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A N-ORDER ITERATIVE SCHEME FOR THE ROBIN PROBLEM FOR A NONLINEAR WAVE EQUATION WITH THE SOURCE TERM CONTAINING THE UNKNOWN BOUNDARY VALUES

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Abstract. In this paper, we consider the Robin problem for a nonlinear wave equation with the source term containing the unknown boundary values. By establishing a high order iterative scheme, we get a convergent sequence at a rate of order N to an unique local weak solution of the model.

1. Introduction

In this paper, we consider the Robin problem for a nonlinear wave equation with the source term containing the unknown boundary values as follows

$$u_{tt} - u_{xx} = f(x, t, u(x, t), u(0, t), u(1, t)), \ 0 < x < 1, \ 0 < t < T, \qquad (1.1)$$

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$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and h_0, $h_1 \geq 0$ are given constants with $h_0 + h_1 > 0$.

One of the methods is used to sovle nonlinear operator equation $F(u) = 0$ is Newton's method. Besides, the variants of Newton's method are also effectively used to solve this equation. Because of difficulties to find the exact solution u, constructing an approximating sequence $\{u_n\}$ and showing its convergence, we can not only estimate the error between the exact solution and the approximating values but also consider speed of convergence. If two consecutive approximating values are estimated by inequality $|u_{n+1} - u| \leq C\,|u_n - u|^N$, for some $C > 0$ and N is large, in this case, one speaks of *convergence of order* N. For the details, it can be found in, for example, [1], [16], [18] and references therein.

In [4], Long and Diem studied the linear recursive scheme associated with the nonlinear wave equation

$$u_{tt} - u_{xx} = f(x,t,u,u_x,u_t),\ 0 < x < 1,\ 0 < t < T, \tag{1.4}$$

associated with (1.2), (1.3).

In [11], a high order iterative scheme was established in order to get a convergent sequence at a rate of order N $(N \geq 1)$ to a local unique weak solution of a nonlinear wave equation as follows

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t)u_x\right) + \lambda u_t = f(x,t,u),\ 0 < x < 1,\ 0 < t < T, \tag{1.5}$$

associated with the homogeneous Dirichlet conditions, where $\lambda \neq 0$ is constant and μ, f are given functions.

By a high convergent method given in [11], Ngoc *et al.* [12] were also established a local unique weak solution of a nonlinear wave equation

$$u_{tt} - u_{xx} = f(x,t,u,\|u(t)\|^2),\ 0 < x < 1,\ 0 < t < T, \tag{1.6}$$

associated with the Dirichlet boundary conditions, where the source term contains a nonlocal term

$$\|u(t)\|^2 = \int_0^1 u^2(x,t)dx. \tag{1.7}$$

In [17], the authors considered a one dimentional nonlocal nonlinear strongly damped wave equation with dynamical boundary conditions. In other words, they looked to the following problem:

$$\begin{cases} u_{tt} - u_{xx} - \alpha u_{txx} + \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) = 0, \\ u(0,t) = 0, \\ u_{tt}(1,t) = -\varepsilon\left[u_x(1,t) + \alpha u_{tx}(1,t) + r u_t(1,t)\right] - \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right), \end{cases} \tag{1.8}$$

with $x \in (0,1)$, $t > 0$, α, $r > 0$ and $\varepsilon \geq 0$. Prob. (1.8) models a spring-mass-damper system, where the term $\varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right)$ represents a control acceleration at $x = 1$. By using the invariant manifold theory, the authors proved that for small values of the parameter ε, the solution of (1.8) attracted to a two dimentional invariant manifold.

Based on the ideas about a high order method for solving the equation $F(u) = 0$ as above and based on Faedo-Galerkin method, recently, in [6], [8]-[13] and in some other works, the authors have constructed a high order iterative scheme in order to obtain a result of existence where recurrent sequences converge at a rate of order N.

In this paper, we consider Prob. (1.1)-(1.3) and associate with Eq. (1.1) a recurrent sequence $\{u_m\}$ defined by

$$\begin{aligned} &\frac{\partial^2 u_m}{\partial t^2} - \Delta u_m \\ &= \sum_{0 \leq i+j+s \leq N-1} D^{ijs} f[u_{m-1}] \left(u_m - u_{m-1}\right)^i \left(u_m(0,t) - u_{m-1}(0,t)\right)^j \\ &\quad \times \left(u_m(1,t) - u_{m-1}(1,t)\right)^s, \ \ 0 < x < 1, \ 0 < t < T, \end{aligned} \tag{1.9}$$

where

$$\begin{aligned} &D^{ijs} f[u_{m-1}](x,t) \\ &= \frac{1}{i!j!s!} D_3^i D_4^j D_5^s f(x,t,u_{m-1}(x,t), u_{m-1}(0,t), u_{m-1}(1,t)), \end{aligned} \tag{1.10}$$

with u_m satisfying (1.2), (1.3). The first term u_0 is chosen as $u_0 \equiv 0$. If $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$ and some other conditions, we prove that the sequence $\{u_m\}$ converges at rate of order N ($N \geq 2$) to a weak unique solution of Prob. (1.1)-(1.3). The main result is given in Theorems 3.1 and 3.3. In our proofs, the fixed point method and Faedo-Galerkin method and the standard compactness argument are employed. This result is a relative generalization of [4]-[15].

2. Preliminaries

First, we put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$ and denote the usual function spaces used in this paper by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let

$\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2, $\|\cdot\|_X$ is the norm in the Banach space X, and X' is the dual space of X.

We denote $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty \quad \text{for} \quad 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess \sup} \|u(t)\|_X \ \text{ for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $f \in C^k([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, $f = f(x,t,y_1,y_2,y_3)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{2+i} f = \frac{\partial f}{\partial y_i}$, $i = 1,2,3$ and $D^\alpha f = D_1^{\alpha_1} \cdots D_5^{\alpha_5} f$; $\alpha = (\alpha_1, \cdots, \alpha_5) \in \mathbb{Z}_+^5$, $|\alpha| = \alpha_1 + \cdots + \alpha_5 = k$, $D^{(0,\cdots,0)} f = D^{(0)} f = f$.

