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GENERALIZED FORM OF TRIPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we introduce a new concept known as FGH-tripled fixed point and prove existence and uniqueness of fixed points in partially ordered complete metric spaces. This concept is a generalization of tripled fixed point and an extension of FG-coupled fixed point. Our results extends and generalizes several results in literature particularly the results of Berinde and Borcut[Vasile Berinde, Marine Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Analysis, 74 (2011), 4889-4897].

1. Introduction

The new trends in fixed point theory is to find multidimensional fixed point results. Guo and Lakshmikantham [11] initiated this idea through coupled fixed points in cone metric spaces. Later in 2006 Gnana Bhaskar and Lakshmikantham [10] defined mixed monotone property and proved existence and uniqueness theorems for coupled fixed points in partially ordered metric spaces. Also as an application they discussed the existence of a unique solution to a

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periodic boundary value problem associated with a first order ordinary differential equation. From this background several authors have studied multidimensional fixed points and established numerous fixed point theorems. In [6]-[8] Berinde and Borcut extended coupled fixed point results to tripled fixed point results using mixed monotone property and thereafter a lots of studies have taken place in this field [1]-[5]. Similarly Karapinar and Loung [12] defined quadrupled fixed points and they proved some fixed point results in this area. As an extension of coupled fixed points, in a natural way Samet and Vetro [14] introduced the concept of fixed points of N-order. Instead of using mixed monotone property they defined F-invariant set and using this concept proved fixed point theorems of N-order.

In the above mentioned multidimensional fixed points the authors have considered fixed points on the finite Cartesian product of the space X with itself. But recently Prajisha and Shaini[13] introduced FG-coupled fixed points in partially ordered metric spaces, where they used the Cartesian product of different spaces as the ambient space. Using this concept Deepa and Shaini[9] proved existence and uniqueness theorems for FG-coupled fixed points on contractive and generalized quasi-contractive mappings.

In this paper, we define a new concept known as FGH-tripled fixed point which is a generalization of tripled fixed point and an extension of FG-coupled fixed point. Now we recall some basic definitions:

Definition 1.1. ([10]) Let X be a partially ordered metric space and $F : X \times X \to X$. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.2. ([6]) Let X be a partially ordered metric space. An element $(x, y, z) \in X \times X \times X$ is said to be tripled fixed point of $F : X \times X \times X \to X$ if $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$.

Definition 1.3. ([13]) Let X and Y be two partially ordered metric spaces, $F : X \times Y \to X$ and $G : Y \times X \to Y$ be two mappings. An element $(x, y) \in X \times Y$ is said to be an FG-coupled fixed point if $F(x, y) = x$ and $G(y, x) = y$.

In next section, we define FGH-tripled fixed point with an example. Also we prove the existence and uniqueness of FGH-tripled fixed point theorems for continuous and discontinuous mappings, which gives extension of the following theorems in [13].

Theorem 1.4. ([13, Theorem 2.4]) *Let* $(X, d_X, \leq_{P_1})$ *and* $(Y, d_Y, \leq_{P_2})$ *be two partially ordered complete metric spaces and* $F : X \times Y \to X$, $G : Y \times X \to Y$ *be two continuous mappings having the mixed monotone property. Assume that there exist* $k, l \in [0, 1); k + l < 1$ *with*

$$d_X(F(x, y), F(u, v)) \leq k\ d_X(x, u) + l\ d_Y(y, v),\ \ \forall\, x \geq_{P_1} u,\ y \leq_{P_2} v,$$
$$d_Y(G(y, x), G(v, u)) \leq k\ d_Y(y, v) + l\ d_X(x, u),\ \ \forall\, x \leq_{P_1} u,\ y \geq_{P_2} v.$$

If there exist $(x_0, y_0) \in X \times Y$ *such that* $x_0 \leq_{P_1} F(x_0, y_0)$ *and* $y_0 \geq_{P_2} G(y_0, x_0)$, *then there exist* $(x, y) \in X \times Y$ *such that* $x = F(x, y)$ *and* $y = G(y, x)$.

Theorem 1.5. ([13, Theorem 2.5]) *Let* $(X, d_X, \leq_{P_1})$ *and* $(Y, d_Y, \leq_{P_2})$ *be two partially ordered complete metric spaces and* $F : X \times Y \to X$, $G : Y \times X \to Y$ *be two continuous mappings having the mixed monotone property. For every* $(x, y), (x_1, y_1) \in X \times Y$ *there exist a* $(z_1, z_2) \in X \times Y$ *that is comparable to both* (x, y) *and* (x_1, y_1). *Assume that there exist* $k, l \in [0, 1);\ k + l < 1$ *with*

$$d_X(F(x, y), F(u, v)) \leq k\ d_X(x, u) + l\ d_Y(y, v),\ \ \forall\, x \geq_{P_1} u,\ y \leq_{P_2} v,$$
$$d_Y(G(y, x), G(v, u)) \leq k\ d_Y(y, v) + l\ d_X(x, u),\ \ \forall\, x \leq_{P_1} u,\ y \geq_{P_2} v.$$

If there exist $x_0 \leq_{P_1} F(x_0, y_0)$ *and* $y_0 \geq_{P_2} G(y_0, x_0)$, *then there exist a unique FG- coupled fixed point.*

Theorem 1.6. ([13, Theorem 2.6]) *Let* $(X, d_X, \leq_{P_1})$ *and* $(Y, d_Y, \leq_{P_2})$ *be two partially ordered complete metric spaces. Assume that* X *and* Y *having the following properties*

(i) *If a non-decreasing sequence* $\{x_n\} \to x$ *then* $x_n \leq_{P_1} x,\ \forall\, n$.
(ii) *If a non-increasing sequence* $\{y_n\} \to y$ *then* $y \leq_{P_2} y_n,\ \forall\, n$.

