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# DEMICLOSED PRINCIPLE AND CONVERGENCE THEOREMS FOR GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CAT($\kappa$) SPACES

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**Abstract.** The aim of this paper is to study modified $S$-iteration process for generalized asymptotically nonexpansive mappings and establish demiclosed principle, existence and convergence theorems for the iteration scheme and mappings in the setting of CAT($\kappa$) spaces with $k > 0$. Our results extend and generalize the previous works from the current existing literature.

## 1. Introduction

A CAT($\kappa$) space is a geodesic metric space whose geodesic triangle is thinner than the corresponding comparison triangle in a model space with curvature $\kappa$ for $\kappa \in \mathbb{R}$. The initials are in honour of E. Cartan, A.D. Alexandrov and V.A. Toponogov, who have made important contribution to the understanding of curvature via inequalities for the distance function.

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Fixed point theory in CAT($\kappa$) space was first studied by Kirk (see [16, 17]). His works were followed by a series of new works by many authors, mainly focusing on CAT(0) spaces (see, *e.g.*, [4, 8, 9, 10, 12, 20, 21]). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT($\kappa$) space with $\kappa \leq 0$ since any CAT($\kappa$) space is a CAT($m$) space for every $m \geq \kappa$ (see [5], "Metric spaces of non-positive curvature").

The class of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [13] in 1972, as an important generalization of the class of nonexpansive mapping and they proved that if $K$ is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self mapping of $K$ has a fixed point.

There are number of papers dealing with the approximation of fixed points of asymptotically nonexpansive mapping and their generalizations in uniformly convex Banach spaces using modified Mann and Ishikawa iteration processes were studied by many authors (see, *e.g.*, [23, 24, 29, 30, 33, 34, 35, 36]).

The concept of $\Delta$-convergence in a general metric space was introduced by Lim [22]. In 2008, Kirk and Panyanak [18] used the notion of $\Delta$-convergence introduced by Lim [22] to prove in the CAT(0) space and analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [11] obtained $\Delta$-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space. Since then, the existence problem and the $\Delta$-convergence problem of iterative sequences to a fixed point for nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically quasi- nonexpansive mapping, total asymptotically nonexpansive mapping, generalized asymptotically quasi-nonexpansive mapping and asymptotically quasi-nonexpansive type mappings through Picard, Mann [25], Ishikawa[14], modified Agarwal *et al.* [2] have been rapidly developed in the framework of CAT(0) space and many papers have appeared in this direction (see, *e.g.*, [1, 8, 11, 15, 19, 26, 28, 31]).

Recently, Kumam, Saluja and Nashine [19] studied modified $S$-iteration process involving two mappings and investigate the existence and convergence theorems in the setting of CAT(0) spaces for a class of mapping which is wider than that of asymptotically nonexpansive mappings.

Very recently, Saluja and Postolache [32] studied modified $S$-iteration process for two asymptotically nonexpansive mappings in the intermediate sense in the framework of CAT(0) spaces and investigate the existence and convergence theorems for the mentioned iteration scheme and mappings.

The purpose of this article is to establish $\Delta$-convergence and strong convergence of modified modified $S$-iteration process for a class of mappings which is wider than that of nonexpansive and asymptotically nonexpansive mappings in CAT($\kappa$) spaces with $\kappa > 0$. Our results extend and generalize several results given in the currnt existing literature.

Let $F(T) = \{x \in K : Tx = x\}$ denotes the set of fixed point of the mapping $T$. We begin with the following definitions.

**Definition 1.1.** Let $(X, d)$ be a metric space and $K$ be its nonempty subset. Then the mapping $T\colon K \to K$ said to be:

(1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;
(2) asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$ for all $x, y \in K$ and $n \geq 1$;
(3) uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L\, d(x, y)$ for all $x, y \in K$ and $n \geq 1$;
(4) semi-compact if for a sequence $\{x_n\}$ in $K$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$ as $k \to \infty$;
(5) a sequence $\{x_n\}$ in $K$ is called approximate fixed point sequence for $T$ (AFPS, in short) if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

It is easy to see that every nonexpansive mapping is asymptotically nonexpansive with the constant sequence $\{1\}$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [13] in 1972, as an important generalization of the class of nonxpansive mappings and they proved that if $K$ is a nonempty closed convex subset of a real uniformly convex Banach space, then every asymptotically nonexpansive self mapping on $K$ has a fixed point.