On H^1, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2 \right)^{1/2}.$$

We put

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + h_0 u(0) v(0) + h_1 u(1) v(1), \ u, v \in H^1. \tag{2.1}$$

We have the following lemmas, the proofs of which are straightforward hence we omit the details.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2} \, \|v\|_{H^1} \quad \textit{for all} \quad v \in H^1. \tag{2.2}$$

Lemma 2.2. *Let h_0, $h_1 \geq 0$, with $h_0 + h_1 > 0$. Then, the symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.1) *is continuous on $H^1 \times H^1$ and coercive on H^1, i.e.,*

$$\begin{aligned} &\text{(i)} \quad |a(u,v)| \leq a_1 \|u\|_{H^1} \|v\|_{H^1}, \\ &\text{(ii)} \quad a(v,v) \geq a_0 \|v\|_{H^1}^2, \end{aligned} \tag{2.3}$$

for all u, $v \in H^1$, where $a_1 = 1 + 2h_0 + 2h_1$ and

$$a_0 = \frac{1}{4} \min\{1, \ \max\{h_0, \ h_1\}\}. \tag{2.4}$$

Remark 2.1. It follows from (2.3) that on H^1, $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v\|_a = \sqrt{a(v,v)}$ are two equivalant norms satisfying

$$\sqrt{a_0}\,\|v\|_{H^1} \le \|v\|_a \le \sqrt{a_1}\,\|v\|_{H^1}\,,\ \forall v \in H^1. \tag{2.5}$$

Lemma 2.3. *Let $h_0 \ge 0$. Then there exists the Hilbert orthonormal base $\{\widetilde{w}_j\}$ of L^2 consisting of the eigenfunctions $\widetilde{w}_j$ corresponding to the eigenvalue λ_j such that*

$$\begin{cases} 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots\,,\ \lim\limits_{j\to+\infty} \lambda_j = +\infty, \\ a(\widetilde{w}_j, v) = \lambda_j \langle \widetilde{w}_j, v\rangle \text{ for all } v \in H^1,\ j = 1,2,\cdots. \end{cases} \tag{2.6}$$

Furthermore, the sequence $\{\widetilde{w}_j/\sqrt{\lambda_j}\}$ is also a Hilbert orthonormal base of H^1 with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we also have $\widetilde{w}_j$ satisfying the following boundary value problem

$$\begin{cases} -\Delta \widetilde{w}_j = \lambda_j \widetilde{w}_j,\ in\ (0,1), \\ \widetilde{w}_{jx}(0) - h_0 \widetilde{w}_j(0) = \widetilde{w}_{jx}(1) + h_1 \widetilde{w}_j(1) = 0,\ \widetilde{w}_j \in C^\infty([0,1]). \end{cases} \tag{2.7}$$

The proof of Lemma 2.3 can be found in ([19, p.87, Theorem 7.7]) with $H = L^2$ and $V = H^1$, $a(\cdot,\cdot)$ as defined by (2.1).

3. The existence of a recurrent sequence and its convergence

We make the following assumptions:

(H_1) $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^1$;
(H_2) $f \in C^0([0,1]\times\mathbb{R}_+\times\mathbb{R}^3)$ such that
(i) $D_3^i D_4^j D_5^s f \in C^0([0,1]\times\mathbb{R}_+\times\mathbb{R}^3)$, $0 \le i+j+s \le N$,

(ii) $D_1 D_3^i D_4^j D_5^s f,\ D_3^{i+1} D_4^j D_5^s f \in C^0([0,1]\times\mathbb{R}_+\times\mathbb{R}^3)$, $1 \le i+j+s \le N-1$.

Fix $T^* > 0$. For each $M > 0$ given, we set two constants $K_M^{[0]}(f)$, $K_M(f)$ as follows

$$\begin{cases} K_M^{[0]}(f) = \sup\{|f(x,t,y_1,y_2,y_3)| : 0 \le x \le 1,\ 0 \le t \le T^*,\ \max\limits_{1\le i\le 3}|y_i| \le M\}, \\ K_M(f) = \sum\limits_{i+j+s\le N} K_M^{[0]}(D_3^i D_4^j D_5^s f) + \sum\limits_{1\le i+j+s\le N-1} K_M^{[0]}(D_1 D_3^i D_4^j D_5^s f) \\ \qquad + \sum\limits_{1\le i+j+s\le N-1} K_M^{[0]}(D_3^{i+1} D_4^j D_5^s f). \end{cases}$$

For every $T \in (0,T^*]$ and $M > 0$, we put

$$W_T = \{v \in L^\infty(0,T;H^2) : v_t \in L^\infty(0,T;H^1),\ v_{tt} \in L^2(Q_T)\}.$$

Then W_T is a Banach space with respect to the norm

$$\|v\|_{W_T} = \max\{\|v\|_{L^\infty(0,T;H^2)}, \|v_t\|_{L^\infty(0,T;H^1)}, \|v_{tt}\|_{L^2(Q_T)}\}$$

(see Lions [2]). We also put

$$\begin{cases} W(M,T) = \{v \in W_T : \|v\|_{W_T} \le M\}, \\ W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T;L^2)\}. \end{cases}$$

Now, we establish the recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W(M,T), \tag{3.1}$$

we associate problem (1.1)-(1.3) with the following problem.

Find $u_m \in W_1(M,T)$ $(m \ge 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + a(u_m(t), w) = \langle F_m(t), w\rangle, \forall w \in H^1, \\ u_m(0) = \tilde{u}_0, \, u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} &F_m(x,t) \\ &= \sum_{i+j+s\le N-1} D^{ijs} f[u_{m-1}] \left(u_m - u_{m-1}\right)^i \left(u_m(0,t) - u_{m-1}(0,t)\right)^j \\ &\quad \times \left(u_m(1,t) - u_{m-1}(1,t)\right)^s, \end{aligned} \tag{3.3}$$

with the notations

$$D^{ijs} f = \frac{1}{i!j!s!} D_3^i D_4^j D_5^s f, \quad 1 \le i+j+s \le N, \;\; D^{000} f = f. \tag{3.4}$$

Then we have the following theorem.

Theorem 3.1. *Let* $(H_1) - (H_2)$ *hold. Then there exist a constant* $M > 0$ *depending on* $\tilde{u}_0$, $\tilde{u}_1$, h_0, h_1 *and a constant* $T > 0$ *depending on* $\tilde{u}_0$, $\tilde{u}_1$, f, h_0, h_1, *such that, for* $u_0 \equiv 0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (3.2) *and* (3.3).

Proof. The proof of Theorem 3.1 consists three steps.