Let $F : X \times Y \to X$, $G : Y \times X \to Y$ *be two mappings satisfying the mixed monotone property. Also assume that there exist* $k, l \in [0, 1)$ *such that* $k + l < 1$ *with*

$$d_X(F(x, y), F(u, v)) \leq k\ d_X(x, u) + l\ d_Y(y, v),\ \ \forall\, x \geq_{P_1} u,\ y \leq_{P_2} v,$$
$$d_Y(G(y, x), G(v, u)) \leq k\ d_Y(y, v) + l\ d_X(x, u),\ \ \forall\, x \leq_{P_1} u,\ y \geq_{P_2} v.$$

If there exist $x_0 \leq_{P_1} F(x_0, y_0)$ *and* $y_0 \geq_{P_2} G(y_0, x_0)$, *then there exist* $(x, y) \in X \times Y$ *such that* $x = F(x, y)$ *and* $y = G(y, x)$.

2. FGH-tripled fixed point theorems

Definition 2.1. Let $(X, \leq_{P_1})$, $(Y, \leq_{P_2})$, and $(Z, \leq_{P_3})$ be three partially ordered sets and $F : X \times Y \times Z \to X$, $G : Y \times X \times Y \to Y$ and $H : Z \times Y \times X \to Z$ be three mappings. An element $(x, y, z) \in X \times Y \times Z$ is said to be an FGH-tripled fixed point if $F(x, y, z) = x, G(y, x, y) = y$, and $H(z, y, x) = z$.

Definition 2.2. Let $(X, \leq_{P_1})$, $(Y, \leq_{P_2})$, and $(Z, \leq_{P_3})$ be three partially ordered sets and $F : X \times Y \times Z \to X$, $G : Y \times X \times Y \to Y$ and $H : Z \times Y \times X \to Z$. We say that F, G and H have mixed monotone property if for any $x \in X$, $y, y^{'} \in Y$ and $z \in Z$ we have

$$\begin{aligned}
x_1, x_2 \in X, \quad x_1 \leq_{P_1} x_2 \quad &\Rightarrow \quad F(x_1, y, z) \leq_{P_1} F(x_2, y, z),\\
&\qquad G(y, x_1, y') \geq_{P_2} G(y, x_2, y'),\\
&\qquad H(z, y, x_1) \leq_{P_3} H(z, y, x_2),\\
y_1, y_2 \in Y, \quad y_1 \leq_{P_2} y_2 \quad &\Rightarrow \quad F(x, y_1, z) \geq_{P_1} F(x, y_2, z),\\
&\qquad G(y_1, x, y') \leq_{P_2} G(y_2, x, y'),\\
&\qquad G(y, x, y_1) \leq_{P_2} G(y, x, y_2),\\
&\qquad H(z, y_1, x) \geq_{P_3} H(z, y_2, x),\\
z_1, z_2 \in Z, \quad z_1 \leq_{P_3} z_2 \quad &\Rightarrow \quad F(x, y, z_1) \leq_{P_1} F(x, y, z_2),\\
&\qquad H(z_1, y, x) \leq_{P_3} H(z_2, y, x).
\end{aligned}$$

Definition 2.3. Partial order $\leq$ in $X \times Y \times Z$ is defined by $(x, y, z) \leq (u, v, w)$ implies that $x \leq_{P_1} u, y \geq_{P_2} v$ and $z \leq_{P_3} w$; where $\leq_{P_1}$, $\leq_{P_2}$, $\leq_{P_3}$ are partial orders in X, Y and Z respectively.

Example 2.4. Let $X = [0, 1]$, $Y = [1, 2]$ and $Z = [2, 4]$ with usual metric and usual ordering. Define $F : X \times Y \times Z \to X$ by $F(x, y, z) = \dfrac{x + y + z + 1}{12}$, $G : Y \times X \times Y \to Y$ by $G(y, x, y) = \dfrac{x + y + 1}{2}$ and $H : Z \times Y \times X \to Z$ by $H(z, y, x) = \dfrac{z + y + x + 1}{2}$ for every $(x, y, z) \in X \times Y \times Z$. Then $\left(\dfrac{1}{2}, \dfrac{3}{2}, 3\right)$ is a FGH-tripled fixed point.

Throughout this paper we use the following notations to prove our results. For $n \geq 1$ define

$$\begin{aligned}
F^{n+1}(x, y, z) &= F(F^n(x, y, z), G^n(y, x, y), H^n(z, y, x)),\\
G^{n+1}(y, x, y) &= G(G^n(y, x, y), F^n(x, y, z), G^n(y, x, y)),\\
H^{n+1}(z, y, x) &= H(H^n(z, y, x), G^n(y, x, y), F^n(x, y, z)),
\end{aligned}$$

for every $(x, y, z) \in X \times Y \times Z$.

Theorem 2.5. *Let $(X, \leq_{P_1}, d_X)$, $(Y, \leq_{P_2}, d_Y)$ and $(Z, \leq_{P_3}, d_Z)$ be three partially ordered complete metric spaces. Let $F : X \times Y \times Z \to X$, $G : Y \times X \times Y \to Y$ and $H : Z \times Y \times X \to Z$ be three continuous functions having the mixed* *monotone property. Assume that there exist constants $j,k,l \in [0,1)$ with $j+k+l<1$ such that*

$$d_X(F(x,y,z),F(u,v,w)) \\ \leq jd_X(x,u)+kd_Y(y,v)+l\ d_Z(z,w);\ \ \forall\, x \geq_{P_1} u,\ y \leq_{P_2} v,\ z \geq_{P_3} w, \tag{2.1}$$

$$d_Y(G(y,x,y'),G(v,u,v')) \\ \leq jd_Y(y,v)+kd_X(x,u)+l\ d_Y(y',v');\ \ \forall\, y \geq_{P_2} v,\ x \leq_{P_1} u,\ y' \geq_{P_2} v', \tag{2.2}$$

$$d_Z(H(z,y,x),H(w,v,u)) \\ \leq jd_Z(z,w)+kd_Y(y,v)+l\ d_X(x,u);\ \ \forall\, x \geq_{P_1} u,\ y \leq_{P_2} v,\ z \geq_{P_3} w. \tag{2.3}$$

If there exist $x_0 \in X,\ y_0 \in Y$ and $z_0 \in Z$ such that $x_0 \leq_{P_1} F(x_0,y_0,z_0),\ y_0 \geq_{P_2} G(y_0,x_0,y_0)$ and $z_0 \leq_{P_3} H(z_0,y_0,x_0)$. Then there exist $(x,y,z) \in X \times Y \times Z$ such that $x=F(x,y,z),\ y=G(y,x,y)$ and $z=H(z,y,x)$.