$T$ is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big(d(T^n x, T^n y) - d(x, y)\Big) \leq 0. \tag{1.1}$$

Putting $c_n = \max\big\{0, \sup_{x,y\in K}(d(T^n x, T^n y) - d(x, y))\big\}$, we see that $c_n \to 0$ as $n \to \infty$. Then (1.1) is reduced to the following:

$$d(T^n x, T^n y) \;\; \leq \;\; d(x, y) + c_n, \; \forall\, x, y \in K, \, n \geq 1.$$

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck *et al.* [6] as a generalization of the class of asymptotically nonexpansive mappings. It is known that if $K$ is a nonempty closed convex and bounded subset of a real Hilbert space, then every asymptotically nonexpansive self-mapping in the intermediate sense has a fixed point (see [38], for more details).

$T$ is said to be generalized asymptotically nonexpansive [3] if it is continuous and there exists a positive sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big( d(T^n x, T^n y) - k_n\, d(x,y) \Big) \leq 0. \tag{1.2}$$

Putting $c_n = \max\big\{0, \sup_{x,y\in K}(d(T^n x, T^n y) - k_n\, d(x,y))\big\}$, we see that $c_n \to 0$ as $n \to \infty$. Then (1.2) is reduced to the following:

$$d(T^n x, T^n y) \quad \leq \quad k_n\, d(x,y) + c_n,\ \forall\, x,y \in K,\ n \geq 1.$$

We remark that if $k_n = 1$ for all $n$, then the class of generalized asymptotically nonexpansive mappings is reduced to the class of asymptotically nonexpansive mappings in the intermediate sense.

Let $(X,d)$ be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0,l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, $c$ is an isometry, and $d(x,y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) *segment* joining $x$ and $y$. We say that $X$ is (i) a *geodesic space* if any two points of $X$ are joined by a geodesic and (ii) a *uniquely geodesic* if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$, which we will denote by $[x,y]$, called the segment joining $x$ to $y$. This means that $z \in [x,\, y]$ if and only if $d(x,z) = (1-\alpha)d(x,y)$ and $d(y,z) = \alpha\, d(x,y)$. In this case, we write $z = \alpha\, x \oplus (1-\alpha)y$. The space $(X,d)$ is said to be a geodesic space (D-geodesic space) if every two points of $X$ (every two points of distance smaller than D) are joined by a geodesic, and $X$ is said to be uniquely geodesic (D-uniquely geodesic) if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$ (for $x, y \in X$ with $d(x,y) < D$). A subset $K$ of $X$ is said to be convex if $K$ includes every geodesic segment joining any two of its points. The set $K$ is said to be bounded if $diam(K) := \sup\{d(x,y) : x, y \in K\} < \infty$.

The model spaces $M^2_\kappa$ are defined as follows.

**Definition 1.2.** Given a real number $\kappa$, we denote by $M^2_\kappa$ the following metric spaces:

(i) if $\kappa = 0$ then $M^2_\kappa$ is Euclidean space $\mathbb{E}^n$;
(ii) if $\kappa > 0$ then $M^2_\kappa$ is obtained from the sphere $\mathbb{S}^n$ by multiplying the distance function by $\frac{1}{\sqrt{\kappa}}$;
(iii) if $\kappa < 0$ then $M^2_\kappa$ is obtained from hyperbolic space $\mathbb{H}^n$ by by multiplying the distance function by $\frac{1}{\sqrt{-\kappa}}$.

The metric space $(X,d)$ is called a CAT($\kappa$) space if it is $D_\kappa$-geodesic and any geodesic triangle in $X$ of perimeter less than $2D_\kappa$ satisfies the CAT($\kappa$) inequality.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $M_k^2$ such that $d(x_1, x_2) = d_{M_k^2}(\overline{x_1}, \overline{x_2})$, $d(x_2, x_3) = d_{M_k^2}(\overline{x_2}, \overline{x_3})$ and $d(x_3, x_1) = d_{M_k^2}(\overline{x_3}, \overline{x_1})$. If $\kappa \leq 0$, then such a comparison triangle always exists in $M_\kappa^2$. If $\kappa > 0$, then such a triangle exists whenever $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = d_{M_\kappa^2}(\bar{x}, \bar{p})$.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in $X$ is said to satisfy the CAT($\kappa$) inequality if for any $p, q \in \triangle(x_1, x_2, x_3)$ and for their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x_1}, \bar{x_2}, \bar{x_3})$, one has $d(p, q) = d_{M_\kappa^2}(\overline{p}, \overline{q})$.

**Definition 1.3.** If $k \leq 0$, then $X$ is called a CAT($\kappa$) space if and only if $X$ is a geodesic space such that all of its geodesic triangles satisfy the CAT($\kappa$) inequality.

