**Nonlinear Functional Analysis and Applications**
**Vol. 22, No. 3 (2017), pp. 595-626**
**ISSN: 1229-1595(print), 2466-0973(online)**

http://nfaa.kyungnam.ac.kr/journal-nfaa


# A DIRICHLET PROBLEM FOR A NONLINEAR WAVE EQUATION OF KIRCHHOFF-LOVE TYPE

**Nguyen Anh Triet[1], Vo Thi Tuyet Mai[2,4], Le Thi Phuong Ngoc[3] and Nguyen Thanh Long[4]**

[1]Department of Mathematics, University of Architecture of Ho Chi Minh City
196 Pasteur Str., Dist.3, HoChiMinh City, Vietnam
e-mail: anhtriet1@gmail.com

[2]University of Natural Resources and Environment of Ho Chi Minh City
236B Le Van Sy Str., Ward 1, Tan Binh Dist., Ho Chi Minh City, Vietnam
e-mail: vttmai@hcmunre.edu.vn

[3]Khanh Hoa University
01 Nguyen Chanh Str., Nha Trang City, Vietnam
e-mail: ngoc1966@gmail.com

[4]Department of Mathematics and Computer Science, VNUHCM-University of Science
227 Nguyen Van Cu Str., Dist.5, HoChiMinh City, Vietnam
e-mail: longnt2@gmail.com

**Abstract.** In this paper, we consider the following Dirichlet problem for a nonlinear Kirchhoff-Love equation

$$\begin{cases} u_{tt} - B_1\left(\|u_x(t)\|^2, \|u_{xt}(t)\|^2\right) u_{xx} - \lambda B_2\left(\|u_x(t)\|^2, \|u_{xt}(t)\|^2\right) u_{xxtt} \\ = f(x,t,u,u_x,u_t,u_{xt}), \quad 0<x<1, \quad 0<t<T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1}$$

where $\lambda > 0$ is a constant, $\tilde{u}_0$, $\tilde{u}_1$, $f$, $B_1$, $B_2$, are given functions and $\|u_x(t)\|^2 = \int_0^1 u_x^2(x,t)\,dx$, $\|u_{xt}(t)\|^2 = \int_0^1 u_{xt}^2(x,t)\,dx$. Combining the linearization method for nonlinear terms, the Faedo-Galerkin method and the weak compact method, a unique weak solution of the problem (1) is obtained. In case of $B_1$, $B_2 \in C^{N+1}(\mathbb{R}_+^2)$, $B_i \geq b_0 > 0$, $f \in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R}^4)$

[0]Received November 28, 2016. Revised June 6, 2017.
[0]2010 Mathematics Subject Classification: 35L20, 35L70, 35Q72.
[0]Keywords: Faedo-Galerkin method, linear recurrent sequence, asymptotic expansion of order $N+1$ in a small parameter.

and $f_1 \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$ we obtain from the following equation

$$u_{tt} - B_1\left(\|u_x(t)\|^2, \|u_{xt}(t)\|^2\right) u_{xx} - \lambda B_2\left(\|u_x(t)\|^2, \|u_{xt}(t)\|^2\right) u_{xxtt}$$
$$= f(x,t,u,u_x,u_t,u_{xt}) + \varepsilon f_1(x,t,u,u_x,u_t,u_{xt}),$$

associated to $(1)_{2,3}$ a weak solution $u_\varepsilon(x,t)$ having an asymptotic expansion of order $N+1$ in $\varepsilon$, for $\varepsilon$ sufficiently small.

## 1. Introduction

In this paper, we consider the following Dirichlet problem for a nonlinear Kirchhoff-Love equation

$$u_{tt} - B_1\left(\|u_x(t)\|^2, \|u_{xt}(t)\|^2\right) u_{xx} - \lambda B_2\left(\|u_x(t)\|^2, \|u_{xt}(t)\|^2\right) u_{xxtt} \quad (1.1)$$
$$= f(x,t,u,u_x,u_t,u_{xt}),\ x \in \Omega = (0,1),\ 0 < t < T,$$

$$u(0,t) = u(1,t) = 0, \quad (1.2)$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \quad (1.3)$$

where $\lambda > 0$ is a constant and $\tilde{u}_0$, $\tilde{u}_1$, $f$, $B_1$, $B_2$, are given functions.

When $\Omega = (0,L)$, $B_1 = B_1\left(\|u_x(t)\|^2\right)$, $\lambda = 0$, $f = 0$, Eq. (1.1) is related to the Kirchhoff equation

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right) u_{xx}, \quad (1.4)$$

presented by Kirchhoff in 1876 (see, [10]). This equation is an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.4) have the following meanings: $u$ is the lateral deflection, $L$ is the length of the string, $h$ is the area of the cross-section, $E$ is the Young modulus of the material, $\rho$ is the mass density, and $P_0$ is the initial tension.

One of the early classical studies dedicated to Kirchhoff equations was given by Pohozaev [26]. After the work of Lions, for example see [12], Eq. (1.4) received much attention where an abstract framework to the problem was proposed. We refer the reader to, *e.g.*, Cavalcanti *et al.* [5]-[7], Ebihara, Medeiros and Miranda [9], Miranda *et al.* [23], Medeiros [20], Menzala [24], Park *et al.* [27], [28], Rabello *et al.* [30], Santos *et al.* [31], for many interesting results and further references. A survey of the results about the mathematical aspects of Kirchhoff model can be found in Medeiros, Limaco and Menezes [21], [22], and the references therein.

When $\Omega = (0, L)$, $B_1 = B_2 = 1$, $f = 0$, Eq. (1.1) is related to the Love equation

$$u_{tt} - \frac{E}{\rho} u_{xx} - 2\mu^2 k^2 u_{xxtt} = 0, \tag{1.5}$$

presented by V. Radochová in 1978 (see, [29]). This equation describes the vertical oscillations of a rod, which was established from Euler's variational equation of an energy function

$$\int_0^T dt \int_0^L \left[ \frac{1}{2} F\rho \left( u_t^2 + \mu^2 k^2 u_{tx}^2 \right) - \frac{1}{2} F \left( E u_x^2 + \rho \mu^2 k^2 u_x u_{xtt} \right) \right] dx. \tag{1.6}$$

The parameters in (1.5) have the following meanings: $u$ is the displacement, $L$ is the length of the rod, $F$ is the area of cross-section, $k$ is the cross-section radius, $E$ is the Young modulus of the material and $\rho$ is the mass density. By using the Fourier method, Radochová [29] obtained a classical solution of Prob. (1.5) associated with the initial conditions (1.3) and boundary conditions

$$u(0, t) = u(L, t) = 0, \tag{1.7}$$

or

$$u(0, t) = 0, \quad \lambda u_{xtt}(L, t) + c^2 u_x(L, t) = 0, \tag{1.8}$$

where $c^2 = \frac{E}{\rho}$, $\lambda = 2\mu^2 k^2$. On the other hand, the asymptotic behaviour of solutions for Prob. (1.3), (1.5), (1.8) as $\lambda \to 0_+$ was also established.

Equations of Love waves or equations for waves of Love types have been studied by many authors, we refer to [3], [8], [19] and references therein.