Proof. We have $x_0 \leq_{P_1} F(x_0,y_0,z_0)=x_1$ (say), $y_0 \geq_{P_2} G(y_0,x_0,y_0)=y_1$ (say) and $z_0 \leq_{P_3} H(z_0,y_0,x_0)=z_1$ (say).
For $n \geq 1$, denote

$$x_n=F(x_{n-1},y_{n-1},z_{n-1}),\ \ y_n=G(y_{n-1},x_{n-1},y_{n-1})$$

and

$$z_n=H(z_{n-1},y_{n-1},x_{n-1}).$$

Then we get

$$F^{n+1}(x_0,y_0,z_0)=x_{n+1},\ \ G^{n+1}(y_0,x_0,y_0)=y_{n+1}$$

and

$$H^{n+1}(z_0,y_0,x_0)=z_{n+1}.$$

Due to the mixed monotone property, it is easy to show that

$$\begin{aligned} x_2&=F(x_1,y_1,z_1) \geq_{P_1} F(x_0,y_1,z_1) \geq_{P_1} F(x_0,y_0,z_1) \geq_{P_1} F(x_0,y_0,z_0)=x_1,\\ y_2&=G(y_1,x_1,y_1) \leq_{P_2} G(y_0,x_1,y_0) \leq_{P_2} G(y_0,x_0,y_0)=y_1,\\ z_2&=H(z_1,y_1,x_1) \geq_{P_3} H(z_0,y_1,x_1) \geq_{P_3} H(z_0,y_0,x_1) \geq_{P_3} H(z_0,y_0,x_0)=z_1. \end{aligned}$$

Thus we get three sequences as follows

$$\begin{aligned} &x_0 \leq_{P_1} x_1 \leq_{P_1} x_2 \leq_{P_1} \cdots \leq_{P_1} x_n \leq_{P_1} \cdots,\\ &y_0 \geq_{P_2} y_1 \geq_{P_2} y_2 \geq_{P_2} \cdots \geq_{P_2} y_n \geq_{P_2} \cdots,\\ &z_0 \leq_{P_3} z_1 \leq_{P_3} z_2 \leq_{P_3} \cdots \leq_{P_3} z_n \leq_{P_3} \cdots. \end{aligned}$$

Denote

$$D_n^x=d_X(x_{n-1},x_n),\ \ D_n^y=d_Y(y_{n-1},y_n),\ \ D_n^z=d_Z(z_{n-1},z_n).$$

Then,

$$\begin{aligned} D_2^x &= d_X(x_1, x_2) \\ &= d_X(F(x_0, y_0, z_0), F(x_1, y_1, z_1)) \\ &\leq j\ d_X(x_0, x_1) + k\ d_Y(y_0, y_1) + l\ d_Z(z_0, z_1) \\ &= j\ D_1^x + k\ D_1^y + l\ D_1^z. \end{aligned}$$

Similarly

$$\begin{aligned} D_2^y &= d_Y(y_1, y_2) \\ &= d_Y(G(y_0, x_0, y_0), G(y_1, x_1, y_1)) \\ &\leq (j+l)\ d_Y(y_0, y_1) + k\ d_X(x_0, x_1) \\ &= (j+l)\ D_1^y + k\ D_1^x \end{aligned}$$

and

$$\begin{aligned} D_2^z &= d_Z(z_1, z_2) \\ &= d_Z(H(z_0, y_0, x_0), H(z_1, y_1, x_1)) \\ &\leq j\ d_Z(z_0, z_1) + k\ d_Y(y_0, y_1) + l\ d_X(x_0, x_1) \\ &= j\ D_1^z + k\ D_1^y + l\ D_1^x. \end{aligned}$$

For the simplicity we will do the calculations by matrix method. Considering the coefficients of $D_1^x,\ D_1^y,\ D_1^z$ from the above inequalities we construct A.

$$A = \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \text{ denoted by } \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & b_1 & h_1 \end{pmatrix},$$

where $a_1 + b_1 + c_1 = d_1 + e_1 + f_1 = g_1 + b_1 + h_1 = j + k + l < 1$. Therefore

$$\begin{pmatrix} D_2^x \\ D_2^y \\ D_2^z \end{pmatrix} \leq \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & b_1 & h_1 \end{pmatrix} \begin{pmatrix} D_1^x \\ D_1^y \\ D_1^z \end{pmatrix}.$$

Also

$$\begin{aligned} D_3^x &= d_X(x_2, x_3) \\ &= d_X(F(x_1, y_1, z_1), F(x_2, y_2, z_2)) \\ &\leq j\ d_X(x_1, x_2) + k\ d_Y(y_1, y_2) + l\ d_Z(z_1, z_2) \\ &= j\ D_2^x + k\ D_2^y + l\ D_2^z \\ &\leq j\ [j\ D_1^x + k\ D_1^y + l\ D_1^z] + k\ [(j+l)\ D_1^y + k\ D_1^x] \\ &\quad + l\ [j\ D_1^z + k\ D_1^y + l\ D_1^x] \\ &= (j^2 + k^2 + l^2) D_1^x + (2jk + 2kl)\ D_1^y + 2jl\ D_1^z. \end{aligned}$$