If $\kappa > 0$, then $X$ is called a CAT($\kappa$) space if and only if $X$ is $D_\kappa$-geodesic and any geodesic triangle $\triangle(x_1, x_2, x_3)$ in $X$ with $d(x_1, x_2)+d(x_2, x_3)+d(x_3, x_1) < 2D_\kappa$ satisfies the CAT($\kappa$) inequality.

Notice that in a CAT(0) space $(X, d)$ if $x, y, z \in X$, then the CAT(0) inequality implies

$$d^2\Big(x, \frac{y \oplus z}{2}\Big) \leq \frac{1}{2}\, d^2(x, y) + \frac{1}{2}\, d^2(x, z) - \frac{1}{4}\, d^2(y, z). \tag{CN}$$

This is the $(CN)$ inequality of Bruhat and Tits [7]. This inequality is extended by Dhompongsa and Panyanak in [11] as

$$\begin{aligned} &d^2(z, \alpha x \oplus (1-\alpha)y) \\ &\leq \alpha d^2(z, x) + (1-\alpha) d^2(z, y) - \alpha(1-\alpha) d^2(x, y), \end{aligned} \tag{CN*}$$

for all $\alpha \in [0, 1]$ and $x, y, z \in X$. In fact, if $X$ is a geodesic space, then the following statements are equivalent:

(i) $X$ is a CAT(0);
(ii) $X$ satisfies $(CN)$ inequality;
(iii) $X$ satisfies $(CN^*)$ inequality.

Let $R \in (0, 2]$. Recall that a geodesic space $(X, d)$ is said to be *R-convex* for $R$ (see [27]) if for any three points $x, y, z \in X$, we have

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha)y) \;\; &\leq \;\; \alpha d^2(z, x) + (1-\alpha) d^2(z, y) \\ &\quad - \frac{R}{2} \alpha(1-\alpha) d^2(x, y). \end{aligned} \tag{1.3}$$

It follows from $(CN^*)$ that a geodesic space $(X, d)$ is a CAT(0) space if and only if $(X, d)$ is $R$-convex for $R$=2.

In the sequel we need the following lemma.

**Lemma 1.4.** ([5, p.176]) *Let $k > 0$ and $(X, d)$ be a complete CAT$(k)$ space with diam$(X) = \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Then*

$$d((1-\alpha)x \oplus \alpha\, y, z) \le (1-\alpha)d(x, z) + \alpha\, d(y, z)$$

*for all $x, y, z \in X$ and $\alpha \in [0, 1]$.*

We now recall some elementary facts about CAT($\kappa$) spaces. Most of them are proved in the framework of CAT(1) spaces. For completeness, we state the results in CAT($\kappa$) with $\kappa > 0$.

Let $\{x_n\}$ be a bounded sequence in a CAT($\kappa$) space $(X, d)$. For $x \in X$, set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \big\{x \in X : r(\{x_n\}) = r(x, \{x_n\})\big\}.$$

It is known from Proposition 4.1 of [12] that in a CAT$(k)$ space with $diam(X) = \frac{\pi}{2\sqrt{\kappa}}$, $A(\{x_n\})$ consists of exactly one point. We now give the concept of $\Delta$-convergence and collect some of its basic properties.

**Definition 1.5.** ([18, 22]) A sequence $\{x_n\}$ in $X$ is said to $\Delta$-converge to $x \in X$ if $x$ is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = x$ and call $x$ is the $\Delta$-limit of $\{x_n\}$.

Recall that a subset $K$ in a metric space $X$ is said to be $\Delta$-compact [22] if every sequence in $K$ has a $\Delta$-convergent subsequence. A mapping $T$ from a metric space $X$ to a metric space $Y$ is said to be completely continuous if $T(K)$ is a compact subset of $Y$ whenever $K$ is a $\Delta$-compact subset of $X$.

**Lemma 1.6.** *Let $\kappa > 0$ and $(X, d)$ be a complete CAT$(\kappa)$ space with diam$(X) = \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Then the following statements hold:*

(i) ([12, Corollary 4.4]) *Every sequence in $X$ has a $\Delta$-convergent subsequence.*

(ii) ([12, Proposition 4.5]) *If $\{x_n\} \subseteq X$ and $\Delta\text{-}\lim_{n\to\infty} x_n = x$, then*

$$x \in \bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, \dots\},$$

*where* $\overline{conv}(A) = \bigcap\{B : B \supseteq A and\ B\ is\ closed\ and\ convex\}.$

By the uniqueness of asymptotic center, we can obtain the following lemma in [11].