Step 1. *The Faedo-Galerkin approximation.*

Let $\{w_j\}$ be a basis of H^1 as in Lemma 2.3, we find an approximate solution of Prob. (3.2), (3.3) in the form

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.5}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the following system of nonlinear differential equations

$$\begin{cases} \left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + a(u_m^{(k)}(t), w_j) = \left\langle F_m^{(k)}(t), w_j \right\rangle, \ 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.6}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \longrightarrow \tilde{u}_0 \text{ strongly in } H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \longrightarrow \tilde{u}_1 \text{ strongly in } H^1, \end{cases} \tag{3.7}$$

and

$$\begin{aligned} & F_m^{(k)}(x,t) \\ & = \sum_{i+j+s \le N-1} D^{ijs} f[u_{m-1}] \left(u_m^{(k)} - u_{m-1}\right)^i \left(u_m^{(k)}(0,t) - u_{m-1}(0,t)\right)^j \\ & \quad \times \left(u_m^{(k)}(1,t) - u_{m-1}(1,t)\right)^s. \end{aligned} \tag{3.8}$$

The system (3.6) can be written in the form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \lambda_j c_{mj}^{(k)}(t) = \left\langle F_m^{(k)}(t), w_j \right\rangle, \ 1 \le j \le k, \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)}, \ \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}. \end{cases} \tag{3.9}$$

It can see that, system (3.9) is equivalent to system of intergal equations

$$\begin{aligned} c_{mj}^{(k)}(t) = {} & \alpha_j^{(k)} \cos(\sqrt{\lambda_j} t) + \beta_j^{(k)} \frac{\sin(\sqrt{\lambda_j} t)}{\sqrt{\lambda_j}} \\ & + \int_0^t \frac{\sin(\sqrt{\lambda_j}(t-s))}{\sqrt{\lambda_j}} F_{mj}^{(k)}(s) ds, \ 1 \le j \le k. \end{aligned} \tag{3.10}$$

Omitting the indexs m, k, it is written as follows

$$c = U[c], \tag{3.11}$$

where $U[c] = (U_1[c], \cdots, U_k[c])$, $c = (c_1, \cdots, c_k)$, and

$$\begin{cases} U_j[c](t) = q_j(t) + L_j[c](t), \\ q_j(t) = \alpha_j \cos(\sqrt{\lambda_j} t) + \beta_j \dfrac{\sin(\sqrt{\lambda_j} t)}{\sqrt{\lambda_j}}, \\ L_j[c](t) = \int_0^t \dfrac{\sin(\sqrt{\lambda_j}(t-s))}{\sqrt{\lambda_j}} \langle F[c](s), w_j \rangle \, ds, \ 1 \leq j \leq k, \\ F[c](t) = \sum\limits_{0 \leq i+j+s \leq N-1} D^{ijs} f[u_{m-1}] \left(u(t) - u_{m-1}\right)^i \\ \qquad \times \left(u(0,t) - u_{m-1}(0,t)\right)^j \left(u(1,t) - u_{m-1}(1,t)\right)^s, \\ u(t) = \sum\limits_{i=1}^{k} c_i(t) w_i. \end{cases} \tag{3.12}$$

Applying the contraction principle, we shall prove that the system (3.10) has an unique solution $c_{mj}^{(k)}(t)$ in $[0, T_m^{(k)}]$, with certain $T_m^{(k)} \in (0, T]$. Indeed, for every $T_m^{(k)} \in (0, T]$ and $\rho > 0$ that will be chosen later, we set

$$X = C^0\left([0, T_m^{(k)}]; \mathbb{R}^k\right), \ S = \{c \in X : \|c\|_X \leq \rho\}, \tag{3.13}$$

where

$$\|c\|_X = \sup_{0 \leq t \leq T_m^{(k)}} |c(t)|_1, \ |c(t)|_1 = \sum_{j=1}^{k} |c_j(t)|. \tag{3.14}$$

Clearly, S is a nonempty closed subset of X and $U : X \to X$. We will choose $\rho > 0$ and $T_m^{(k)} > 0$ such that $U : S \to S$ is contractive as follows.
(a) First we note that, for all $c = (c_1, \cdots, c_k) \in S$,

$$\begin{aligned} &\|u(t)\| \leq |c(t)|_1 \leq \|c\|_X \leq \rho, \\ &\|u(t)\|_{C^0(\overline{\Omega})} \leq \sqrt{2} \|u(t)\|_{H^1} \leq \sqrt{\frac{2}{a_0}} \|u(t)\|_a \leq \sqrt{\frac{2\lambda_k}{a_0}} |c(t)|_1 \leq \sqrt{\frac{2\lambda_k}{a_0}} \rho, \\ &\|u(t)\|_{H^1} \leq \sqrt{\frac{1}{a_0}} \|u(t)\|_a \leq \sqrt{\frac{\lambda_k}{a_0}} |c(t)|_1 \leq \sqrt{\frac{\lambda_k}{a_0}} \rho, \end{aligned} \tag{3.15}$$

so

$$|L[c](t)|_1 \leq \frac{k}{\sqrt{\lambda_1}} \int_0^t \|F[c](s)\| \, ds. \tag{3.16}$$

On the other hand, by the formula

$$\sum_{i+j+s=p} \frac{1}{i!j!s!} = \frac{3^p}{p!}, \text{ for all } p \in \mathbb{Z}_+, \tag{3.17}$$

and the inequality

$$
\begin{aligned}
&|F[c](x,t)| \\
&\le K_M(f) \sum_{0\le i+j+s\le N-1} \frac{1}{i!j!s!} \left|u(x,t)-u_{m-1}\right|^i \left|u(0,t)-u_{m-1}(0,t)\right|^j \\
&\quad \times \left|u(1,t)-u_{m-1}(1,t)\right|^s \\
&\le K_M(f) \sum_{0\le i+j+s\le N-1} \frac{1}{i!j!s!} \left(\|u(t)\|_{C^0(\bar{\Omega})}+\sqrt{2}M\right)^i \left(\|u(t)\|_{C^0(\bar{\Omega})}+\sqrt{2}M\right)^j \\
&\quad \times \left(\|u(t)\|_{C^0(\bar{\Omega})}+\sqrt{2}M\right)^s \\
&\le K_M(f) \sum_{0\le i+j+s\le N-1} \frac{1}{i!j!s!} \left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{i+j+s} \\
&\le K_M(f) \sum_{p=0}^{N-1} \sum_{i+j+s=p} \frac{1}{i!j!s!} \left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{p} \\
&\le K_M(f) \sum_{p=0}^{N-1} \frac{3^p}{p!} \left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{p}, \qquad (3.18)
\end{aligned}
$$

it follows from (3.16) and (3.18) that

$$
|L[c](t)|_1 \le \frac{k}{\sqrt{\lambda_1}} T_m^{(k)} K_M(f) \sum_{p=0}^{N-1} \frac{3^p}{p!} \left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{p}. \qquad (3.19)
$$