On the other hand, in [32], a symmetric version of the regularized long wave equation (SRLW)

$$\begin{cases} u_{xxt} - u_t = \rho_x + u u_x, \\ \rho_t + u_x = 0, \end{cases} \tag{1.9}$$

has been proposed to describe weakly nonlinear ion acoustic and space - charge waves. Eliminating $\rho$ from (1.9), a class of SRLWE is obtained as follows

$$u_{tt} - u_{xx} - u_{xxtt} = -u u_{xt} - u_x u_t. \tag{1.10}$$

Eq. (1.10) is explicitly symmetric in the $x$ and $t$ derivatives and it is very similar to the regularized long wave equation that describes shallow water waves and plasma drift waves [1], [2]. The SRLW equation also arises in many other areas of mathematical physics [4], [18], [25]. It is clear that Eq. (1.10) is a special form of Equation (1.1), in which $f(x, t, u, u_x, u_t, u_{xt}) = -u u_{xt} - u_x u_t$.

Motivated by the problems in the above mentioned works, in this paper, we consider Prob. (1.1)-(1.3) with $f \in C^1([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $B_1$, $B_2 \in C^1(\mathbb{R}_+^2)$. Since $f$, $B_1$, $B_2$ are arbitrary, the methods used in [29] or in [32] are no longer suitable, here we will combine the linearization method for a nonlinear term, the Faedo-Galerkin method and the weak compactness method.

The paper consists of four sections. At first, some preliminaries are done in Section 2. With the technique presented as above, we begin Section 3 by establishing a sequence of approximate solutions of Prob. (1.1) - (1.3) based on the Faedo-Galerkin's method. Thanks to a priori estimates, this sequence is bounded in an appropriate space, from which, using compact imbedding theorems and Gronwall's Lemma, one deduces the existence of a unique weak solution of Prob. (1.1)-(1.3). In particular, an asymptotic expansion of a weak solution $u = u_\varepsilon$ of order $N + 1$ in a small parameter $\varepsilon$ for the equation

$$u_{tt} - B_1\left(\|u_x(t)\|^2, \|u_{xt}(t)\|^2\right) u_{xx} - \lambda B_2\left(\|u_x(t)\|^2, \|u_{xt}(t)\|^2\right) u_{xxtt} \tag{1.11}$$
$$= f(x,t,u,u_x,u_t,u_{xt}) + \varepsilon f_1(x,t,u,u_x,u_t,u_{xt}),$$

$0 < x < 1,\ 0 < t < T$, associated to (1.2), (1.3), with $B_1,\ B_2 \in C^{N+1}(\mathbb{R}_+^2)$, $B_i(y,z) \geq b_0 > 0$, $(i = 1,2)$, for all $(y,z) \in \mathbb{R}_+^2$, $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $f_1 \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$ is established in Section 4. This result is a relative generalization of [13]-[17].

## 2. Preliminaries

We put $\Omega = (0,1)$ and denote the usual function spaces used in this paper by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle\cdot,\cdot\rangle$ be either the scalar product in $L^2$ or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in $L^2$ and we denote by $\|\cdot\|_X$ the norm in the Banach space $X$. We call $X'$ the dual space of $X$.

We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable with the norm defined by

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\, dt\right)^{1/p} < \infty \quad \text{for} \quad 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \quad \text{for} \quad p = \infty.$$

On $H^1$, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}.$$

The following lemma is known.

**Lemma 2.1.** *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\, \|v\|_{H^1} \quad \textit{for all} \quad v \in H^1.$$

**Remark 2.2.** On $H_0^1$, $v \longmapsto \|v\|_{H^1}$ and $v \longmapsto \|v_x\|$ are equivalent norms. Furthermore,

$$\|v\|_{C^0(\overline{\Omega})} \leq \|v_x\| \quad \text{for all} \quad v \in H_0^1. \tag{2.1}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. With $f \in C^N([0,1]\times\mathbb{R}_+\times\mathbb{R}^4)$, $f = f(x,t,u,v,w,z)$, we define $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial u}$, $D_4 f = \frac{\partial f}{\partial v}$, $D_5 f = \frac{\partial f}{\partial w}$, $D_6 f = \frac{\partial f}{\partial z}$ and $D^\alpha f = D_1^{\alpha_1} \cdots D_6^{\alpha_6} f$; $\alpha = (\alpha_1, \cdots, \alpha_6) \in \mathbb{Z}_+^6$, $|\alpha| = \alpha_1 + \cdots + \alpha_6 = N$; $D^{(0,\cdots,0)} f = f$.

Similarly, with $B \in C^N(\mathbb{R}_+^2)$, $B = B(y,z)$, we define $D_1 B = \frac{\partial B}{\partial y}$, $D_2 B = \frac{\partial B}{\partial z}$ and $D^\beta B = D_1^{\beta_1} D_2^{\beta_2} B$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2$, $|\beta| = \beta_1 + \beta_2 = N$; $D^{(0,0)} B = B$.

## 3. The existence and uniqueness of solution

We make the following assumptions:

$(H_1)$ $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$;
$(H_2)$ $B_1, B_2 \in C^1(\mathbb{R}_+^2)$ and $B_i(y,z) \geq b_0 > 0$, for all $(y,z) \in \mathbb{R}_+^2$, $i = 1,2$;
$(H_3)$ $f \in C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^4)$ and
$f(0,t,0,v,0,z) = f(1,t,0,v,0,z) = 0$, for all $(t,v,z) \in \mathbb{R}_+ \times \mathbb{R}^2$.

Let $T^* > 0$ fixed and $M > 0$. Put

$$\begin{cases} \tilde{K}_M(B_i) = \sup\limits_{0 \leq y,z \leq M^2} \left(B_i(y,z) + |D_1 B_i(y,z)| + |D_2 B_i(y,z)|\right), \; i = 1,2, \\ K_M(f) = \sup\limits_{(x,t,u,v,w,z) \in A_M} \left( |f(x,t,u,v,w,z)| + \sum\limits_{i=1}^{6} |D_i f(x,t,u,v,w,z)| \right), \end{cases} \tag{3.1}$$

where $A_M = [0,1] \times [0,T^*] \times [-M,M]^4$. For every $T \in (0,T^*]$ and $M > 0$, we put

$$V_T = \{v \in L^\infty(0,T; H_0^1 \cap H^2) : v_t \in L^\infty(0,T; H_0^1 \cap H^2), \; v_{tt} \in L^\infty(0,T; H_0^1)\}.$$

Then $V_T$ is a Banach space with respect to the norm

$$\|v\|_{V_T} = \max\left\{ \|v\|_{L^\infty(0,T;H_0^1 \cap H^2)}, \; \|v_t\|_{L^\infty(0,T;H_0^1 \cap H^2)}, \; \|v_{tt}\|_{L^\infty(0,T;H_0^1)} \right\}$$

(see, Lions [11]). We also put

$$\begin{cases} W(M,T) = \{v \in V_T : \|v\|_{V_T} \leq M\}, \\ W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T; H_0^1 \cap H^2)\}. \end{cases} \tag{3.2}$$

We establish the linear recurrent sequence $\{u_m\}$ as follows.