**Lemma 1.7.** ([11, Lemma 2.8]) *Let* $\kappa > 0$ *and* $(X, d)$ *be a complete* $CAT(\kappa)$ *space with* $diam(X) = \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some* $\varepsilon \in (0, \pi/2)$. *If* $\{x_n\}$ *is a bounded sequence in* $X$ *with* $A(\{x_n\}) = \{x\}$ *and* $\{u_n\}$ *is a subsequence of* $\{x_n\}$ *with* $A(\{u_n\}) = \{u\}$ *and the sequence* $\{d(x_n, u)\}$ *converges, then* $x = u$.

**Lemma 1.8.** ([35]) *Let* $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *be sequences of non-negative numbers satisfying the inequality*

$$p_{n+1} \le (1 + q_n)p_n + r_n,\ \forall n \ge 1.$$

*If* $\sum_{n=1}^{\infty} q_n < \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$, *then* $\lim_{n\to\infty} p_n$ *exists.*

**Algorithm 1.** The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \\ x_{n+1} &= (1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \quad n \ge 1, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in (0,1) is called modified S-iterative sequence (see [2]).

If $T^n = T$ for all $n \ge 1$, then Algorithm 1 reduces to the following.

**Algorithm 2.** The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n)T x_n \oplus \alpha_n T y_n, \quad n \ge 1, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in (0,1) is called S-iterative sequence (see [2]).

**Algorithm 3.** The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{aligned} y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \\ x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \quad n \ge 1, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in [0,1] is called an Ishikawa iterative sequence (see [14]).

If $\beta_n = 0$ for all $n \ge 1$, then Algorithm 3 reduces to the following.

**Algorithm 4.** The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \quad n \ge 1, \tag{1.7}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in (0,1) is called a Mann iterative sequence (see [25]).

## 2. Main results

### 2.1. Existence theorems.

**Theorem 2.1.** *Let $\kappa > 0$ and $(X, d)$ be a complete CAT($\kappa$) space with diam($X$) $= \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let $K$ be a nonempty closed convex subset of $X$ and let $T\colon K \to K$ be a generalized asymptotically nonexpansive mapping. Then $T$ has a fixed point.*

*Proof.* Fix $x \in K$. We can consider the sequence $\{T^n x\}_{n=1}^{\infty}$ as a bounded sequence in $K$. Let $\phi$ be a function defined by

$$\phi\colon K \to [0, \infty), \quad \phi(u) = \limsup_{n\to\infty} d(T^n x, u), \quad \text{for all } u \in K.$$

Then there exists $z \in K$ such that $\phi(z) = \inf\{\Phi(u) : u \in K\}$. Since $T$ is generalized asymptotically nonexpansive mapping, for each $n, m \in \mathbb{N}$, we have

$$d(T^{n+m}x, T^m z) \leq k_m\, d(T^n x, z) + c_m.$$

On taking limit as $n \to \infty$, we obtain

$$\phi(T^m z) \quad \leq \quad k_m\, \phi(z) + c_m \tag{2.1}$$

for any $m \in \mathbb{N}$. This implies that

$$\lim_{m\to\infty} \phi(T^m z) \quad \leq \quad \phi(z). \tag{2.2}$$

In view of inequality (1.3), we obtain

$$\begin{aligned} d\Big(T^n x, \frac{T^m z \oplus T^h z}{2}\Big)^2 \quad &\leq \quad \frac{1}{2} d(T^n x, T^m z)^2 + \frac{1}{2} d(T^n x, T^h z)^2 \\ &\quad -\frac{R}{8} d(T^m z, T^h z)^2 \end{aligned}$$

which on taking limit as $n \to \infty$ gives

$$\begin{aligned} \phi(z)^2 \quad &\leq \quad \Phi\Big(\frac{T^m z \oplus T^h z}{2}\Big)^2 \\ &\leq \quad \frac{1}{2}\, \phi(T^m z)^2 + \frac{1}{2}\, \phi(T^h z)^2 - \frac{R}{8}\, d(T^m z, T^h z)^2. \end{aligned} \tag{2.3}$$

The above inequality yields

$$\frac{R}{8}\, d(T^m z, T^h z)^2 \quad \leq \quad \frac{1}{2}\, \phi(T^m z)^2 + \frac{1}{2}\, \phi(T^h z)^2 - \phi(z)^2. \tag{2.4}$$

By (2.2) and (2.4), we have $\limsup_{m,h\to\infty} d(T^m z, T^h z) \leq 0$. Therefore, $\{T^n z\}_{n=1}^{\infty}$ is a Cauchy sequence in $K$ and hence converges to some point $v \in K$. Since $T$ is continuous,

$$Tv = T\Big(\lim_{n\to\infty} T^n z\Big) = \lim_{n\to\infty} T^{n+1} z = v.$$

This shows that $T$ has a fixed point in $K$. This completes the proof. □

From Theorem 2.1 we shall now derive a result for CAT(0) space as follows.