Hence

$$
|U[c](t)|_1 \le |\alpha|_1 + \frac{1}{\sqrt{\lambda_1}} |\beta|_1 + \overline{C}_\rho^{(1)} T_m^{(k)}, \ \forall t\in[0,T_m^{(k)}], \qquad (3.20)
$$

in which

$$
\overline{C}_\rho^{(1)} = \frac{k}{\sqrt{\lambda_1}} K_M(f) \sum_{p=0}^{N-1} \frac{3^p}{p!} \left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{p}. \qquad (3.21)
$$

Consequently

$$
\|U[c]\|_X \le |\alpha|_1 + \frac{1}{\sqrt{\lambda_1}} |\beta|_1 + \overline{C}_\rho^{(1)} T_m^{(k)}. \qquad (3.22)
$$

(b) Next, with $c=(c_1,\cdots,c_k)\in S$, $d=(d_1,\cdots,d_k)\in S$ and $t\in[0,T_m^{(k)}]$, considering

$$
u(t)=\sum_{j=1}^{k} c_j(t) w_j, \quad v(t)=\sum_{j=1}^{k} d_j(t) w_j, \qquad (3.23)
$$

we prove that

$$\|U[c]-U[d]\|_X \leq \frac{k}{\sqrt{\lambda_1}}\overline{C}_\rho^{(2)} T_m^{(k)} \|c-d\|_X\,, \ \forall c,d \in S, \tag{3.24}$$

where

$$\overline{C}_\rho^{(2)} = 3K_M(f)\sqrt{\frac{2\lambda_k}{a_0}}\sum\nolimits_{p=0}^{N-2}\frac{3^p}{p!}\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^p. \tag{3.25}$$

Indeed

$$\begin{aligned} |U[c](t)-U[d](t)|_1 &= |L[c](t)-L[d](t)|_1 \\ &\leq \frac{k}{\sqrt{\lambda_1}}\int_0^t \|F[c](s)-F[d](s)\|\,ds. \end{aligned} \tag{3.26}$$

On the other hand

$$\begin{aligned} &F[c](x,t)-F[d](x,t) \\ &= \sum_{1\leq i+j+s\leq N-1} D^{ijs}f[u_{m-1}]\,(u(t)-u_{m-1})^i \\ &\quad \times (u(0,t)-u_{m-1}(0,t))^j\,(u(1,t)-u_{m-1}(1,t))^s \\ &\quad - \sum_{1\leq i+j+s\leq N-1} D^{ijs}f[u_{m-1}]\,(v(t)-u_{m-1})^i \\ &\quad \times (v(0,t)-u_{m-1}(0,t))^j\,(v(1,t)-u_{m-1}(1,t))^s \\ &= \sum_{1\leq i+j+s\leq N-1} D^{ijs}f[u_{m-1}]\left[(u(t)-u_{m-1})^i-(v(t)-u_{m-1})^i\right] \\ &\quad \times (u(0,t)-u_{m-1}(0,t))^j\,(u(1,t)-u_{m-1}(1,t))^s \\ &\quad + \sum_{1\leq i+j+s\leq N-1} D^{ijs}f[u_{m-1}]\left[(u(0,t)-u_{m-1}(0,t))^j-(v(0,t)-u_{m-1}(0,t))^j\right] \\ &\quad \times (v(t)-u_{m-1})^i\,(u(1,t)-u_{m-1}(1,t))^s \\ &\quad + \sum_{1\leq i+j+s\leq N-1} D^{ijs}f[u_{m-1}]\left[(u(1,t)-u_{m-1}(1,t))^s-(v(1,t)-u_{m-1}(1,t))^s\right] \\ &\quad \times (v(t)-u_{m-1})^i\,(v(0,t)-u_{m-1}(0,t))^j\,. \end{aligned} \tag{3.27}$$

We also note that $a^i-b^i=(a-b)\sum\limits_{\nu=0}^{i-1}a^\nu b^{i-1-\nu}$ for all $a,\,b\in\mathbb{R}$, $i=1,2,\cdots,$ we deduce from (3.15) that

$$
\begin{aligned}
&\left|(u(t)-u_{m-1})^i-(v(t)-u_{m-1})^i\right| \\
&=|u(t)-v(t)|\left|\sum_{\nu=0}^{i-1}(u(t)-u_{m-1})^{\nu}(v(t)-u_{m-1})^{i-1-\nu}\right| \\
&\leq |u(t)-v(t)|\sum_{\nu=0}^{i-1}|u(t)-u_{m-1}|^{\nu}|v(t)-u_{m-1}|^{i-1-\nu} \\
&\leq \sqrt{\frac{2\lambda_k}{a_0}}\,\|c-d\|_X\sum_{\nu=0}^{i-1}\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{\nu}\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{i-1-\nu} \\
&=\sqrt{\frac{2\lambda_k}{a_0}}i\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{i-1}\|c-d\|_X\,. \qquad (3.28)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\left|(u(0,t)-u_{m-1}(0,t))^j-(v(0,t)-u_{m-1}(0,t))^j\right| \\
&\leq \sqrt{\frac{2\lambda_k}{a_0}}j\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{j-1}\|c-d\|_X\,; \\
&|(u(1,t)-u_{m-1}(1,t))^s-(v(1,t)-u_{m-1}(1,t))^s| \\
&\leq \sqrt{\frac{2\lambda_k}{a_0}}s\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{s-1}\|c-d\|_X\,. \qquad (3.29)
\end{aligned}
$$