We choose the first term $u_0 \equiv \tilde{u}_0$. Suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.3}$$

and associate with Prob. (1.1)-(1.3) the following problem:

Find $u_m \in W_1(M,T)$ $(m \geq 1)$ which satisfies the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + C_m(t)\langle u_{mx}(t), w_x\rangle + \lambda D_m(t)\langle u_{mx}''(t), w_x\rangle \\ = \langle F_m(t), w\rangle\,, \quad \forall\, w \in H_0^1, \\ u_m(0) = \tilde{u}_0, \quad u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.4}$$

in which

$$\begin{cases} C_m(t) = B_1[u_{m-1}](t) = B_1\left(\|\nabla u_{m-1}(t)\|^2, \left\|\nabla u_{m-1}'(t)\right\|^2\right), \\ D_m(t) = B_2[u_{m-1}](t) = B_2\left(\|\nabla u_{m-1}(t)\|^2, \left\|\nabla u_{m-1}'(t)\right\|^2\right), \\ F_m(x,t) = f[u_{m-1}](x,t) \\ \qquad\qquad = f(x,t,u_{m-1}(t), \nabla u_{m-1}(t), u_{m-1}'(t), \nabla u_{m-1}'(t)). \end{cases} \tag{3.5}$$

Then we have the following theorem.

**Theorem 3.1.** *Under assumptions* $(H_1)-(H_3)$, *there exist positive constants* $M$, $T > 0$ *such that, for* $u_0 \equiv \tilde{u}_0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (3.4) *and* (3.5).

*Proof.* The proof consists of several steps.

**Step 1.** *The Faedo-Galerkin approximation*(introduced by Lions [11]).

Consider a special orthonormal basis $\{w_j\}$ on $H_0^1 : w_j(x) = \sqrt{2}\sin(j\pi x)$, $j \in \mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta = -\frac{\partial^2}{\partial x^2}$. Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.6}$$

where the coefficients $c_{mj}^{(k)}$ satisfy a system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + C_m(t)\langle u_{mx}^{(k)}(t), w_{jx}\rangle + \lambda D_m(t)\langle \ddot{u}_{mx}^{(k)}(t), w_{jx}\rangle \\ = \langle F_m(t), w_j\rangle,\ 1 \leq j \leq k, \\ u_m^{(k)}(0) = \tilde{u}_{0k},\ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.7}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum\limits_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{u}_0 \quad \text{strongly in} \quad H_0^1 \cap H^2, \\ \tilde{u}_{1k} = \sum\limits_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{u}_1 \quad \text{strongly in} \quad H_0^1 \cap H^2. \end{cases} \tag{3.8}$$

System (3.7) can be rewritten in form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \mu_{mj}(t) c_{mj}^{(k)}(t) = f_{mj}(t), \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)}, \quad \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \quad 1 \leq j \leq k, \end{cases} \tag{3.9}$$

where

$$f_{mj}(t) = \frac{1}{1+\lambda_j \lambda D_m(t)} \langle F_m(t), w_j \rangle, \tag{3.10}$$
$$\mu_{mj}(t) = \frac{\lambda_j C_m(t)}{1+\lambda_j \lambda D_m(t)}, \ \lambda_j = (j\pi)^2, \ 1 \le j \le k.$$

Hence

$$c_{mj}^{(k)}(t) = \alpha_j^{(k)} + t\beta_j^{(k)} + \int_0^t dr \int_0^r f_{mj}(s)ds - \int_0^t dr \int_0^r \mu_{mj}(s) c_{mj}^{(k)}(s)ds, \ \ 1 \le j \le k. \tag{3.11}$$

By (3.3), it is not difficult to prove that the system (3.11) has a unique solution $c_{mj}^{(k)}(t)$, $1 \le j \le k$ on interval $[0, T]$. The details are omitted.

**Step 2.** *A priori estimates.*

Put

$$S_m^{(k)}(t) = p_m^{(k)}(t) + q_m^{(k)}(t) + r_m^{(k)}(t), \tag{3.12}$$

where

$$\begin{cases} p_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|^2 + C_m(t) \left\| u_{mx}^{(k)}(t) \right\|^2 + \lambda D_m(t) \left\| \dot{u}_{mx}^{(k)}(t) \right\|^2, \\ q_m^{(k)}(t) = \left\| \dot{u}_{mx}^{(k)}(t) \right\|^2 + C_m(t) \left\| \Delta u_m^{(k)}(t) \right\|^2 + \lambda D_m(t) \left\| \Delta \dot{u}_m^{(k)}(t) \right\|^2, \\ r_m^{(k)}(t) = \left\| \ddot{u}_m^{(k)}(t) \right\|^2 + C_m(t) \left\| \dot{u}_{mx}^{(k)}(t) \right\|^2 + \lambda D_m(t) \left\| \ddot{u}_{mx}^{(k)}(t) \right\|^2. \end{cases} \tag{3.13}$$

Then, it follows from (3.7), (3.12), (3.13) that

$$\begin{aligned} S_m^{(k)}(t) = {} & S_m^{(k)}(0) + 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s) \rangle ds \\ & + 2\int_0^t \langle F_{mx}(s), \dot{u}_{mx}^{(k)}(s) \rangle ds + 2\int_0^t \langle F_m'(s), \ddot{u}_m^{(k)}(s) \rangle ds \\ & + \int_0^t C_m'(s) \Big( \left\| u_{mx}^{(k)}(s) \right\|^2 + \left\| \Delta u_m^{(k)}(s) \right\|^2 \\ & + \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 - 2\langle u_{mx}^{(k)}(s), \ddot{u}_{mx}^{(k)}(s) \rangle \Big) ds \\ & + \lambda \int_0^t D_m'(s) \left( \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 + \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 - \left\| \ddot{u}_{mx}^{(k)}(s) \right\|^2 \right) ds \\ = {} & S_m^{(k)}(0) + \sum_{j=1}^5 I_j. \end{aligned} \tag{3.14}$$

First, we estimate $\xi_m^{(k)} = \left\|\ddot{u}_m^{(k)}(0)\right\|^2 + \lambda D_m(0)\left\|\ddot{u}_{mx}^{(k)}(0)\right\|^2$. Letting $t \to 0_+$ in Eq. $(3.7)_1$, multiplying the result by $\ddot{c}_{mj}^{(k)}(0)$, we get

$$\begin{aligned}&\left\|\ddot{u}_m^{(k)}(0)\right\|^2 + \lambda D_m(0)\left\|\ddot{u}_{mx}^{(k)}(0)\right\|^2 + C_m(0)\langle \tilde{u}_{0kx}, \ddot{u}_{mx}^{(k)}(0)\rangle \\ &= \langle F_m(0), \ddot{u}_m^{(k)}(0)\rangle. \end{aligned} \tag{3.15}$$

This implies that

$$\begin{aligned}\xi_m^{(k)} &= \left\|\ddot{u}_m^{(k)}(0)\right\|^2 + \lambda D_m(0)\left\|\ddot{u}_{mx}^{(k)}(0)\right\|^2 \\ &= -C_m(0)\langle \tilde{u}_{0kx}, \ddot{u}_{mx}^{(k)}(0)\rangle + \langle F_m(0), \ddot{u}_m^{(k)}(0)\rangle \\ &\le \left(C_m(0)\left\|\tilde{u}_{0kx}\right\| + \left\|F_m(0)\right\|\right)\left\|\ddot{u}_{mx}^{(k)}(0)\right\| \\ &\le \left(C_m(0)\left\|\tilde{u}_{0kx}\right\| + \left\|F_m(0)\right\|\right)\sqrt{\frac{\xi_m^{(k)}}{\lambda b_0}} \\ &\le \frac{1}{\lambda b_0}\left(C_m(0)\left\|\tilde{u}_{0kx}\right\| + \left\|F_m(0)\right\|\right)^2 \le \overline{X}_0 \quad \text{for all} \quad m, k \in \mathbb{N}, \end{aligned} \tag{3.16}$$