Similarly

$$\begin{aligned}
D_3^y &= d_Y(y_2, y_3) \\
&= d_Y(G(y_1, x_1, y_1), G(y_2, x_2, y_2)) \\
&\leq (j+l)\ d_Y(y_1, y_2) + k\ d_X(x_1, x_2) \\
&= (j+l)\ D_2^y + k\ D_2^x \\
&\leq (j+l)\ [(j+l)\ D_1^y + k\ D_1^x] + k\ [j\ D_1^x + k\ D_1^y + l\ D_1^z] \\
&= (2jk + kl)\ D_1^x + [(j+l)^2 + k^2]\ D_1^y + kl\ D_1^z,
\end{aligned}$$

$$\begin{aligned}
D_3^z &= d_Z(z_2, z_3) \\
&= d_Z(H(z_1, y_1, x_1), H(z_2, y_2, x_2)) \\
&\leq j\ d_Z(z_1, z_2) + k\ d_Y(y_1, y_2) + l\ d_X(x_1, x_2) \\
&= j\ D_2^z + k\ D_2^y + l\ D_2^x \\
&\leq j\ [j\ D_1^z + k\ D_1^y + l\ D_1^x] + k\ [(j+l)\ D_1^y + k\ D_1^x] \\
&\quad + l\ [j\ D_1^x + k\ D_1^y + l\ D_1^z] \\
&= (jl + jl + k^2)\ D_1^x + [jk + k(j+l) + kl]\ D_1^y + (j^2 + l^2)\ D_1^z \\
&= (2jl + k^2)\ D_1^x + (2jk + 2kl)\ D_1^y + (j^2 + l^2)\ D_1^z.
\end{aligned}$$

Considering the coefficients of $D_1^x,\ D_1^y,\ D_1^z$ above inequalities we get A^2.

$$A^2 = \begin{pmatrix} j^2+k^2+l^2 & 2jk+2kl & 2jl \\ 2jk+kl & (j+l)^2+k^2 & kl \\ 2jl+k^2 & 2jk+2kl & j^2+l^2 \end{pmatrix} \text{denoted by} \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & b_2 & h_2 \end{pmatrix},$$

where $a_2 + b_2 + c_2 = d_2 + e_2 + f_2 = g_2 + b_2 + h_2 = (j+k+l)^2 < j+k+l < 1$. Therefore

$$\begin{pmatrix} D_3^x \\ D_3^y \\ D_3^z \end{pmatrix} \leq \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & b_2 & h_2 \end{pmatrix} \begin{pmatrix} D_1^x \\ D_1^y \\ D_1^z \end{pmatrix}.$$

Now we have to prove by induction that

$$A^n = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix},$$

where $a_n + b_n + c_n = d_n + e_n + f_n = g_n + b_n + h_n = (j+k+l)^n < j+k+l < 1$. We already have the result true for $n = 1$. Now we will assume that the result is true upto $n = m$.

Consider

$$A^{m+1} = A^m \cdot A = \begin{pmatrix} a_m & b_m & c_m \\ d_m & e_m & f_m \\ g_m & b_m & h_m \end{pmatrix} \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix}$$

$$= \begin{pmatrix} a_m j + b_m k + c_m l & a_m k + b_m(j+l) + c_m k & a_m l + c_m j \\ d_m j + e_m k + f_m l & d_m k + e_m(j+l) + f_m k & d_m l + f_m j \\ g_m j + b_m k + h_m l & g_m k + b_m(j+l) + h_m k & g_m l + h_m j \end{pmatrix},$$

where

$$\begin{aligned} & a_{m+1} + b_{m+1} + c_{m+1} \\ & = a_m j + b_m k + c_m l + a_m k + b_m(j+l) + c_m k + a_m l + c_m j \\ & = a_m(j+k+l) + b_m(k+j+l) + c_m(l+k+j) \\ & = (a_m + b_m + c_m)\ (j+k+l) \\ & = (j+k+l)^{m+1} < j+k+l < 1, \end{aligned}$$

$$\begin{aligned} & d_{m+1} + e_{m+1} + f_{m+1} \\ & = d_m j + e_m k + f_m l + d_m k + e_m(j+l) + f_m k + d_m l + f_m j \\ & = d_m(j+k+l) + e_m(k+j+l) + f_m(l+k+j) \\ & = (d_m + e_m + f_m)(j+k+l) \\ & = (j+k+l)^{m+1} < j+k+l < 1, \end{aligned}$$

$$\begin{aligned} & g_{m+1} + b_{m+1} + h_{m+1} \\ & = g_m j + b_m k + h_m l + g_m k + b_m(j+l) + h_m k + g_m l + h_m j \\ & = g_m(j+k+l) + b_m(k+j+l) + h_m(l+k+j) \\ & = (g_m + b_m + h_m)(k+j+l) \\ & = (j+k+l)^{m+1} < j+k+l < 1. \end{aligned}$$

Hence the result is true for every $n \in \mathbb{N}$. Therefore, we will get

$$\begin{pmatrix} D^x_{n+1} \\ D^y_{n+1} \\ D^z_{n+1} \end{pmatrix} \leq \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix} \begin{pmatrix} D^x_1 \\ D^y_1 \\ D^z_1 \end{pmatrix},$$

which implies that

$$D^x_{n+1} \leq a_n\ D^x_1 + b_n\ D^y_1 + c_n\ D^z_1, \tag{2.4}$$

$$D^y_{n+1} \leq d_n\ D^x_1 + e_n\ D^y_1 + f_n\ D^z_1, \tag{2.5}$$

$$D^z_{n+1} \leq g_n\ D^x_1 + b_n\ D^y_1 + h_n\ D^z_1. \tag{2.6}$$

Now we have to prove that $\{F^n(x_0,y_0,z_0)\}$, $\{G^n(y_0,x_0,y_0)\}$ and $\{H^n(z_0,y_0,x_0)\}$ are Cauchy sequences in X,Y and Z respectively. For $m>n$, using (2.4) we will get