**Corollary 2.2.** *Let $(X, d)$ be a complete CAT$(0)$ space and $K$ be a nonempty bounded, closed convex subset of $X$. If $T\colon K \to K$ is a generalized asymptotically nonexpansive mapping, then $T$ has a fixed point.*

*Proof.* It is well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT($\kappa$) space (see, [5]). Then $(K, d)$ is a CAT(0) space and hence it is a CAT($\kappa$) space for all $\kappa > 0$. Also note that $K$ is $R$-convex for $R = 2$. Since $K$ is bounded, we can chose $\varepsilon \in (0, \pi/2)$ and $\kappa > 0$ so that $diam(K) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$. The conclusion follows from Theorem 2.1. This completes the proof. □

2.2. **Demiclosed principle.**

**Theorem 2.3.** *Let $\kappa > 0$ and $(X, d)$ be a complete CAT$(\kappa)$ space with $diam(X) = \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let $K$ be a nonempty closed convex subset of $X$ and let $T\colon K \to K$ be a generalized asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for $T$ such that $\Delta$-$\lim_{n\to\infty} x_n = z$, then $z \in K$ and $z = Tz$.*

*Proof.* By Lemma 1.6, we get that $z \in K$. As in Theorem 2.1, we define $\phi(u) = \limsup_{n\to\infty} d(x_n, u)$ for each $u \in K$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, by induction we can show that $\lim_{n\to\infty} d(x_n, T^m x_n) = 0$ for some $m \in \mathbb{N}$ (cf. [37]). This implies that

$$\phi(u) \;=\; \limsup_{n\to\infty} d(T^m x_n, u) \quad \text{for each } u \in K \text{ and } m \in \mathbb{N}. \tag{2.5}$$

Taking $u = T^m z$ in (2.5), we have

$$\begin{aligned} \phi(T^m z) &= \limsup_{n\to\infty} d(T^m x_n, T^m z) \\ &\leq \limsup_{n\to\infty} [k_m\, d(x_n, z) + c_m]. \end{aligned} \tag{2.6}$$

Hence

$$\limsup_{m\to\infty} \phi(T^m z) \;\leq\; \phi(z). \tag{2.7}$$

In view of inequality (1.3), we have

$$d\Big(x_n, \frac{z \oplus T^m z}{2}\Big)^2 \leq \frac{1}{2}\, d(x_n, z)^2 + \frac{1}{2}\, d(x_n, T^m z)^2 - \frac{R}{8} d(z, T^m z)^2,$$

where $R = (\pi - 2\varepsilon) tan(\varepsilon)$. Since $\Delta - \lim_{n\to\infty} x_n = z$, letting $n \to \infty$, we get

$$\begin{aligned} \phi(z)^2 &\leq \Phi\Big(\frac{z \oplus T^m z}{2}\Big)^2 \\ &\leq \frac{1}{2}\, \phi(z)^2 + \frac{1}{2}\, \phi(T^m z)^2 - \frac{R}{8}\, d(z, T^m z)^2. \end{aligned} \tag{2.8}$$

This yields

$$d(z, T^m z)^2 \leq \frac{4}{R}\, [\phi(T^m z)^2 - \phi(z)^2]. \tag{2.9}$$

By (2.7) and (2.9), we have $\lim_{m\to\infty} d(z, T^m z) = 0$. Since $T$ is continuous,

$$Tz = T\Big(\lim_{m\to\infty} T^m z\Big) = \lim_{n\to\infty} T^{m+1} z = z.$$

This shows that $T$ has a fixed point in $K$. This completes the proof. □

From Theorem 2.3 we can derive the following result as follows.

**Corollary 2.4.** *Let $(X, d)$ be a complete CAT(0) space, $K$ be a nonempty bounded, closed convex subset of $X$ and $T: K \to K$ be a generalized asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for $T$ such that $\Delta$-$\lim_{n\to\infty} x_n = z$, then $z \in K$ and $z = Tz$.*

2.3. **Convergence theorems.** In this section, we prove the following lemmas using iteration scheme (1.4) needed in the sequel.