where $\overline{X}_0$ is a constant depending only on $\lambda$, $f$, $\tilde{u}_0$, $\tilde{u}_1$, $B_1$ and $B_2$. By (3.3), (3.8), (3.12), (3.13) and (3.16)

$$\begin{aligned}S_m^{(k)}(0) = \xi_m^{(k)} &+ \left\|\tilde{u}_{1k}\right\|^2 + \left\|\tilde{u}_{1kx}\right\|^2 \\ &+ B_1\left(\left\|\tilde{u}_{0x}\right\|^2, \left\|\tilde{u}_{1x}\right\|^2\right)\left(\left\|\tilde{u}_{0kx}\right\|^2 + \left\|\Delta\tilde{u}_{0k}\right\|^2 + \left\|\tilde{u}_{1kx}\right\|^2\right) \\ &+ \lambda B_2\left(\left\|\tilde{u}_{0x}\right\|^2, \left\|\tilde{u}_{1x}\right\|^2\right)\left(\left\|\tilde{u}_{1kx}\right\|^2 + \left\|\Delta\tilde{u}_{1k}\right\|^2\right) \\ &\le S_0, \quad \text{for all} \quad m \in \mathbb{N}, \end{aligned} \tag{3.17}$$

with a constant $S_0$ depending only on $\lambda$, $f$, $\tilde{u}_0$, $\tilde{u}_1$, $B_1$ and $B_2$.

Next, we shall estimate three terms $I_j$ on the right-hand side of (3.14) as follows.
*First term* $I_1$. By the Cauchy-Schwartz inequality, we have

$$I_1 = 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds \le TK_M^2(f) + \int_0^t p_m^{(k)}(s)ds. \tag{3.18}$$

*Second term* $I_2$. It is known that

$$\begin{aligned}\nabla F_m(t) = &D_1 f[u_{m-1}] + D_3 f[u_{m-1}]\nabla u_{m-1}(t) + D_4 f[u_{m-1}]\Delta u_{m-1}(t) \\ &+ D_5 f[u_{m-1}]\nabla u'_{m-1}(t) + D_6 f[u_{m-1}]\Delta u'_{m-1}(t), \end{aligned} \tag{3.19}$$

with $D_i f[u_{m-1}] = D_i f(x, t, u_{m-1}(t), \nabla u_{m-1}(t), u'_{m-1}(t), \nabla u'_{m-1}(t))$, $i = 1, \cdots, 6$. Combining $(3.1)_2$, (3.3) and (3.19), we obtain

$$\begin{aligned} &\|\nabla F_m(t)\| \\ &\leq \left[1 + \|\nabla u_{m-1}(t)\| + \|\Delta u_{m-1}(t)\| + \left\|\nabla u'_{m-1}(t)\right\| + \left\|\Delta u'_{m-1}(t)\right\| \ \right] K_M(f) \\ &\leq \gamma_M K_M(f), \end{aligned} \tag{3.20}$$

where $\gamma_M = 1 + 4M$, so it implies that

$$\begin{aligned} I_2 &= 2\int_0^t \langle \nabla F_m(s), \dot{u}_{mx}^{(k)}(s)\rangle ds \\ &\leq 2\int_0^t \|\nabla F_m(s)\| \left\|\dot{u}_{mx}^{(k)}(s)\right\| ds \\ &\leq T\gamma_M^2 K_M^2(f) + \int_0^t q_m^{(k)}(s) ds. \end{aligned} \tag{3.21}$$

*Third term* $I_3$. Similarly, by the following equality

$$\begin{aligned} F'_m(t) = D_2 f[u_{m-1}] &+ D_3 f[u_{m-1}] u'_{m-1}(t) + D_4 f[u_{m-1}] \nabla u'_{m-1}(t) \\ &+ D_5 f[u_{m-1}] u''_{m-1}(t) + D_6 f[u_{m-1}] \nabla u''_{m-1}(t), \end{aligned} \tag{3.22}$$

we obtain

$$\begin{aligned} &\left\|F'_m(t)\right\| \\ &\leq \left[1 + \left\|u'_{m-1}(t)\right\| + \left\|\nabla u'_{m-1}(t)\right\| + \left\|u''_{m-1}(t)\right\| + \left\|\nabla u''_{m-1}(t)\right\| \ \right] K_M(f) \\ &\leq \gamma_M K_M(f). \end{aligned} \tag{3.23}$$

Thus

$$\begin{aligned} I_3 &= 2\int_0^t 2\langle F'_m(s), \ddot{u}_m^{(k)}(s)\rangle ds \\ &\leq 2\int_0^t \left\|F'_m(s)\right\| \left\|\ddot{u}_m^{(k)}(s)\right\| ds \\ &\leq T\gamma_M^2 K_M^2(f) + \int_0^t r_m^{(k)}(s) ds. \end{aligned} \tag{3.24}$$

*Fourth term* $I_4$. It is obviously that

$$\begin{aligned} C'_m(t) = 2D_1 B_1[u_{m-1}] &\langle \nabla u_{m-1}(t), \nabla u'_{m-1}(t)\rangle \\ &+ 2D_2 B_1[u_{m-1}] \langle \nabla u'_{m-1}(t), \nabla u''_{m-1}(t)\rangle, \end{aligned} \tag{3.25}$$

with $D_jB_i[u_{m-1}] = D_jB_i\left(\|\nabla u_{m-1}(t)\|^2, \left\|\nabla u'_{m-1}(t)\right\|^2\right)$, $i,\ j = 1, 2$. Hence, by the Cauchy-Schwartz inequality, and (3.3), we have

$$\begin{aligned}
&\left|C'_m(t)\right| \\
&\leq 2\left[\|\nabla u_{m-1}(t)\| \left\|\nabla u'_{m-1}(t)\right\| + \left\|\nabla u'_{m-1}(t)\right\| \left\|\nabla u''_{m-1}(t)\right\| \ \right] \tilde{K}_M(B_1) \\
&\leq 4M^2 \tilde{K}_M(B_1). \qquad (3.26)
\end{aligned}$$

Similarly

$$\left|D'_m(t)\right| \leq 4M^2 \tilde{K}_M(B_2). \qquad (3.27)$$

On the other hand, from assumption $(H_2)$ we obtain from (3.12), (3.13) that

$$\begin{aligned}
S_m^{(k)}(t) \geq b_* \Big[ &\left\|\ddot{u}_m^{(k)}(t)\right\|^2 + \left\|\ddot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 \\
&+ \left\|\Delta \dot{u}_m^{(k)}(t)\right\|^2 + \left\|u_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2 \Big], \qquad (3.28)
\end{aligned}$$

where $b_* = \min\{1,\, b_0,\, \lambda b_0\} > 0$. Since

$$-2\langle u_{mx}^{(k)}(s), \ddot{u}_{mx}^{(k)}(s)\rangle \leq \left\|u_{mx}^{(k)}(s)\right\|^2 + \left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2, \qquad (3.29)$$

it follows from (3.26), (3.28), (3.29), that

$$\begin{aligned}
I_4 &= \int_0^t C'_m(s) \Big( \left\|u_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2 \\
&\quad + \left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 - 2\langle u_{mx}^{(k)}(s), \ddot{u}_{mx}^{(k)}(s)\rangle \Big) ds \\
&\leq \int_0^t \left|C'_m(s)\right| \left( 2\left\|u_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2 \right) ds \\
&\leq \frac{8}{b_*} M^2 \tilde{K}_M(B_1) \int_0^t S_m^{(k)}(s) ds. \qquad (3.30)
\end{aligned}$$