It implies that

$$
\begin{aligned}
&|F[c](x,t)-F[d](x,t)| \\
&\leq K_M(f)\sum_{1\leq i+j+s\leq N-1}\frac{1}{i!j!s!}\sqrt{\frac{2\lambda_k}{a_0}}i\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{i+j+s-1}\|c-d\|_X \\
&\quad+K_M(f)\sum_{1\leq i+j+s\leq N-1}\frac{1}{i!j!s!}\sqrt{\frac{2\lambda_k}{a_0}}j\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{i+j+s-1}\|c-d\|_X \\
&\quad+K_M(f)\sum_{1\leq i+j+s\leq N-1}\frac{1}{i!j!s!}\sqrt{\frac{2\lambda_k}{a_0}}s\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{i+j+s-1}\|c-d\|_X \\
&\leq K_M(f)\sqrt{\frac{2\lambda_k}{a_0}}\|c-d\|_X\sum_{1\leq i+j+s\leq N-1}\frac{1}{i!j!s!}(i+j+s)\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{i+j+s-1}
\end{aligned}
$$

$$
\begin{aligned}
&= K_M(f)\sqrt{\frac{2\lambda_k}{a_0}}\,\|c-d\|_X \sum_{p=1}^{N-1}\sum_{i+j+s=p}\frac{1}{i!j!s!}p\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{p-1} \\
&= K_M(f)\sqrt{\frac{2\lambda_k}{a_0}}\,\|c-d\|_X \sum_{p=1}^{N-1}\frac{3^p}{p!}p\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{p-1} \\
&= 3K_M(f)\sqrt{\frac{2\lambda_k}{a_0}}\,\|c-d\|_X \sum_{p=0}^{N-2}\frac{3^p}{p!}\left(\sqrt{\frac{2\lambda_k}{a_0}}\rho+\sqrt{2}M\right)^{p} \\
&= \overline{C}_\rho^{(2)}\,\|c-d\|_X\,, \qquad (3.30)
\end{aligned}
$$

where $\overline{C}_\rho^{(2)}$ is defined as in (3.25). Thus

$$
|U[c](t)-U[d](t)|_1 \le \frac{k}{\sqrt{\lambda_1}}\overline{C}_\rho^{(2)}T_m^{(k)}\,\|c-d\|_X\,. \qquad (3.31)
$$

It follows from (3.31), that (3.24) holds. By choosing $\rho > |\alpha|_1 + \frac{1}{\sqrt{\lambda_1}}|\beta|_1$ and $T_m^{(k)} \in (0,T]$ with the properties

$$
0 < \overline{C}_\rho^{(1)}T_m^{(k)} \le \rho - |\alpha|_1 - \frac{1}{\sqrt{\lambda_1}}|\beta|_1 \quad \text{and} \quad \frac{k}{\sqrt{\lambda_1}}\overline{C}_\rho^{(2)}T_m^{(k)} < 1, \qquad (3.32)
$$

thanks to (3.22), (3.24) and (3.32), it is easy to see that $U : S \longrightarrow S$ is contractive. Then, system (3.10) has an unique solution $c_{mj}^{(k)}(t)$ in $[0,T_m^{(k)}]$. We deduce that system (3.6) has an unique solution $u_m^{(k)}(t)$ in $[0,T_m^{(k)}]$.

The following estimates allow one to take $T_m^{(k)} = T$ independent of m and k. By such a priori estimate of $u_m^{(k)}(t)$, it can be extended outside $[0,T_m^{(k)}]$ and then, a solution defined in $[0,T]$ will be obtained.

Step 2. *A priori estimate.*

First, we put

$$
\begin{aligned}
S_m^{(k)}(t) = \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \left\|u_m^{(k)}(t)\right\|_a^2 \\
+ \left\|\Delta u_m^{(k)}(t)\right\|^2 + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds. \qquad (3.33)
\end{aligned}
$$

Then, it follows from (3.6) and (3.33) that

$$
S_m^{(k)}(t) = S_m^{(k)}(0) + 2\int_0^t \langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle ds
$$

$$
\begin{aligned}
&+2\int_0^t a(F_m^{(k)}(s),\dot{u}_m^{(k)}(s))ds+\int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds \\
&\equiv S_m^{(k)}(0)+\sum_{j=1}^{3} I_j. \qquad (3.34)
\end{aligned}
$$

We now estimate the integrals and $S_m^{(k)}(0)$ on the right-hand side of (3.34) as follows.

First integral I_1: Using the inequalities $(a+b)^p \le 2^{p-1}(a^p+b^p)$, for all a, $b \ge 0$, $p \ge 1$ and

$$
s^q \le 1+s^p,\ \forall s \ge 0,\ \forall q \in (0,p], \qquad (3.35)
$$

we get from (3.8) that

$$
\begin{aligned}
&\left|F_m^{(k)}(x,t)\right| \\
&\le K_M(f)\sum_{0\le i+j+s\le N-1}\frac{1}{i!j!s!}\left|u_m^{(k)}(x,t)-u_{m-1}\right|^i \\
&\quad\times\left|u_m^{(k)}(0,t)-u_{m-1}(0,t)\right|^j\times\left|u_m^{(k)}(1,t)-u_{m-1}(1,t)\right|^s \\
&\le K_M(f)\sum_{0\le i+j+s\le N-1}\frac{1}{i!j!s!}\left(\left|u_m^{(k)}(x,t)\right|+\sqrt{2}M\right)^i\left(\left|u_m^{(k)}(0,t)\right|+\sqrt{2}M\right)^j \\
&\quad\times\left(\left|u_m^{(k)}(0,t)\right|+\sqrt{2}M\right)^s \\
&\le K_M(f)\sum_{0\le i+j+s\le N-1}\frac{1}{i!j!s!}\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{i+j+s} \\
&= K_M(f)\sum_{p=0}^{N-1}\sum_{i+j+s=p}\frac{1}{i!j!s!}\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{p} \\
&= K_M(f)\sum_{p=0}^{N-1}\frac{3^p}{p!}\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{p} \\
&\le K_M(f)\sum_{p=0}^{N-1}\frac{3^p}{p!}2^{p-1}\left[\left(\sqrt{\frac{2}{a_0}}\right)^p\left(\sqrt{S_m^{(k)}(t)}\right)^p+\left(\sqrt{2}M\right)^p\right] \\
&\le K_M(f)\sum_{p=0}^{N-1}\frac{3^p}{p!}2^{p-1}\left[\left(\sqrt{\frac{2}{a_0}}\right)^p\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right]+\left(\sqrt{2}M\right)^p\right]
\end{aligned}
$$

$$\leq K_M(f)\sum_{p=0}^{N-1}\frac{3^p}{p!}2^{p-1}\left[\left(\sqrt{\frac{2}{a_0}}\right)^p+\left(\sqrt{2}M\right)^p\right]\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right]$$
$$\equiv A_1(M)\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right]. \tag{3.36}$$