**Lemma 2.5.** *Let $\kappa > 0$ and $(X, d)$ be a complete CAT($\kappa$) space with diam($X$) $= \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let $K$ be a nonempty closed convex subset of $X$ and let $T: K \to K$ be a generalized asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{x_n\}$ be a sequence in $K$ defined by* (1.4). *Then* $\lim_{n\to\infty} d(x_n, p)$ *exists for each $p \in F(T)$.*

*Proof.* It follows from Theorem 2.1 that $F(T) \neq \emptyset$. Let $p \in F(T)$ and since $T$ is generalized asymptotically nonexpansive, by (1.4) and Lemma 1.4, we have

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n T^n x_n, p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d(T^n x_n, p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n [k_n\, d(x_n, p) + c_n] \\ &\leq (1-\beta_n) k_n d(x_n, p) + \beta_n k_n d(x_n, p) + \beta_n c_n \\ &\leq k_n\, d(x_n, p) + c_n. \end{aligned} \tag{2.10}$$

Finally, using (1.4), (2.10) and Lemma 1.4, we get

$$\begin{aligned} d(x_{n+1},p) &= d((1-\alpha_n)T^nx_n \oplus \alpha_n T^n y_n, p) \\ &\le (1-\alpha_n)d(T^nx_n,p) + \alpha_n d(T^ny_n,p) \\ &\le (1-\alpha_n)[k_n\, d(x_n,p)+c_n] + \alpha_n[k_n\, d(y_n,p)+c_n] \\ &= (1-\alpha_n)k_n\, d(x_n,p) + \alpha_n k_n\, d(y_n,p) + c_n \\ &\le (1-\alpha_n)k_n\, d(x_n,p) + \alpha_n k_n[k_n\, d(x_n,p)+c_n] + c_n \\ &\le k_n^2[(1-\alpha_n)d(x_n,p) + \alpha_n\, d(x_n,p)] + (1+k_n)c_n \\ &= k_n^2\, d(x_n,p) + (1+k_n)c_n \\ &= (1+v_n)d(x_n,p) + p_n, \end{aligned} \tag{2.11}$$

where $v_n = (k_n^2-1) = (k_n+1)(k_n-1)$ and $p_n = (1+k_n)c_n$. Since $\sum_{n=1}^{\infty}(k_n-1) < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, it follows that $\sum_{n=1}^{\infty} v_n < \infty$ and $\sum_{n=1}^{\infty} p_n < \infty$. Hence by Lemma 1.8, we get that $\lim_{n\to\infty} d(x_n,p)$ exists. This completes the proof. □

**Lemma 2.6.** *Let $\kappa > 0$ and $(X,d)$ be a complete $CAT(\kappa)$ space with $diam(X) = \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. Let $K$ be a nonempty closed convex subset of $X$ and let $T\colon K \to K$ be a generalized asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{x_n\}$ be a sequence in $K$ defined by (1.4). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

*Proof.* It follows from Theorem 2.1 that $F(T) \neq \emptyset$. Let $p \in F(T)$. From Lemma 2.5, we obtain $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F(T)$. We claim that $\lim_{n\to\infty} d(Tx_n,x_n) = 0$. Since $\{x_n\}$ is bounded, there exists $R' > 0$ such that $\{x_n\},\{y_n\} \subset B_{R'}(p)$ for all $n \ge 1$ with $R' < D_\kappa/2$. In view of (1.3), we have

$$\begin{aligned} &d(y_n,p)^2 \\ &= d^2((1-\beta_n)x_n \oplus \beta_n T^n x_n, p)^2 \\ &\le \beta_n d(T^nx_n,p)^2 + (1-\beta_n)d^2(x_n,p)^2 - \frac{R}{2}\beta_n(1-\beta_n)d(T^nx_n,x_n)^2 \\ &\le \beta_n[k_n\, d(x_n,p)+c_n]^2 + (1-\beta_n)d(x_n,p)^2 - \frac{R}{2}\beta_n(1-\beta_n)d(T^nx_n,x_n)^2 \\ &\le k_n^2\beta_n d(x_n,p)^2 + M\, c_n + (1-\beta_n)d(x_n,p)^2 \\ &\quad -\frac{R}{2}\beta_n(1-\beta_n)d(T^nx_n,x_n)^2 \\ &\le k_n^2\, d^2(x_n,p) + M\, c_n - \frac{R}{2}\beta_n(1-\beta_n)d(T^nz_n,x_n)^2 \end{aligned} \tag{2.12}$$