*Fifth term* $I_5$. Similarly, It follows from (3.27), (3.28), that

$$\begin{aligned}
I_5 &= \lambda \int_0^t D'_m(s) \left( \left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta \dot{u}_m^{(k)}(s)\right\|^2 - \left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2 \right) ds \\
&\leq \lambda \int_0^t \left|D'_m(s)\right| \left( \left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta \dot{u}_m^{(k)}(s)\right\|^2 + \left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2 \right) ds \\
&\leq \frac{4}{b_*} \lambda M^2 \tilde{K}_M(B_2) \int_0^t S_m^{(k)}(s) ds. \qquad (3.31)
\end{aligned}$$

Finally, from (3.14), (3.17), (3.18), (3.21), (3.24), (3.30) and (3.31), the following inequality is fulfilled

$$S_m^{(k)}(t) \le S_0 + \left(1 + 2\gamma_M^2\right) T K_M^2(f) + \bar{D}_M \int_0^t S_m^{(k)}(s) ds, \tag{3.32}$$

where $\bar{D}_M = 1 + \frac{4M^2}{b_*}\left(2\tilde{K}_M(B_1) + \lambda \tilde{K}_M(B_2)\right)$. We can choose $M > 0$ sufficiently large such that

$$S_0 \le \frac{1}{2} M^2. \tag{3.33}$$

Choose $T \in (0, T^*]$ small enough such that

$$\left(\frac{1}{2} M^2 + \left(1 + 2\gamma_M^2\right) T K_M^2(f)\right) \exp\left(T \bar{D}_M\right) \le M^2, \tag{3.34}$$

and

$$\begin{aligned} k_T = & \sqrt{\frac{16}{b_*}\left[K_M^2(f) + \frac{M^4}{b_*}\left(\tilde{K}_M^2(B_1) + \lambda^2 \tilde{K}_M^2(B_2)\right)\right]} \\ & \times \sqrt{T \exp\left[T\left(3 + \frac{4M^2}{b_*}\left(\tilde{K}_M(B_1) + \lambda \tilde{K}_M(B_2)\right)\right)\right]} < 1. \end{aligned} \tag{3.35}$$

It follows from (3.32)-(3.34) that

$$S_m^{(k)}(t) \le M^2 \exp\left(-T \bar{D}_M\right) + \bar{D}_M \int_0^t S_m^{(k)}(s) ds. \tag{3.36}$$

By using Gronwall's Lemma, (3.36) yields

$$S_m^{(k)}(t) \le M^2 \exp\left(-T \bar{D}_M\right) \exp\left(t \bar{D}_M\right) \le M^2, \tag{3.37}$$

for all $t \in [0, T]$, and for all $m$, $k \in \mathbb{N}$. Therefore

$$u_m^{(k)} \in W(M, T) \quad \text{for all} \quad m \quad \text{and} \quad k \in \mathbb{N}. \tag{3.38}$$

**Step 3.** *Limiting process.*

From (3.37), we deduce the existence of a subsequence of $\{u_m^{(k)}\}$ denoted by the same symbol, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0, T; H_0^1 \cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0, T; H_0^1 \cap H^2) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^\infty(0, T; H_0^1) \text{ weakly*}, \\ u_m \in W(M, T). \end{cases} \tag{3.39}$$

Passing to the limit in (3.7), (3.8), we have $u_m$ satisfying (3.4), (3.5) in $L^2(0,T)$. On the other hand, it follows from $(3.4)_1$ and $(3.39)_4$ that

$$\Delta u''_m = \frac{1}{\lambda D_m(t)} \left( u''_m - F_m(t) - C_m(t)\Delta u_m \right) \in L^\infty(0,T;L^2). \tag{3.40}$$

Consequently

$$u''_m \in L^\infty(0,T;H_0^1 \cap H^2), \tag{3.41}$$

hence $u_m \in W_1(M,T)$ and the proof of Theorem 3.1 is complete. □

We apply Theorem 3.1 and the compact imbedding theorems to get the existence and uniqueness of a weak solution of Prob. (1.1)-( 1.3), the main result of this section.

**Theorem 3.2.** *Suppose assumptions* $(H_1)-(H_3)$. *Then*

(i) *Prob.* (1.1)-(1.3) *has a unique weak solution* $u \in W_1(M,T)$, *where the constants* $M>0$ *and* $T>0$ *are chosen as in Theorem* 3.1.

(ii) *The linear recurrent sequence* $\{u_m\}$ *defined by* (3.4) *and* (3.5) *converges to the solution* $u$ *of Prob.* (1.1)-(1.3) *strongly in the space*

$$W_1(T) = \{v \in L^\infty(0,T;H_0^1) : v' \in L^\infty(0,T;H_0^1)\}.$$

*And we have the estimate*

$$\|u_m - u\|_{L^\infty(0,T;H_0^1)} + \left\|u'_m - u'\right\|_{L^\infty(0,T;H_0^1)} \le C k_T^m, \quad \text{for all } m \in \mathbb{N}, \tag{3.42}$$

*where the constant* $k_T \in (0,1)$ *is defined as in* (3.35) *and* $C$ *is a constant only depending on* $T$, $\tilde{u}_0$, $\tilde{u}_1$ *and* $k_T$.

*Proof.* **(a)** *Existence.* First, we note that $W_1(T)$ is a Banach space with respect to the norm (see Lions [11]).

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;H_0^1)} + \left\|v'\right\|_{L^\infty(0,T;H_0^1)}.$$

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then $w_m$ satisfies the variational problem

$$\begin{cases} \langle w''_m(t), w\rangle + C_{m+1}(t)\langle w_{mx}(t), w_x\rangle + \lambda D_{m+1}(t)\langle w''_{mx}(t), w_x\rangle \\ = -\left[C_{m+1}(t) - C_m(t)\right]\langle u_{mx}(t), w_x\rangle \\ \quad -\lambda\left[D_{m+1}(t) - D_m(t)\right]\langle u''_{mx}(t), w_x\rangle \\ \quad + \langle F_{m+1}(t) - F_m(t), w\rangle, \quad \forall w \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u'_m(0) = \tilde{u}_1, \end{cases} \tag{3.43}$$

Taking $w = w_m'$ in (3.43), after integrating in $t$, we get

$$\begin{aligned} Z_m(t) = & \int_0^t \left( C_{m+1}'(s) \left\| w_{mx}(s) \right\|^2 + \lambda D_{m+1}'(s) \left\| w_{mx}'(s) \right\|^2 \right) ds \\ & -2 \int_0^t \left[ C_{m+1}(s) - C_m(s) \right] \langle u_{mx}(s), w_{mx}'(s) \rangle \, ds \\ & -2\lambda \int_0^t \left[ D_{m+1}(s) - D_m(s) \right] \langle u_{mx}''(s), w_{mx}'(s) \rangle \, ds \\ & +2 \int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s) \rangle \, ds = J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{3.44}$$

where

$$Z_m(t) = \left\| w_m'(t) \right\|^2 + C_{m+1}(t) \left\| w_{mx}(t) \right\|^2 + \lambda D_{m+1}(t) \left\| w_{mx}'(t) \right\|^2. \tag{3.45}$$

We shall estimate three integrals $J_1$, $J_2$, $J_3$, $J_4$ on the right–hand side of (3.44) as follows.

