D_{*a_1}^{\nu_1} f_1\right) \in L_q([a_1, b_1], X), \quad \left(D_{*a_2}^{\nu_2} f_2\right) \in L_p([a_2, b_2], X). \quad (157)$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{\|f_1(x_1)\| \|f_2(x_2)\| dx_1 dx_2}{\left(\frac{(x_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)} + \frac{(x_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}$$
$$\leq \frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \left\|D_{*a_1}^{\nu_1} f_1\right\|_{L_q([a_1,b_1],X)} \left\|D_{*a_2}^{\nu_2} f_2\right\|_{L_p([a_2,b_2],X)}. \quad (158)$$

Proof. We have that

$$f_i(x_i) = \frac{1}{\Gamma(\nu_i)} \int_{a_i}^{x_i} (x_i - z_i)^{\nu_i - 1} \left(D_{*a_i}^{\nu_i} f_i\right)(z_i)\, dz_i, \quad (159)$$

$\forall\ x_i \in [a_i, b_i]$, $i = 1, 2$. Then

$$\|f_i(x_i)\| \leq \frac{1}{\Gamma(\nu_i)} \int_{a_i}^{x_i} (x_i - z_i)^{\nu_i - 1} \left\|\left(D_{*a_i}^{\nu_i} f_i\right)(z_i)\right\| dz_i,$$

$i = 1, 2$, $\forall\ x_i \in [a_i, b_i]$. We get as before,

$$\|f_1(x_1)\| \leq \frac{1}{\Gamma(\nu_1)} \frac{(x_1 - a_1)^{\frac{p(\nu_1-1)+1}{p}}}{(p(\nu_1 - 1) + 1)^{\frac{1}{p}}} \left\|D_{*a_1}^{\nu_1} f_1\right\|_{L_q([a_1,b_1],X)} \quad (160)$$

and

$$\|f_2(x_2)\| \leq \frac{1}{\Gamma(\nu_2)} \frac{(x_2 - a_2)^{\frac{q(\nu_2-1)+1}{q}}}{(q(\nu_2 - 1) + 1)^{\frac{1}{q}}} \left\|D_{*a_2}^{\nu_2} f_2\right\|_{L_p([a_2,b_2],X)}. \tag{161}$$

Hence we have

$$\begin{aligned} &\|f_1(x_1)\| \|f_2(x_2)\| \\ &\leq \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)(p(\nu_1-1)+1)^{\frac{1}{p}}(q(\nu_2-1)+1)^{\frac{1}{q}}} \\ &\quad \times (x_1 - a_1)^{\frac{p(\nu_1-1)+1}{p}} (x_2 - a_2)^{\frac{q(\nu_2-1)+1}{q}} \left\|D_{*a_1}^{\nu_1} f_1\right\|_{L_q([a_1,b_1],X)} \\ &\quad \times \left\|D_{*a_2}^{\nu_2} f_2\right\|_{L_p([a_2,b_2],X)} \\ &\qquad \left(\text{using Young's inequality for } a, b \geq 0,\ a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}\right) \\ &\leq \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \left(\frac{(x_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)} + \frac{(x_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right) \\ &\quad \times \left\|D_{*a_1}^{\nu_1} f_1\right\|_{L_q([a_1,b_1],X)} \left\|D_{*a_2}^{\nu_2} f_2\right\|_{L_p([a_2,b_2],X)}, \end{aligned} \tag{162}$$

$\forall\ x_i \in [a_i, b_i]$; $i = 1, 2$. So far we have

$$\begin{aligned} &\frac{\|f_1(x_1)\| \|f_2(x_2)\|}{\left(\frac{(x_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)} + \frac{(x_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \\ &\leq \frac{\left\|D_{*a_1}^{\nu_1} f_1\right\|_{L_q([a_1,b_1],X)} \left\|D_{*a_2}^{\nu_2} f_2\right\|_{L_p([a_2,b_2],X)}}{\Gamma(\nu_1)\Gamma(\nu_2)}, \quad \forall\ x_i \in [a_i, b_i];\ i = 1, 2. \end{aligned} \tag{163}$$

The denominator in (163) can be zero only when $x_1 = a_1$ and $x_2 = a_2$. Therefore, we obtain (158), by integrating (163) over $[a_1, b_1] \times [a_2, b_2]$. ☐

Remark 2.42. Many variations and generalizations of the above inequalities are possible, however due to lack of space we stop here.

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