$$\begin{aligned} d_X(x_n,x_m) &\le d_X(x_n,x_{n+1})+d_X(x_{n+1},x_{n+2})+\cdots+d_X(x_{m-1},x_m)\\ &= D^x_{n+1}+D^x_{n+2}+\cdots+D^x_m\\ &\le a_n\ D^x_1+b_n\ D^y_1+c_n\ D^z_1+a_{n+1}\ D^x_1+b_{n+1}\ D^y_1+c_{n+1}\ D^z_1\\ &\quad+\cdots+a_{m-1}\ D^x_1+b_{m-1}\ D^y_1+c_{m-1}\ D^z_1\\ &=(a_n+a_{n+1}+\cdots+a_{m-1})\ D^x_1+(b_n+b_{n+1}+\cdots+b_{m-1})D^y_1\\ &\quad+(c_n+c_{n+1}+\cdots+c_{m-1})D^z_1\\ &\le(\alpha^n+\alpha^{n+1}+\cdots+\alpha^{m-1})\ D^x_1+(\alpha^n+\alpha^{n+1}+\cdots+\alpha^{m-1})D^y_1\\ &\quad+(\alpha^n+\alpha^{n+1}+\cdots+\alpha^{m-1})D^z_1\quad \text{where } \alpha=j+k+l<1\\ &=(\alpha^n+\alpha^{n+1}+\cdots+\alpha^{m-1})\ (D^x_1+D^y_1+D^z_1)\\ &\le\frac{\alpha^n}{1-\alpha}\ (D^x_1+D^y_1+D^z_1)\\ &\to 0\ \text{ as }\ n\to\infty, \end{aligned}$$

which implies that $\{F^n(x_0,y_0,z_0)\}$ is a Cauchy sequence in X. Similarly using (2.5) and (2.6) we can prove that $\{G^n(y_0,x_0,y_0)\}$ and $\{H^n(z_0,y_0,x_0)\}$ are Cauchy sequences in Y and Z respectively. Since X,Y and Z are complete metric spaces, there exist $(x,y,z)\in X\times Y\times Z$ such that $\lim_{n\to\infty}F^n(x_0,y_0,z_0)=x$, $\lim_{n\to\infty}G^n(y_0,x_0,y_0)=y$ and $\lim_{n\to\infty}H^n(z_0,y_0,x_0)=z$.

Now we have to prove the existence of FGH-tripled fixed points. Consider,

$$\begin{aligned} &d_X(F(x,y,z),x)\\ &=\lim_{n\to\infty} d_X(F(F^n(x_0,y_0,z_0),G^n(y_0,x_0,y_0),H^n(z_0,y_0,x_0)),F^n(x_0,y_0,z_0))\\ &=\lim_{n\to\infty} d_X(F^{n+1}(x_0,y_0,z_0),F^n(x_0,y_0,z_0))\\ &=0. \end{aligned}$$

Therefore, $F(x,y,z)=x$. Similarly we can prove

$$G(y,x,y)=y\quad\text{and}\quad H(z,y,x)=z.$$

□

Setting $X=Y=Z$ and $F=G=H$ in Theorem 2.5 we get the following theorem of Berinde and Borcut as a corollary to our result.

Corollary 2.6. ([6, Theorem 7]) *Let $(X,\le)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space.*

Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist constants $j, k, l \in [0, 1)$ with $j+k+l < 1$ for which

$$d(F(x, y, z), F(u, v, w)) \leq j\ d(x, u) + k\ d(y, v) + l\ d(z, w);$$

for all $x \geq u,\ y \leq v,\ z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0),\ y_0 \geq F(y_0, x_0, y_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $x = F(x, y, z),\ y = F(y, x, y)$ and $z = F(z, y, x)$.

In order to prove the next theorem we define a metric d on the Cartesian product $X \times Y \times Z$ as follows:

$$d((x, y, z), (u, v, w)) = d_X(x, u) + d_Y(y, v) + d_Z(z, w)$$

for every $(x, y, z), (u, v, w) \in X \times Y \times Z$.

Theorem 2.7. *Let $(X, \leq_{P_1}, d_X), (Y, \leq_{P_2}, d_Y)$ and $(Z, \leq_{P_3}, d_Z)$ be three partially ordered complete metric spaces. Let $F : X\times Y\times Z \to X,\ G : Y\times X\times Y \to Y$ and $H : Z\times Y\times X \to Z$ be three continuous functions having the mixed monotone property. Assume that there exist constants $j, k, l \in [0, 1)$ with $j+k+l < 1$ such that*

$$\begin{aligned}
&d_X(F(x, y, z), F(u, v, w))\\
&\leq j\ d_X(x, u) + k\ d_Y(y, v) + l\ d_Z(z, w);\ \ \forall\, x \geq_{P_1} u,\ y \leq_{P_2} v,\ z \geq_{P_3} w,\\
&d_Y(G(y, x, y'), G(v, u, v'))\\
&\leq j\ d_Y(y, v) + k\ d_X(x, u) + l\ d_Y(y', v');\ \ \forall\, y \geq_{P_2} v,\ x \leq_{P_1} u,\ y' \geq_{P_2} v',\\
&d_Z(H(z, y, x), H(w, v, u))\\
&\leq j\ d_Z(z, w) + k\ d_Y(y, v) + l\ d_X(x, u);\ \ \forall\, x \geq_{P_1} u,\ y \leq_{P_2} v,\ z \geq_{P_3} w.
\end{aligned}$$

If there exist $(x_0, y_0, z_0) \in X \times Y \times Z$ such that $x_0 \leq_{P_1} F(x_0, y_0, z_0), y_0 \geq_{P_2} G(y_0, x_0, y_0)$ and $z_0 \leq_{P_3} H(z_0, y_0, x_0)$, and for every $(x, y, z), (x^*, y^*, z^*) \in X \times Y \times Z$ there exist a $(u, v, w) \in X \times Y \times Z$ that is comparable to both (x, y, z) and (x^*, y^*, z^*), then there exist a unique FGH-tripled fixed point.