*Estimation of* $J_1$. By

$$\begin{aligned} & \left| C_{m+1}'(t) \right| \leq 4M^2 \tilde{K}_M(B_1), \left| D_{m+1}'(t) \right| \leq 4M^2 \tilde{K}_M(B_2), \\ & \quad Z_m(t) \geq b_* \left( \left\| w_m'(t) \right\|^2 + \left\| w_{mx}(t) \right\|^2 + \left\| w_{mx}'(t) \right\|^2 \right), \end{aligned} \tag{3.46}$$

we have

$$\begin{aligned} J_1 & = \int_0^t \left( C_{m+1}'(s) \left\| w_{mx}(s) \right\|^2 + \lambda D_{m+1}'(s) \left\| w_{mx}'(s) \right\|^2 \right) ds \\ & \leq \int_0^t \left( 4M^2 \tilde{K}_M(B_1) \left\| w_{mx}(s) \right\|^2 + 4\lambda M^2 \tilde{K}_M(B_2) \left\| w_{mx}'(s) \right\|^2 \right) ds \\ & \leq \frac{4M^2}{b_*} \left( \tilde{K}_M(B_1) + \lambda \tilde{K}_M(B_2) \right) \int_0^t Z_m(s) ds. \end{aligned} \tag{3.47}$$

*Estimation of* $J_2$. We have

$$\begin{aligned} \left| C_{m+1}(t) - C_m(t) \right| & \leq 2M \tilde{K}_M(B_1) \left[ \left\| \nabla w_{m-1}(t) \right\| + \left\| \nabla w_{m-1}'(t) \right\| \right] \\ & \leq 2M \tilde{K}_M(B_1) \left\| w_{m-1} \right\|_{W_1(T)}. \end{aligned} \tag{3.48}$$

Similarly

$$\left| D_{m+1}(t) - D_m(t) \right| \leq 2M \tilde{K}_M(B_2) \left\| w_{m-1} \right\|_{W_1(T)}. \tag{3.49}$$

Hence

$$\begin{aligned} J_2 & = -2 \int_0^t \left[ C_{m+1}(s) - C_m(s) \right] \langle u_{mx}(s), w_{mx}'(s) \rangle \, ds \\ & \leq 4M \tilde{K}_M(B_1) \left\| w_{m-1} \right\|_{W_1(T)} \int_0^t \left\| u_{mx}(s) \right\| \left\| w_{mx}'(s) \right\| ds \\ & \leq \frac{4}{b_*} T M^4 \tilde{K}_M^2(B_1) \left\| w_{m-1} \right\|_{W_1(T)}^2 + \int_0^t Z_m(s) ds. \end{aligned} \tag{3.50}$$

*Estimation of* $J_3$. Similarly

$$J_3 = -2\lambda \int_0^t \left[D_{m+1}(s) - D_m(s)\right] \left\langle u''_{mx}(s), w'_{mx}(s)\right\rangle ds$$
$$\leq \frac{4}{b_*} T\lambda^2 M^4 \tilde{K}_M^2(B_2) \left\|w_{m-1}\right\|^2_{W_1(T)} + \int_0^t Z_m(s)ds. \qquad (3.51)$$

*Estimation of* $J_4$. From $(H_3)$ we obtain from $(3.1)_2$, (3.3), (3.5), $(3.39)_4$, that

$$\left\|F_{m+1}(t) - F_m(t)\right\|$$
$$\leq K_M(f)\left[\left\|w_{m-1}(t)\right\| + \left\|\nabla w_{m-1}(t)\right\| + \left\|w'_{m-1}(t)\right\| + \left\|\nabla w'_{m-1}(t)\right\|\right]$$
$$\leq 2K_M(f)\left[\left\|\nabla w_{m-1}(t)\right\| + \left\|\nabla w'_{m-1}(t)\right\|\ \right] \leq 2K_M(f)\left\|w_{m-1}\right\|_{W_1(T)}. \qquad (3.52)$$

Hence

$$J_4 = 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w'_m(s)\right\rangle ds$$
$$\leq 4K_M(f)\left\|w_{m-1}\right\|_{W_1(T)} \int_0^t \left\|w'_m(s)\right\| ds$$
$$\leq 4TK_M^2(f)\left\|w_{m-1}\right\|^2_{W_1(T)} + \int_0^t Z_m(s)ds. \qquad (3.53)$$

Combining (3.44), (3.47), (3.50), (3.51) and (3.53), we obtain

$$Z_m(t) \leq 4T\left[K_M^2(f) + \frac{M^4}{b_*}\left(\tilde{K}_M^2(B_1) + \lambda^2 \tilde{K}_M^2(B_2)\right)\right]\left\|w_{m-1}\right\|^2_{W_1(T)}$$
$$+ \left[3 + \frac{4M^2}{b_*}\left(\tilde{K}_M(B_1) + \lambda \tilde{K}_M(B_2)\right)\right]\int_0^t Z_m(s)ds. \qquad (3.54)$$

Using Gronwall's Lemma, we deduce from (3.54) that

$$\left\|w_m\right\|_{W_1(T)} \leq k_T \left\|w_{m-1}\right\|_{W_1(T)}, \quad \forall\, m \in \mathbb{N}, \qquad (3.55)$$

where $0 < k_T < 1$ is defined as in (3.35), which implies that

$$\left\|u_m - u_{m+p}\right\|_{W_1(T)} \leq \left\|u_0 - u_1\right\|_{W_1(T)} (1 - k_T)^{-1} k_T^m, \ \ \forall\, m, p \in \mathbb{N}. \qquad (3.56)$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strongly in} \quad W_1(T). \qquad (3.57)$$

Since $u_m \in W_1(M,T)$, there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T; H_0^1 \cap H^2) \text{ weakly}^*, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T; H_0^1 \cap H^2) \text{ weakly}^*, \\ u''_{m_j} \to u'' & \text{in} \quad L^\infty(0,T; H_0^1) \text{ weakly}^*, \\ u \in W(M,T). \end{cases} \qquad (3.58)$$

By (3.1), (3.3), (3.5) and $(3.58)_4$, we obtain

$$\begin{aligned} &\left\|F_m(t)-f(\cdot,t,u(t),u_x(t),u'(t),u'_x(t))\right\| \leq 2K_M(f)\left\|u_{m-1}-u\right\|_{W_1(T)}, \\ &\left|C_m(t)-B_1\left(\|\nabla u(t)\|^2,\left\|\nabla u'(t)\right\|^2\right)\right| \leq 4M\tilde{K}_M(B_1)\left\|u_{m-1}-u\right\|_{W_1(T)}, \\ &\left|D_m(t)-B_2\left(\|\nabla u(t)\|^2,\left\|\nabla u'(t)\right\|^2\right)\right| \leq 4M\tilde{K}_M(B_2)\left\|u_{m-1}-u\right\|_{W_1(T)}. \end{aligned} \tag{3.59}$$

Hence, from (3.57) and (3.59), we obtain

$$\begin{cases} F_m(t) \to f(\cdot,t,u(t),u_x(t),u'(t),u'_x(t)) \text{ strongly in } L^\infty(0,T;L^2), \\ C_m(t) \to B_1\left(\|\nabla u(t)\|^2,\|\nabla u'(t)\|^2\right) \text{ strongly in } L^\infty(0,T), \\ D_m(t) \to B_2\left(\|\nabla u(t)\|^2,\|\nabla u'(t)\|^2\right) \text{ strongly in } L^\infty(0,T). \end{cases} \tag{3.60}$$