Hence

$$\left\|F_m^{(k)}(t)\right\|\leq A_1(M)\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right], \tag{3.37}$$

where $A_1(M)=K_M(f)\sum_{p=0}^{N-1}\frac{3^p}{p!}2^{p-1}\left[\left(\sqrt{\frac{2}{a_0}}\right)^p+\left(\sqrt{2}M\right)^p\right]$. By (3.37), the integral I_1 is estimated as follows

$$\begin{aligned}
I_1 &= 2\int_0^t\langle F_m^{(k)}(s),\dot{u}_m^{(k)}(s)\rangle ds\leq 2\int_0^t\left\|F_m^{(k)}(s)\right\|\left\|\dot{u}_m^{(k)}(s)\right\|ds\\
&\leq 2A_1(M)\int_0^t\left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{S_m^{(k)}(s)}ds\\
&= 2A_1(M)\int_0^t\left[\sqrt{S_m^{(k)}(s)}+\left(\sqrt{S_m^{(k)}(s)}\right)^{N}\right]ds\\
&\leq 4A_1(M)\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds\\
&\equiv \tilde{A}_1(M)\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds.
\end{aligned} \tag{3.38}$$

Second integral I_2 : We need to estimate $\left\|F_m^{(k)}(t)\right\|_{H^1}$. By (3.8), we have

$$\begin{aligned}
&F_{mx}^{(k)}(x,t)\\
&= D_1f[u_{m-1}]+D_3f[u_{m-1}]\nabla u_{m-1}\\
&\quad+\sum_{1\leq i+j+s\leq N-1}\left(D_1D^{ijs}f[u_{m-1}]+D_3D^{ijs}f[u_{m-1}]\nabla u_{m-1}\right)\\
&\qquad\times\left(u_m^{(k)}(x,t)-u_{m-1}\right)^i\left(u_m^{(k)}(0,t)-u_{m-1}(0,t)\right)^j\left(u_m^{(k)}(1,t)-u_{m-1}(1,t)\right)^s\\
&\quad+\sum_{1\leq i+j+s\leq N-1}D^{ijs}f[u_{m-1}]i\left(u_m^{(k)}(x,t)-u_{m-1}\right)^{i-1}\left(u_{mx}^{(k)}(x,t)-\nabla u_{m-1}\right)\\
&\qquad\times\left(u_m^{(k)}(0,t)-u_{m-1}(0,t)\right)^j\left(u_m^{(k)}(1,t)-u_{m-1}(1,t)\right)^s.
\end{aligned} \tag{3.39}$$

It follows that

$$
\begin{aligned}
&\left\|F_{mx}^{(k)}(t)\right\| \\
&\le K_M(f)(1+\sqrt{2}M)\left[1+\sum_{1\le i+j+s\le N-1}\frac{1}{i!j!s!}\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{i+j+s}\right] \\
&\quad + K_M(f)\sum_{1\le i+j+s\le N-1}\frac{1}{i!j!s!}i\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{i-1} \\
&\quad\ \times\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{j} \\
&\quad\ \times\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{s} \\
&\le K_M(f)(1+\sqrt{2}M)\sum_{i+j+s\le N-1}\frac{1}{i!j!s!}\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{i+j+s} \\
&\quad + (N-1)K_M(f)\sum_{i+j+s\le N-1}\frac{1}{i!j!s!}\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{i+j+s} \\
&\le K_M(f)(N+\sqrt{2}M)\sum_{i+j+s\le N-1}\frac{1}{i!j!s!}\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{i+j+s} \\
&= K_M(f)(N+\sqrt{2}M)\sum_{p=0}^{N-1}\sum_{i+j+s=p}\frac{1}{i!j!s!}\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{p} \\
&= K_M(f)(N+\sqrt{2}M)\sum_{p=0}^{N-1}\frac{3^p}{p!}\left(\sqrt{\frac{2}{a_0}}\sqrt{S_m^{(k)}(t)}+\sqrt{2}M\right)^{p} \\
&\le K_M(f)(N+\sqrt{2}M)\sum_{p=0}^{N-1}\frac{3^p}{p!}2^{p-1}\left[\left(\sqrt{\frac{2}{a_0}}\right)^{p}+\left(\sqrt{2}M\right)^{p}\right] \\
&\quad\times\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right] \\
&= (N+\sqrt{2}M)A_1(M)\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right] \\
&= A_2(M)\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right],
\end{aligned}
\tag{3.40}
$$

where $A_2(M) = (N + \sqrt{2}M)A_1(M)$.

On the other hand, by (2.5), we get

$$\begin{aligned}\left\|F_m^{(k)}(t)\right\|_a &\le \sqrt{a_1}\left\|F_m^{(k)}(t)\right\|_{H^1} \le \sqrt{a_1}\left[\left\|F_m^{(k)}(t)\right\| + \left\|F_{mx}^{(k)}(t)\right\|\right] \\ &\le \sqrt{a_1}\left[A_1(M) + A_2(M)\right]\left[1 + \left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right]. \qquad (3.41)\end{aligned}$$

It leads to

$$\begin{aligned} I_2 &= 2\int_0^t a(F_m^{(k)}(s), \dot{u}_m^{(k)}(s))ds \le 2\int_0^t \left\|F_m^{(k)}(s)\right\|_a \left\|\dot{u}_m^{(k)}(s)\right\|_a ds \\ &\le 2\sqrt{a_1}\left[A_1(M) + A_2(M)\right]\int_0^t \left[1 + \left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{S_m^{(k)}(s)}ds \\ &= 2\sqrt{a_1}\left[A_1(M) + A_2(M)\right]\int_0^t \left[\sqrt{S_m^{(k)}(s)} + \left(\sqrt{S_m^{(k)}(s)}\right)^{N}\right] ds \\ &\le 4\sqrt{a_1}\left[A_1(M) + A_2(M)\right]\int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N-1}\right] ds \\ &\equiv \tilde{A}_2(M)\int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N-1}\right] ds, \qquad (3.42)\end{aligned}$$

where $\tilde{A}_2(M) = 4\sqrt{a_1}\left[A_1(M) + A_2(M)\right]$.