Proof. Following the proof of Theorem 2.5 we get existence of FGH-tripled fixed point. If $(x^*, y^*, z^*) \in X \times Y \times Z$ is another FGH-tripled fixed point, then we have to show that $d((x, y, z), (x^*, y^*, z^*)) = 0$, where

$$x = \lim_{n\to\infty} F^n(x_0, y_0, z_0),\ \ y = \lim_{n\to\infty} G^n(y_0, x_0, y_0)$$

and

$$z = \lim_{n\to\infty} H^n(z_0, y_0, x_0).$$

Consider two cases.

Case 1: If (x, y, z) is comparable to (x^*, y^*, z^*) with respect to the ordering in $X \times Y \times Z$, then for every $n = 0, 1, 2, \cdots$, $(F^n(x, y, z), G^n(y, x, y), H^n(z, y, x)) = (x, y, z)$ is comparable to

$$(F^n(x^*, y^*, z^*), G^n(y^*, x^*, y^*), H^n(z^*, y^*, x^*)) = (x^*, y^*, z^*).$$

Also,

$$\begin{aligned}
&d((x, y, z), (x^*, y^*, z^*)) \\
&= d_X(x, x^*) + d_Y(y, y^*) + d_Z(z, z^*) \\
&= d_X(F^n(x, y, z), F^n(x^*, y^*, z^*)) + d_Y(G^n(y, x, y), G^n(y^*, x^*, y^*)) \\
&\quad + d_Z(H^n(z, y, x), H^n(z^*, y^*, x^*)) \\
&= D_n^x + D_n^y + D_n^z,
\end{aligned}$$

where

$$\begin{aligned}
D_n^x &= d_X(F^n(x, y, z), F^n(x^*, y^*, z^*)), \\
D_n^y &= d_Y(G^n(y, x, y), G^n(y^*, x^*, y^*)), \\
D_n^z &= d_Z(H^n(z, y, x), H^n(z^*, y^*, x^*)).
\end{aligned}$$

For $n = 1$,

$$\begin{aligned}
D_1^x &= d_X(F(x, y, z), F(x^*, y^*, z^*)) \leq j\ d_X(x, x^*) + k\ d_Y(y, y^*) + l\ d_Z(z, z^*), \\
D_1^y &= d_Y(G(y, x, y), G(y^*, x^*, y^*)) \leq (j + l)\ d_Y(y, y^*) + k\ d_X(x, x^*), \\
D_1^z &= d_Z(H(z, y, x), H(z^*, y^*, x^*)) \leq j\ d_Z(z, z^*) + k\ d_Y(y, y^*) + l\ d_X(x, x^*).
\end{aligned}$$

For the simplicity of calculations we use matrix method. Let

$$A = \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \text{ and denote it by } \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & b_1 & h_1 \end{pmatrix},$$

where $a_1 + b_1 + c_1 = d_1 + e_1 + f_1 = g_1 + b_1 + h_1 = j + k + l < 1$. Therefore

$$\begin{pmatrix} D_1^x \\ D_1^y \\ D_1^z \end{pmatrix} \leq \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \begin{pmatrix} D_1^x \\ D_1^y \\ D_1^z \end{pmatrix}.$$

For $n = 2$,

$$\begin{aligned}
D_2^x &= d_X(F^2(x, y, z), F^2(x^*, y^*, z^*)) \\
&= d_X((F(F(x, y, z), G(y, x, y), H(z, y, x)), \\
&\qquad F(F(x^*, y^*, z^*), G(y^*, x^*, y^*), H(z^*, y^*, x^*))) \\
&\leq j\ d_X(F(x, y, z), F(x^*, y^*, z^*)) + k\ d_Y(G(y, x, y), G(y^*, x^*, y^*)) \\
&\quad + l\ d_Z(H(z, y, x), H(z^*, y^*, x^*))
\end{aligned}$$

$$
\begin{aligned}
&\leq j[j\ d_X(x,x^*) + k\ d_Y(y,y^*) + l\ d_Z(z,z^*)]\\
&\quad + k\ [(j+l)\ d_Y(y,y^*) + k\ d_X(x,x^*)]\\
&\quad + l\ [j\ d_Z(z,z^*) + k\ d_Y(y,y^*) + l\ d_X(x,x^*)]\\
&= (j^2+k^2+l^2)\ d_X(x,x^*) + (2kj+2kl)\ d_Y(y,y^*) + 2jl\ d_Z(z,z^*),
\end{aligned}
$$

$$
\begin{aligned}
D_2^y &= d_Y(G^2(y,x,y), G^2(y^*,x^*,y^*))\\
&= d_Y((G(G(y,x,y), F(x,y,z), G(y,x,y)),\\
&\qquad G(G(y^*,x^*,y^*), F(x^*,y^*,z^*), G(y^*,x^*,y^*)))\\
&\leq (j+l)\ d_Y(G(y,x,y), G(y^*,x^*,y^*)) + k\ d_X(F(x,y,z), F(x^*,y^*,z^*))\\
&\leq (j+l)\ [(j+l)\ d_Y(y,y^*) + k\ d_X(x,x^*)]\\
&\quad + k\ [j\ d_X(x,x^*) + k\ d_Y(y,y^*) + l\ d_Z(z,z^*)]\\
&= (2kj+lk)\ d_X(x,x^*) + [(j+l)^2 + k^2]\ d_Y(y,y^*) + kl\ d_Z(z,z^*),
\end{aligned}
$$

$$
\begin{aligned}
D_2^z &= d_Z(H^2(z,y,x), H^2(z^*,y^*,x^*))\\
&= d_Z((H(H(z,y,x), G(y,x,y), F(x,y,z)),\\
&\qquad H(H(z^*,y^*,x^*), G(y^*,x^*,y^*), F(x^*,y^*,z^*)))\\
&\leq j\ d_Z(H(z,y,x), H(z^*,y^*,x^*)) + k\ d_Y(G(y,x,y), G(y^*,x^*,y^*))\\
&\quad + l\ d_X(F(x,y,z), F(x^*,y^*,z^*))\\
&\leq j\ [j\ d_Z(z,z^*) + k\ d_Y(y,y^*) + l\ d_X(x,x^*)]\\
&\quad + k\ [(j+l)\ d_Y(y,y^*) + k\ d_X(x,x^*)]\\
&\quad + l\ [j\ d_X(x,x^*) + k\ d_Y(y,y^*) + l\ d_Z(z,z^*)]\\
&= (2jl+k^2)\ d_X(x,x^*) + (2jk+2kl)\ d_Y(y,y^*) + (j^2+l^2)\ d_Z(z,z^*).
\end{aligned}
$$