for some $M > 0$. This implies that

$$d(y_n,p)^2 \leq k_n^2\, d^2(x_n,p) + M\, c_n. \tag{2.13}$$

Finally, from (1.3) and using (2.13), we have

$$\begin{aligned} d(x_{n+1},p)^2 &= d((1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, p)^2 \\ &\leq \alpha_n d(T^n y_n,p)^2 + (1-\alpha_n) d(T^n x_n,p)^2 \\ &\quad -\frac{R}{2}\alpha_n(1-\alpha_n) d(T^n x_n, T^n y_n)^2 \\ &\leq \alpha_n[k_n\, d(y_n,p)+c_n]^2 + (1-\alpha_n)[k_n\, d(x_n,p)+c_n]^2 \\ &\quad -\frac{R}{2}\alpha_n(1-\alpha_n) d(T^n x_n, T^n y_n)^2 \\ &\leq \alpha_n k_n^2 d^2(y_n,p) + Pc_n + (1-\alpha_n)k_n^2 d^2(x_n,p) \\ &\quad +Qc_n - \frac{R}{2}\alpha_n(1-\alpha_n) d(T^n x_n, T^n y_n)^2 \\ &\leq \alpha_n k_n^2[k_n^2\, d^2(x_n,p) + M\, c_n] \\ &\quad +(P+Q)c_n + (1-\alpha_n)k_n^2 d^2(x_n,p) \\ &\quad -\frac{R}{2}\alpha_n(1-\alpha_n) d(T^n x_n, T^n y_n)^2 \\ &\leq k_n^4\, d^2(x_n,p) + (P+Q+M)k_n^2 c_n \\ &\quad -\frac{R}{2}\alpha_n(1-\alpha_n) d(T^n x_n, T^n y_n)^2 \\ &= [1+(k_n-1)\delta] d^2(x_n,p) + (P+Q+M)k_n^2 c_n \\ &\quad -\frac{R}{2}\alpha_n(1-\alpha_n) d(T^n x_n, T^n y_n)^2 \end{aligned} \tag{2.14}$$

for some $P$, $Q$, $\delta > 0$. This implies that

$$\begin{aligned} &\frac{R}{2}\alpha_n(1-\alpha_n) d(T^n x_n, T^n y_n)^2 \\ &\leq d(x_n,p)^2 - d(x_{n+1},p)^2 + (k_n-1)\delta\, d(x_n,p)^2 + (P+Q+M)k_n^2 c_n. \end{aligned}$$

Since $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty}(k_n-1) < \infty$ and $d(x_n,p) < R'$, we have

$$\frac{R}{2}\alpha_n(1-\alpha_n) d(T^n x_n, T^n y_n)^2 < \infty.$$

Hence by the fact that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} d(T^n x_n, T^n y_n) = 0. \tag{2.15}$$

Now, consider (2.12), we have

$$\begin{aligned}
&d(y_n,p)^2\\
&\le [1+(k_n^4-1)]d(x_n,p)^2 + Mc_n - \frac{R}{2}\beta_n(1-\beta_n)d(T^n x_n, x_n)^2\\
&\le [1+(k_n-1)\nu]d(x_n,p)^2 + Mc_n - \frac{R}{2}\beta_n(1-\beta_n)d(T^n x_n, x_n)^2 \qquad (2.16)
\end{aligned}$$

for some $\nu > 0$. Equation (2.16) yields

$$\begin{aligned}
&\frac{R}{2}\beta_n(1-\beta_n)d(T^n x_n, x_n)^2\\
&\le d(x_n,p)^2 - d(y_n,p)^2 + (k_n-1)\nu\, d(x_n,p)^2 + Mc_n.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty}(k_n-1) < \infty$, $d(x_n,p) < R'$ and $d(y_n,p) < R'$, we have

$$\frac{R}{2}\beta_n(1-\beta_n)d(T^n x_n, x_n)^2 < \infty.$$

Thus by the fact that $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty} d(T^n x_n, x_n) = 0. \qquad (2.17)$$

By the uniform continuity of $T$, we have

$$\lim_{n\to\infty} d(T^{n+1} x_n, Tx_n) = 0. \qquad (2.18)$$

It follows from (2.17) and the definition of $x_{n+1}$ and $y_n$, we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\le d(x_n, T^n y_n)\\
&\le d(x_n, T^n x_n) + d(T^n x_n, T^n y_n)\\
&\le d(x_n, T^n x_n) + k_n\, d(x_n, y_n) + c_n\\
&\le (1+k_n\beta_n)d(x_n, T^n x_n) + c_n\\
&\to 0 \quad \text{as } n\to\infty. \qquad (2.19)
\end{aligned}$$

By (2.17), (2.19) and uniform continuity of $T$, we have

$$\begin{aligned}
d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1})\\
&\quad + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n)\\
&\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1})\\
&\quad + k_{n+1}\, d(x_{n+1}, x_n) + c_{n+1} + d(T^{n+1}x_n, Tx_n)\\
&= (1+k_{n+1})\, d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1})\\
&\quad + d(T^{n+1}x_n, Tx_n) + c_{n+1} \to 0 \quad \text{as } n\to\infty. \qquad (2.20)
\end{aligned}$$

This completes the proof. □

Now, we are in a position to prove the $\Delta$-convergence and strong convergence theorems.