Finally, passing to the limit in (3.4), (3.5) as $m=m_j\to\infty$, it follows from (3.57), $(3.58)_{1,3}$, and (3.60) that there exists $u\in W(M,T)$ satisfying the equation

$$\begin{aligned} &\langle u''(t),w\rangle + B_1\left(\|\nabla u(t)\|^2,\left\|\nabla u'(t)\right\|^2\right)\langle u_x(t),w_x\rangle \\ &+\lambda B_2\left(\|\nabla u(t)\|^2,\left\|\nabla u'(t)\right\|^2\right)\langle u''_x(t),w_x\rangle \\ &=\left\langle f(\cdot,t,u(t),u_x(t),u'(t),u'_x(t)),w\right\rangle, \end{aligned} \tag{3.61}$$

for all $w\in H_0^1$ and the initial conditions

$$u(0)=\tilde{u}_0,\ u'(0)=\tilde{u}_1. \tag{3.62}$$

On the other hand, from assumptions $(H_2)$, $(H_3)$ we obtain from $(3.58)_4$, (3.60) and (3.61) that

$$\begin{aligned} \Delta u'' &= \frac{u''-f(\cdot,t,u,u_x,u',u'_x)-B_1\left(\|\nabla u(t)\|^2,\|\nabla u'(t)\|^2\right)\Delta u}{\lambda B_2\left(\|\nabla u(t)\|^2,\|\nabla u'(t)\|^2\right)} \\ &\in L^\infty(0,T;L^2). \end{aligned} \tag{3.63}$$

Hence

$$u''\in L^\infty(0,T;H_0^1\cap H^2), \tag{3.64}$$

so $u\in W_1(M,T)$ and the existence follows.

**(b)** *Uniqueness.* Let $u_1$, $u_2$ be two weak solutions of Prob. (1.1)-(1.3), such that $u_i\in W_1(M,T)$, $i=1,2$. Then $w=u_1-u_2$ verifies

$$\begin{cases} \langle w''(t),w\rangle+\bar{C}_1(t)\langle w_x(t),w_x\rangle+\lambda\bar{D}_1(t)\langle w''_x(t),w_x\rangle \\ =-\left(\bar{C}_1(t)-\bar{C}_2(t)\right)\langle u_{2x}(t),w_x\rangle-\lambda\left(\bar{D}_1(t)-\bar{D}_2(t)\right)\langle u''_{2x}(t),w_x\rangle \\ \quad+\left\langle\bar{F}_1(t)-\bar{F}_2(t),w\right\rangle, \quad \text{for all } v\in H_0^1, \\ w(0)=w'(0)=0, \end{cases} \tag{3.65}$$

where

$$\begin{cases} \bar{C}_i(t) = B_1\left(\|\nabla u_i(t)\|^2, \|\nabla u_i'(t)\|^2\right), \\ \bar{D}_i(t) = B_2\left(\|\nabla u_i(t)\|^2, \|\nabla u_i'(t)\|^2\right), \\ \bar{F}_i(t) = f(\cdot, t, u_i(t), u_{ix}(t), u_i'(t), u_{ix}'(t)), \quad i = 1, 2. \end{cases}$$

Taking $v = w' = u_1' - u_2'$ in $(3.65)_1$ and integrating with respect to $t$, we obtain

$$\begin{aligned} \sigma(t) = & \ 2\int_0^t \left\langle \bar{F}_1(s) - \bar{F}_2(s), w'(s)\right\rangle ds \\ & + \int_0^t \left(\bar{C}_1'(s)\|w_x(s)\|^2 + \lambda \bar{D}_1'(s)\left\|w_x'(s)\right\|^2\right) ds \\ & - 2\int_0^t \left[\bar{C}_1(s) - \bar{C}_2(s)\right]\left\langle u_{2x}(s), w_x'(s)\right\rangle ds \\ & - 2\lambda \int_0^t \left[\bar{D}_1(s) - \bar{D}_2(s)\right]\left\langle u_{2x}''(s), w_x'(s)\right\rangle ds, \end{aligned} \tag{3.66}$$

where

$$\sigma(t) = \left\|w'(t)\right\|^2 + \bar{C}_1(t)\|w_x(t)\|^2 + \lambda \bar{D}_1(t)\left\|w_x'(t)\right\|^2.$$

Put $\hat{K}_M = \frac{4}{\sqrt{b_*}}K_M(f) + \frac{12M^2}{b_*}\left[\tilde{K}_M(B_1) + \lambda \tilde{K}_M(B_2)\right]$. Then it follows from (3.66) that

$$\sigma(t) \le \hat{K}_M \int_0^t \sigma(s) ds.$$

By Gronwall's Lemma, we deduce $\sigma(t) = 0$, *i.e.*, $u_1 \equiv u_2$. This completes the proof of the theorem. □

## 4. Asymptotic expansion of the solution with respect to a small parameter

In this section, let $(H_1) - (H_3)$ hold. We also make the following assumptions:

$(H_4)$ $f_1 \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, and
$f_1(0, t, 0, v, 0, z) = f_1(1, t, 0, v, 0, z) = 0, \quad \text{for all } (t, v, z) \in \mathbb{R}_+ \times \mathbb{R}^2.$

We consider the following perturbed problem, where $\varepsilon$ is a small parameter, with $|\varepsilon| < 1$ :

$$(P_\varepsilon) \begin{cases} u_{tt} - B_1[u](t)u_{xx} - \lambda B_2[u](t)u_{xxtt} = F_\varepsilon[u], \ 0 < x < 1, \ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x), \\ B_i[u](t) = B_i\left(\|u_x(t)\|^2, \|u_{xt}(t)\|^2\right), \ i = 1, 2, \\ F_\varepsilon[u] = F_\varepsilon(x,t,u,u_x,u_t,u_{xt}) \\ \qquad = f(x,t,u,u_x,u_t,u_{xt}) + \varepsilon f_1(x,t,u,u_x,u_t,u_{xt}). \end{cases}$$

By the assumptions $(H_1) - (H_3)$, $(H_4)$ and theorem 3.2, Prob. $(P_\varepsilon)$ has a unique weak solution $u$ depending on $\varepsilon : u = u_\varepsilon$. When $\varepsilon = 0$, $(P_\varepsilon)$ is denoted by $(\tilde{P}_0)$. We shall study the asymptotic expansion of the solution $u_\varepsilon$ of Prob. $(P_\varepsilon)$ with respect to a small parameter $\varepsilon$.

We use the following notations. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{Z}_+^N$, and $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, we put

$$\begin{cases} |\alpha| = \alpha_1 + \cdots + \alpha_N, \ \alpha! = \alpha_1! \cdots \alpha_N!, \\ x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \\ \alpha, \ \beta \in \mathbb{Z}_+^N, \ \alpha \le \beta \iff \alpha_i \le \beta_i, \quad \forall i = 1, \cdots, N. \end{cases} \tag{4.1}$$

First, we shall need the following lemma.