Third integral I_3 : We note that the equation $(3.6)_1$ can be written as follows

$$\left\langle \ddot{u}_m^{(k)}(t), w_j\right\rangle - \left\langle \Delta u_m^{(k)}(t), w_j\right\rangle = \left\langle F_m^{(k)}(t), w_j\right\rangle, \ 1 \le j \le k. \qquad (3.43)$$

Hence, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating in t, we have

$$\begin{aligned} I_3 &= \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds \\ &\le 2\int_0^t \left\|\Delta u_m^{(k)}(s)\right\|^2 ds + 2\int_0^t \left\|F_m^{(k)}(s)\right\|^2 ds \\ &\le 2\int_0^t S_m^{(k)}(s)ds + 2A_1^2(M)\int_0^t \left[1 + \left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]^2 ds \\ &\le 2\int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N-1}\right] ds + 4A_1^2(M)\int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N-1}\right] ds \\ &\equiv \tilde{A}_3(M)\int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N-1}\right] ds, \qquad (3.44)\end{aligned}$$

where $\tilde{A}_3(M) = 2 + 4A_1^2(M)$. Combining (3.34), (3.38), (3.42) and (3.44) lead to

$$S_m^{(k)}(t) \leq S_m^{(k)}(0) + T\tilde{A}(M) + \tilde{A}(M) \int_0^t \left(S_m^{(k)}(s)\right)^{N-1} ds, \tag{3.45}$$

where $\tilde{A}(M) = \tilde{A}_1(M) + \tilde{A}_2(M) + \tilde{A}_3(M)$. By means of the convergences (3.7), we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$S_m^{(k)}(0) \leq \frac{M^2}{2}, \quad \forall\, m, k \in \mathbb{N}. \tag{3.46}$$

Finally, it follows from (3.45) and (3.46) that

$$S_m^{(k)}(t) \leq \frac{M^2}{2} + T\tilde{A}(M) + \tilde{A}(M) \int_0^t \left(S_m^{(k)}(s)\right)^{N-1} ds, \tag{3.47}$$

for $0 \leq t \leq T_m^{(k)} \leq T$. Then, by solving a nonlinear Volterra integral inequality (3.47) (based on the methods in [3]), we prove that there exists a constant $T > 0$ independent of k and m such that

$$S_m^{(k)}(t) \leq M^2, \quad \forall\, t \in [0, T], \quad \forall\, m,\ k \in \mathbb{N}. \tag{3.48}$$

So, we can take constant $T_m^{(k)} = T$ for all k and $m \in \mathbb{N}$. Thus, we have

$$u_m^{(k)} \in W(M, T), \ \ \forall\, m,\ k \in \mathbb{N}. \tag{3.49}$$

Step 3. Limiting process.

Thanks to (3.49), there exists a subsequence of $\{u_m^{(k)}\}$, still denoted by $\{u_m^{(k)}\}$ such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0, T; H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to \dot{u}_m & \text{in} \quad L^\infty(0, T; H^1) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to \ddot{u}_m & \text{in} \quad L^2(Q_T) \text{ weakly}, \\ u_m \in W(M, T). \end{cases} \tag{3.50}$$

Thanks to (3.49) and (3.50), we can check from (3.6) and (3.7) that u_m satisfies (3.2), (3.3) in $L^2(0, T)$.

On the other hand, it follows from (3.2) and (3.50) that

$$u_m'' = \Delta u_m + F_m \in L^\infty(0, T; L^2). \tag{3.51}$$

Hence, $u_m \in W_1(M, T)$ and Theorem 3.1 is proved. □

Next, the main result is given by the following theorem. We consider the space $W_1(T)$, defined by

$$W_1(T) = \{v \in L^\infty(0, T; H^1) : v' \in L^\infty(0, T; L^2)\}, \tag{3.52}$$

then $W_1(T)$ is a Banach space with respect to the norm

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;H^1)} + \|v'\|_{L^\infty(0,T;L^2)} . \tag{3.53}$$

Theorem 3.2. *Let* $(H_1) - (H_2)$ *hold. Then, there exist constants* $M > 0$ *and* $T > 0$ *such that the problem* (1.1)-(1.3) *has an unique weak solution* $u \in W_1(M,T)$ *and the recurrent sequence* $\{u_m\}$*, defined by* (3.2)-(3.3)*, converges at a rate of order* N *to the solution* u *strongly in the space* $W_1(T)$ *in sense*

$$\|u_m - u\|_{W_1(T)} \leq C \|u_{m-1} - u\|_{W_1(T)}^N , \tag{3.54}$$

for all $m \geq 1$*, where* C *is a suitable constant. On the other hand, the following estimate is fulfilled*

$$\|u_m - u\|_{W_1(T)} \leq C_T (k_T)^{N^m}, \text{ for all } m \in \mathbb{N}, \tag{3.55}$$

where $C_T > 0$ *and* $0 < k_T < 1$ *are the constants depending only on* T.

Proof. Existence of a solution. We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Indeed, we put $v_m = u_{m+1} - u_m$. Then v_m satisfies the variational problem

$$\begin{cases} \langle v_m''(t), w\rangle + a(v_m(t), w) = \langle F_{m+1}(t) - F_m(t), w\rangle, \ \forall w \in H^1, \\ v_m(0) = v_m'(0) = 0. \end{cases} \tag{3.56}$$

Taking $w = v_m'$ in (3.56), after integrating in t, we have

$$\rho_m(t) \leq 2\int_0^t \|F_{m+1}(s) - F_m(s)\| \|v_m'(s)\| ds, \tag{3.57}$$

where

$$\rho_m(t) = \|v_m'(t)\|^2 + \|v_m(t)\|_a^2 \geq \|v_m'(t)\|^2 + a_0 \|v_m(t)\|_{H^1}^2 . \tag{3.58}$$

Next, we shall estimate the integral on the right side of (3.57) as follows. Using Taylor's expansion of the functions $f(x,t,u_m,u_m(0,t),u_m(1,t)) = f[u_m] = f[u_{m-1} + v_{m-1}]$ around the point $(x,t,u_{m-1},u_{m-1}(0,t),u_{m-1}(1,t))$ up to order N, we obtain

$$\begin{aligned} & f[u_m] - f[u_{m-1}] \\ & = f(x,t,u_m,u_m(0,t),u_m(1,t)) - f(x,t,u_{m-1},u_{m-1}(0,t),u_{m-1}(1,t)) \\ & = \sum_{1\leq i+j+s\leq N-1} D^{ijs} f[u_{m-1}] v_{m-1}^i v_{m-1}^j(0,t) v_{m-1}^s(1,t) \\ & \quad + \sum_{i+j+s=N} D^{ijs} f[u_{m-1} + \theta v_{m-1}] v_{m-1}^i v_{m-1}^j(0,t) v_{m-1}^s(1,t), \end{aligned} \tag{3.59}$$

where $0 < \theta < 1$. Hence, it follows from (3.3) and (3.59) that

$$\begin{aligned} &F_{m+1}(x,t) - F_m(x,t) \\ &= \sum_{1\le i+j+s\le N-1} D^{ijs} f[u_m] v_m^i v_m^j(0,t) v_m^s(1,t) \\ &\quad + \sum_{i+j+s=N} D^{ijs} f[u_{m-1} + \theta v_{m-1}] v_{m-1}^i v_{m-1}^j(0,t) v_{m-1}^s(1,t). \end{aligned} \tag{3.60}$$