**Theorem 2.7.** *Let $\kappa > 0$ and $(X, d)$ be a complete CAT($\kappa$) space with diam($X$) $= \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let $K$ be a nonempty closed convex subset of $X$ and let $T \colon K \to K$ be a uniformly continuous generalized asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{x_n\}$ be a sequence in $K$ defined by (1.4). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. Then $\{x_n\}$ $\Delta$-converges to a fixed point of $T$.*

*Proof.* Let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We can complete the proof by showing that $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ consists of exactly one point. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 1.6, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta$-$\lim_n v_n = v \in K$. Hence $v \in F(T)$ by Lemma 2.5 and Theorem 2.3. Since $\lim_{n\to\infty} d(x_n, v)$ exists, so by Lemma 1.7, $v = u$, *i.e.*, $\omega_w(x_n) \subseteq F(T)$.

To show that $\{x_n\}$ $\Delta$-converges to a fixed point of $T$, it is sufficient to show that $\omega_w(x_n)$ consists of exactly one point.

Let $\{w_n\}$ be a subsequence of $\{x_n\}$ with $A(\{w_n\}) = \{w\}$ and let $A(\{x_n\}) = \{x\}$. Since $w \in \omega_w(x_n) \subseteq F(T)$ and by Lemma 2.5, $\lim_{n\to\infty} d(x_n, w)$ exists. Again by Lemma 2.5, we have $x = w \in F(T)$. Thus $\omega_w(x_n) = \{x\}$. This shows that $\{x_n\}$ $\Delta$-converges to a fixed point of $T$. This completes the proof. □

**Theorem 2.8.** *Let $\kappa > 0$ and $(X, d)$ be a complete CAT($\kappa$) space with diam($X$) $= \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let $K$ be a nonempty closed convex subset of $X$ and let $T \colon K \to K$ be a uniformly continuous generalized asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{x_n\}$ be a sequence in $K$ defined by (1.4). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Suppose that $T^m$ is semi-compact for some $m \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* By Lemma 2.6, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since $T$ is uniformly continuous, we have

$$d(x_n, T^m x_n) \leq d(x_n, Tx_n) + d(Tx_n, T^2 x_n) + \cdots + d(T^{m-1}x_n, T^m x_n) \to 0$$

as $n \to \infty$. That is, $\{x_n\}$ is an AFPS for $T^m$. By the semi-compactness of $T^m$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in K$ such that $\lim_{j\to\infty} x_{n_j} = p$. Again, by the uniform continuity of $T$, we have

$$d(Tp, p) \leq d(Tp, Tx_{n_j}) + d(Tx_{n_j}, x_{n_j}) + d(x_{n_j}, p) \to 0 \quad \text{as } j \to \infty.$$

That is $p \in F(T)$. By Lemma 2.5, $\lim_{n\to\infty} d(x_n, p)$ exists, thus $p$ is the strong limit of the sequence $\{x_n\}$ itself. This shows that the sequence $\{x_n\}$ converges strongly to a fixed point of $T$. This completes the proof. □

From Theorem 2.8 we can derive the following result as corollary.

**Corollary 2.9.** *Let $(X, d)$ be a complete CAT$(0)$ space, $K$ be a nonempty bounded, closed convex subset of $X$ and $T : K \to K$ be a uniformly continuous generalized asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{x_n\}$ be a sequence in $K$ defined by (1.4). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. Suppose that $T^m$ is semi-compact for some $m \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.*

**Example 2.10.** Let $X = \mathbb{R}$, $K = [-1, 1]$ and $T : K \to K$ be a mapping defined by

$$T(x) = \frac{x}{2}, \quad \text{if } x \in [-1, 1].$$

Thus, $T$ is a nonexpansive mapping and hence it is asymptotically nonexpansive mapping with constant sequence $\{1\}$. Also, $T$ is uniformly continuous on $[-1, 1]$. Thus $T$ is asymptotically nonexpansive mapping in the intermediate sene and hence it is generalized asymptotically nonexpansive mapping.

## 3. Conclusion

In this paper, we prove an existence result and a demiclosed principle for generalized asymptotically nonexpansive mapping. Also we establish a $\Delta$ convergence and some strong convergence theorems using iteration scheme (1.4) which contains modified Mann iteration scheme for a wider class of nonexpansive and asymptotically nonexpansive mappings in the setting of CAT$(\kappa)$ space with $\kappa > 0$. The results presented in this paper extend and generalize the previous works from the current existing literature.

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