So we get,

$$
A^2 = \begin{pmatrix} j^2+k^2+l^2 & 2jk+2kl & 2jl \\ 2kj+lk & (j+l)^2+k^2 & kl \\ 2jl+k^2 & 2jk+2kl & j^2+l^2 \end{pmatrix} \text{denoted by} \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & b_2 & h_2 \end{pmatrix},
$$

where $a_2+b_2+c_2 = d_2+e_2+f_2 = g_2+b_2+h_2 = (j+k+l)^2 < j+k+l < 1$. Therefore

$$
\begin{pmatrix} D_2^x \\ D_2^y \\ D_2^z \end{pmatrix} \leq \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix}^2 \begin{pmatrix} D_1^x \\ D_1^y \\ D_1^z \end{pmatrix}.
$$

As by the same lines of Theorem 2.5 we can prove

$$
A^n = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix},
$$

where $a_n+b_n+c_n = d_n+e_n+f_n = g_n+b_n+h_n = (j+k+l)^n < j+k+l < 1$. Therefore we have

$$\begin{pmatrix} D_n^x \\ D_n^y \\ D_n^z \end{pmatrix} \le \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix}^n \begin{pmatrix} D_1^x \\ D_1^y \\ D_1^z \end{pmatrix},$$

that is,

$$\begin{aligned} \begin{pmatrix} D_n^x \\ D_n^y \\ D_n^z \end{pmatrix} &\le \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix} \begin{pmatrix} D_1^x \\ D_1^y \\ D_1^z \end{pmatrix} \\ &= \begin{pmatrix} a_n\ D_1^x + b_n\ D_1^y + c_n\ D_1^z \\ d_n\ D_1^x + e_n\ D_1^y + f_n\ D_1^z \\ g_n\ D_1^x + b_n\ D_1^y + h_n\ D_1^z \end{pmatrix} \\ &\le \begin{pmatrix} a_n + b_n + c_n \end{pmatrix} \begin{pmatrix} D_1^x + D_1^y + D_1^z \\ D_1^x + D_1^y + D_1^z \\ D_1^x + D_1^y + D_1^z \end{pmatrix} \\ &= \begin{pmatrix} j+k+l \end{pmatrix}^n \begin{pmatrix} D_1^x + D_1^y + D_1^z \\ D_1^x + D_1^y + D_1^z \\ D_1^x + D_1^y + D_1^z \end{pmatrix}, \end{aligned}$$

that is,

$$\begin{aligned} D_n^x &\le (j+k+l)^n\ (D_1^x + D_1^y + D_1^z), \\ D_n^y &\le (j+k+l)^n\ (D_1^x + D_1^y + D_1^z), \\ D_n^z &\le (j+k+l)^n\ (D_1^x + D_1^y + D_1^z). \end{aligned}$$

Therefore

$$\begin{aligned} D_n^x + D_n^y + D_n^z &\le 3\ (j+k+l)^n\ (D_1^x + D_1^y + D_1^z) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore $d((x,y,z),(x^*,y^*,z^*)) = 0$.

Case 2: If (x,y,z) are not comparable to (x^*,y^*,z^*) then there exist an upper bound or a lower bound $(u,v,w) \in X \times Y \times Z$ of (x,y,z) and (x^*,y^*,z^*). Then for every $n = 1,2,3,\cdots$, $(F^n(u,v,w), G^n(v,u,v), H^n(w,v,u))$ is comparable to

$$(F^n(x,y,z), G^n(y,x,y), H^n(z,y,x)) = (x,y,z)$$

and to

$$(F^n(x^*,y^*,z^*), G^n(y^*,x^*,y^*), H^n(z^*,y^*,x^*)) = (x^*,y^*,z^*).$$

$$
\begin{aligned}
&d((x,y,z),(x^*,y^*,z^*))\\
&= d((F^n(x,y,z),G^n(y,x,y),H^n(z,y,x)),\\
&\qquad (F^n(x^*,y^*,z^*),G^n(y^*,x^*,y^*),H^n(z^*,y^*,x^*)))\\
&\le d((F^n(x,y,z),G^n(y,x,y),H^n(z,y,x)),\\
&\qquad (F^n(u,v,w),G^n(v,u,v),H^n(w,v,u)))\\
&\quad + d((F^n(u,v,w),G^n(v,u,v),H^n(w,v,u)),\\
&\qquad\quad (F^n(x^*,y^*,z^*),G^n(y^*,x^*,y^*),H^n(z^*,y^*,x^*)))\\
&= d_X((F^n(x,y,z),F^n(u,v,w)) + d_Y(G^n(y,x,y),G^n(v,u,v))\\
&\quad + d_Z(H^n(z,y,x),H^n(w,v,u)) + d_X((F^n(u,v,w),F^n(x^*,y^*,z^*))\\
&\quad + d_Y(G^n(v,u,v),G^n(y^*,x^*,y^*)) + d_Z(H^n(w,v,u),H^n(z^*,y^*,x^*))\\
&\le 3\ (j+k+l)^n\ [(d_X(x,u)+d_Y(y,v)+d_Z(z,w)) + d_X(u,x^*)\\
&\quad + d_Y(v,y^*) + d_Z(w,z^*)]\\
&\to 0 \quad \text{as} \ \ n\to\infty.
\end{aligned}
$$

Therefore, $d((x,y,z),(x^*,y^*,z^*)) = 0$. □

Setting $X = Y = Z$ and $F = G = H$ in Theorem 2.7 we get the following theorem of Berinde and Borcut as a corollary to our result.