Therefore, we have

$$\begin{aligned} &\|F_{m+1}(t) - F_m(t)\| \\ &\le K_M(f) \sum_{1\le i+j+s\le N-1} \frac{1}{i!j!s!} (\sqrt{2})^{i+j+s} \|v_m(t)\|_{H^1}^{i+j+s} \\ &\quad + K_M(f) \sum_{i+j+s=N} \frac{1}{i!j!s!} (\sqrt{2})^{i+j+s} \|v_{m-1}(t)\|_{H^1}^{i+j+s} \\ &= K_M(f) \sum_{p=1}^{N-1} \sum_{i+j+s=p} \frac{1}{i!j!s!} (\sqrt{2})^p \|v_m(t)\|_{H^1}^p \\ &\quad + K_M(f) \sum_{i+j+s=N} \frac{1}{i!j!s!} (\sqrt{2})^N \|v_{m-1}(t)\|_{H^1}^N \\ &= K_M(f) \sum_{p=1}^{N-1} \frac{(3\sqrt{2})^p}{p!} \|v_m(t)\|_{H^1}^p + K_M(f) \frac{(3\sqrt{2})^N}{N!} \|v_{m-1}(t)\|_{H^1}^N \\ &\le \frac{1}{\sqrt{a_0}} K_M(f) \sum_{p=1}^{N-1} \frac{(3\sqrt{2})^p (2M)^{p-1}}{p!} \sqrt{\rho_m(t)} + K_M(f) \frac{(3\sqrt{2})^N}{N!} \|v_{m-1}\|_{W_1(T)}^N \\ &\equiv \eta_T^{(1)} \sqrt{\rho_m(t)} + \eta_T^{(2)} \|v_{m-1}\|_{W_1(T)}^N, \end{aligned} \tag{3.61}$$

where $\eta_T^{(1)} = \frac{1}{\sqrt{a_0}} K_M(f) \sum_{p=1}^{N-1} \dfrac{(3\sqrt{2})^p (2M)^{p-1}}{p!}$, $\eta_T^{(2)} = \dfrac{(3\sqrt{2})^N}{N!} K_M(f)$. Then we deduce from (3.57) and (3.61) that

$$\rho_m(t) \le T\eta_T^{(2)} \|v_{m-1}\|_{W_1(T)}^{2N} + \eta_T^{(3)} \int_0^t \rho_m(s) ds, \tag{3.62}$$

where $\eta_T^{(3)} = 2\eta_T^{(1)} + \eta_T^{(2)}$. By using Gronwall's lemma, (3.62) leads to

$$\|v_m\|_{W_1(T)} \le \mu_T \|v_{m-1}\|_{W_1(T)}^N, \tag{3.63}$$

where $\mu_T = \left(1+\frac{1}{\sqrt{a_0}}\right)\sqrt{T\eta_T^{(2)}\exp(T\eta_T^{(3)})}$. Choosing $T>0$ enough small such that $k_T = M\mu_T^{\frac{-1}{N-1}} < 1$, it follows from (3.63) that, for all m and p,

$$\|u_m - u_{m+p}\|_{W_1(T)} \leq (1-k_T)^{-1}(\mu_T)^{\frac{-1}{N-1}}(k_T)^{N^m}. \tag{3.64}$$

Hence, $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strong in} \quad W_1(T). \tag{3.65}$$

Note that $u_m \in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;H^2) \text{ weakly*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T;H^1) \text{ weakly*}, \\ u''_{m_j} \to u'' & \text{in} \quad L^2(Q_T) \text{ weakly}, \\ u \in W(M,T). \end{cases} \tag{3.66}$$

On the other hand

$$\begin{aligned}
&\|F_m(\cdot,t) - f(\cdot,t,u(t),u(0,t),u(1,t))\| \\
&\leq \|f(\cdot,t,u_{m-1}(t),u_{m-1}(0,t),u_{m-1}(1,t)) - f(\cdot,t,u(t),u(0,t),u(1,t))\| \\
&\quad + \sum_{1\leq i+j+s\leq N-1} \Big\| D^{ijs}f[u_{m-1}]\,(u_m(t)-u_{m-1})^i\,(u_m(0,t)-u_{m-1}(0,t))^j \\
&\qquad \times (u_m(1,t)-u_{m-1}(1,t))^s \| \\
&\leq K_M(f)\,\|u_{m-1}-u\|_{W_1(T)} \\
&\quad + K_M(f) \sum_{1\leq i+j+s\leq N-1} \frac{1}{i!j!s!}(\sqrt{2})^{i+j+s}\,\|u_m - u_{m-1}\|_{W_1(T)}^{i+j+s} \\
&\leq K_M(f)\,\|u_{m-1}-u\|_{W_1(T)} + K_M(f)\sum_{p=1}^{N-1}\frac{3^p}{p!}(\sqrt{2})^p\,\|u_m-u_{m-1}\|_{W_1(T)}^p.
\end{aligned} \tag{3.67}$$

Therefore, it implies from (3.65) and (3.67) that

$$F_m(t) \to f(\cdot,t,u(t),u(0,t),u(1,t)) \quad \text{strong in} \quad L^\infty(0,T;L^2). \tag{3.68}$$

Finally, passing to limit in (3.2) and (3.3) as $m = m_j \to \infty$, there exists $u \in W(M,T)$ satisfying the equation

$$\langle u''(t), w\rangle + a(u(t),w) = \langle f(\cdot,t,u(t),u(0,t),u(1,t)), w\rangle, \tag{3.69}$$

for all $w \in H^1$ and the initial condition

$$u(0) = \tilde{u}_0, \; u'(0) = \tilde{u}_1. \tag{3.70}$$

On the other hand, it follows from $(3.66)_4$ and (3.69) that

$$u'' = \Delta u + f(x,t,u(t),u(0,t),u(1,t)) \in L^\infty(0,T;L^2), \tag{3.71}$$

hence, $u \in W_1(M,T)$.

Uniqueness. Applying a similar argument used in the proof of Theorem 3.1, $u \in W_1(M,T)$ is an unique local weak solution of Pro. (1.1)-(1.3).

Passing to the limit in (3.64) as $p \to \infty$ for fixed m, we get (3.55). Also with a similar argument, (3.54) follows. Theorem 3.2 is proved completely. □

Remark 3.1. In order to construct a N-order iterative scheme, we need the assumption (H_2). Then, we get a convergent sequence at a rate of order N to a local unique weak solution of problem and the existence follows. This condition of f can be relaxed if we only consider the existence of solutions, see [4], [5], [7].

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