Corollary 2.8. ([6, Theorem 9]) *Let $(X,\le)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X\times X\times X \to X$ be a continuous mapping having the mixed monotone mapping on X. Assume that there exist constants $j,k,l \in [0,1)$ with $j+k+l < 1$ for which*

$$d(F(x,y,z),F(u,v,w)) \le j\ d(x,u) + k\ d(y,v) + l\ d(z,w);$$

for all $x \ge u,\ y \le v,\ z \ge w$. If there exist $x_0,y_0,z_0 \in X$ such that

$$x_0 \le F(x_0,y_0,z_0),\ \ y_0 \ge F(y_0,x_0,y_0),\ \ z_0 \le F(z_0,y_0,x_0)$$

and for every $(x,y,z),(x_1,y_1,z_1) \in X\times X\times X$, there exist a $(u,v,w) \in X\times X\times X$ that is comparable to (x,y,z) and (x_1,y_1,z_1) then we obtain a unique tripled fixed point of F.

Remark 2.9. We can replace the continuity of F,G and H in Theorem 2.5 by other properties in order to get the existence of FGH-tripled fixed point as we see in the following theorem.

Theorem 2.10. *Let $(X,\le_{P_1},d_X),(Y,\le_{P_2},d_Y)$ and $(Z,\le_{P_3},d_Z)$ be three partially ordered complete metric spaces and $F : X\times Y\times Z \to X, G : Y\times X\times Y \to$*

Y and $H : Z\times Y\times X \to Z$ be three mappings having the mixed monotone property on X. Assume that there exist constants $j,k,l \in [0,1)$ with $j+k+l<1$ such that

$$\begin{aligned}&d_X(F(x,y,z),F(u,v,w))\\ &\le j\ d_X(x,u)+k\ d_Y(y,v)+l\ d_Z(z,w);\ \ \forall\, x\ge_{P_1}u,\ y\le_{P_2}v,\ z\ge_{P_3}w,\\ &d_Y(G(y,x,y'),G(v,u,v'))\\ &\le j\ d_Y(y,v)+k\ d_X(x,u)+l\ d_Y(y',v');\ \ \forall\, y\ge_{P_2}v,\ x\le_{P_1}u,\ y'\ge_{P_2}v',\\ &d_Z(H(z,y,x),H(w,v,u))\\ &\le j\ d_Z(z,w)+k\ d_Y(y,v)+l\ d_X(x,u);\ \ \forall\, x\ge_{P_1}u,\ y\le_{P_2}v,\ z\ge_{P_3}w.\end{aligned}$$

Further assume that X,Y and Z have the following properties:

(i) *if a non-decreasing sequence $\{x_n\}\to x$, then $x_n\le_{P_1}x$ for every n.*
(ii) *if a non-increasing sequence $\{y_n\}\to y$, then $y_n\ge_{P_2}y$ for every n.*
(iii) *if a non-decreasing sequence $\{z_n\}\to z$, then $z_n\le_{P_3}z$ for every n.*

If there exist $(x_0,y_0,z_0)\in X\times Y\times Z$ such that

$$x_0\le_{P_1}F(x_0,y_0,z_0),\ \ y_0\ge_{P_2}G(y_0,x_0,y_0)\ \ \textit{and}\ \ z_0\le_{P_3}H(z_0,y_0,x_0),$$

then there exist FGH-tripled fixed point.

Proof. Following as in the proof of Theorem 2.5, we get

$$\lim_{n\to\infty}F^n(x_0,y_0,z_0)=x,\ \lim_{n\to\infty}G^n(y_0,x_0,y_0)=y\ \text{ and }\ \lim_{n\to\infty}H^n(z_0,y_0,x_0)=z.$$

We have,

$$\begin{aligned}&d_X(F(x,y,z),x)\\ &\le d_X(F(x,y,z),F^{n+1}(x_0,y_0,z_0))+d_X(F^{n+1}(x_0,y_0,z_0),x)\\ &=d_X(F(x,y,z),F(F^n(x_0,y_0,z_0),G^n(y_0,x_0,y_0),H^n(z_0,y_0,x_0)))\\ &\quad +d_X(F^{n+1}(x_0,y_0,z_0),x)\\ &\le j\ d_X(x,F^n(x_0,y_0,z_0))+k\ d_Y(y,G^n(y_0,x_0,y_0))\\ &\quad +l\ d_Z(z,H^n(z_0,y_0,x_0))+d_X(F^{n+1}(x_0,y_0,z_0),x)\\ &\to 0\quad\text{as}\ \ n\to\infty.\end{aligned}$$

Therefore $F(x,y,z)=x$. Similarly we can prove that

$$G(y,x,y)=y\quad\text{and}\quad H(z,y,x)=z.$$

□

Setting $X=Y=Z$ and $F=G=H$ in Theorem 2.10 we get following result as a corollary.

Corollary 2.11. ([6, Theorem 8]) *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a mapping having the mixed monotone mapping on X. Assume that there exist the constants $j, k, l \in [0, 1)$ with $j + k + l < 1$ for which*

$$d(F(x, y, z), F(u, v, w)) \leq j\ d(x, u) + k\ d(y, v) + l\ d(z, w);$$

for all $x \geq u$, $y \leq v$, $z \geq w$. Assume that X has the following properties:

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for every n.*
(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \geq y$ for every n.*

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \;\; y_0 \geq F(y_0, x_0, y_0) \;\text{ and }\; z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$x = F(x, y, z), \;\; y = F(y, x, y) \;\text{ and }\; z = F(z, y, x).$$

Remark 2.12. By adding the following condition to Theorem 2.10 we get the uniqueness of FGH-tripled fixed point: "for every $(x, y, z), (x^*, y^*, z^*) \in X \times Y \times Z$ there exist a $(u, v, w) \in X \times Y \times Z$ that is comparable to both (x, y, z) and (x^*, y^*, z^*)".

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