**Lemma 4.1.** *Let $m$, $N \in \mathbb{N}$, $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, and $\varepsilon \in \mathbb{R}$. Then*

$$\left(\sum_{i=1}^{N} x_i \varepsilon^i\right)^m = \sum_{k=m}^{mN} P_N^{[m]}[x]_k \varepsilon^k, \tag{4.2}$$

*where the coefficients $P_N^{[m]}[x]_k$, $m \le k \le mN$ depending on $x = (x_1, \cdots, x_N)$ are defined by the formula*

$$P_N^{[m]}[x]_k = \begin{cases} x_k, & 1 \le k \le N, \ m = 1, \\ \sum\limits_{\alpha \in A_k^{[m]}(N)} \dfrac{m!}{\alpha!} x^\alpha, & m \le k \le mN, \ m \ge 2, \end{cases} \tag{4.3}$$

*where* $A_k^{[m]}(N) = \left\{ \alpha \in \mathbb{Z}_+^N : |\alpha| = m, \ \sum\limits_{i=1}^{N} i\alpha_i = k \right\}$.

*Proof.* The proof of this lemma is easy, hence we omit the details. □

Now, we assume that

$(H_5)$ $B_1$, $B_2 \in C^{N+1}(\mathbb{R}_+^2)$, $B_i(y,z) \ge b_0 > 0$, $\forall\,(y,z) \in \mathbb{R}_+^2$, $(i = 1,2)$,

$(H_6)$ $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $f_1 \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, and $f(0,t,0,v,0,z) = f(1,t,0,v,0,z) = f_1(0,t,0,v,0,z)$ $= f_1(1,t,0,v,0,z) = 0$, $\forall\,(t,v,z) \in \mathbb{R}_+ \times \mathbb{R}^2$.

We also use the notations $f[u]=f(x,t,u,u_x,u_t,u_{xt})$, $B[u]=B(\|u_x\|^2,\|u_{xt}\|^2)$.

Let $u_0$ be a unique weak solution of problem $(P_0)$ (as in Theorem 3.2) corresponding to $\varepsilon=0$, *i.e.*,

$$(P_0)\begin{cases} u_0''-B_1[u_0](t)\Delta u_0-\lambda B_2[u_0](t)\Delta u_0''=f[u_0],\ 0<x<1,\ 0<t<T,\\ u_0(0,t)=u_0(1,t)=0,\\ u_0(x,0)=\tilde{u}_0(x),\ u_0'(x,0)=\tilde{u}_1(x),\\ u_0\in W_1(M,T).\end{cases}$$

Considering the sequence of weak solutions $u_r$, $1\le r\le N$, of the following problems:

$$(\tilde{P}_r)\begin{cases} u_r''-B_1[u_0](t)\Delta u_r-\lambda B_2[u_0](t)\Delta u_r''=F_r,\ 0<x<1,\ 0<t<T,\\ u_r(0,t)=u_r(1,t)=0,\\ u_r(x,0)=u_r'(x,0)=0,\\ u_r\in W_1(M,T),\end{cases}$$

where $F_r$, $1\le r\le N$, are defined by the recurrence formulas

$$F_r=\begin{cases} f[u_0], & r=0,\\ \pi_r[N,f]+\pi_{r-1}[N-1,f_1] & \\ +\sum_{i=1}^{r}\left(\rho_i[B_1]\Delta u_{r-i}+\lambda\rho_i[B_2]\Delta u_{r-i}''\right), & 1\le r\le N,\end{cases} \tag{4.4}$$

with $\rho_r[B]=\rho_r[B;\sigma^{(1)},\sigma^{(2)}]$, $\pi_r[N,f]=\pi_r[N,f;u_0,u_1,\cdots,u_r]$, $0\le r\le N$, defined by the formulas:

**(a)** *Formula* $\rho_r[B]$ :

$$\rho_r[B]=\begin{cases} B[u_0], & r=0,\\ \sum_{|\gamma|\le r}\frac{1}{\gamma!}D^\gamma B[u_0]\sum_{\substack{\gamma_1\le i\le 2\gamma_1 N,\\ \gamma_2\le r-i\le 2\gamma_2 N}} P_{2N}^{[\gamma_1]}[\sigma^{(1)}]_i P_{2N}^{[\gamma_2]}[\sigma^{(2)}]_{r-i}, & 1\le r\le N,\end{cases} \tag{4.5}$$

where $\sigma^{(1)}=(\sigma_1^{(1)},\cdots,\sigma_{2N}^{(1)})$, $\sigma^{(2)}=(\sigma_1^{(2)},\cdots,\sigma_{2N}^{(2)})$, are defined by

$$\begin{aligned} \sigma_i^{(1)}&=\begin{cases} 2\langle\nabla u_0,\nabla u_1\rangle, & i=1,\\ 2\langle\nabla u_0,\nabla u_i\rangle+\sum_{j=1}^{i}\langle\nabla u_j,\nabla u_{i-j}\rangle, & 2\le i\le N,\\ \sum_{j=1}^{i}\langle\nabla u_j,\nabla u_{i-j}\rangle, & N+1\le i\le 2N,\end{cases}\\ \sigma_i^{(2)}&=\begin{cases} 2\langle\nabla u_0',\nabla u_1'\rangle, & i=1,\\ 2\langle\nabla u_0',\nabla u_i'\rangle+\sum_{j=1}^{i}\langle\nabla u_j',\nabla u_{i-j}'\rangle, & 2\le i\le N,\\ \sum_{j=1}^{i}\langle\nabla u_j',\nabla u_{i-j}'\rangle, & N+1\le i\le 2N;\end{cases}\end{aligned} \tag{4.6}$$

**(b)** *Formula* $\pi_r[N,f]$ :

$$\pi_r[N,f]=\begin{cases} f[u_0], & r=0,\\ \sum\limits_{\substack{1\le|m|\le r\\ m=(m_1,...,m_4)\in\mathbb{Z}_+^4}} \frac{1}{m!}D^m f[u_0] \sum\limits_{\substack{k=(k_1,...,k_4)\in A(m,N)\\ |k|=r}} P_N^{[m_1]}[u]_{k_1} & \\ \quad\times P_N^{[m_2]}[\nabla u]_{k_2} P_N^{[m_3]}[u']_{k_3} P_N^{[m_4]}[\nabla u']_{k_4}, & 1\le r\le N, \end{cases} \tag{4.7}$$

in which $m=(m_1,...,m_4)\in\mathbb{Z}_+^4$, $|m|=m_1+\cdots+m_4$, $m!=m_1!\cdots m_4!$, $D^m f=D_3^{m_1}D_4^{m_2}D_5^{m_3}D_6^{m_4}f$, $A(m,N)=\{k=(k_1,\cdots,k_4)\in\mathbb{Z}_+^4 : m_i\le k_i\le m_iN,\ i=1,2,3,4\}$.

Then, we have the following lemma.

**Lemma 4.2.** *Let* $\rho_r[B]=\rho_r[B,\sigma^{(1)},\sigma^{(2)}]$, $\pi_r[N,f]=\pi_r[N,f;u_0,u_1,\cdots,u_r]$, $0\le r\le N$, *be the functions defined by formulas* (4.5) *and* (4.7). *Let* $h=\sum\limits_{r=0}^{N}u_r\varepsilon^r$. *Then we have*

$$B[h]=\sum_{r=0}^{N}\rho_r[B]\varepsilon^r+|\varepsilon|^{N+1}\widetilde{R}_N^{(1)}[B,\varepsilon], \tag{4.8}$$

$$f[h]=\sum_{r=0}^{N}\pi_r[N,f]\varepsilon^r+|\varepsilon|^{N+1}\bar{R}_N^{(1)}[f,\varepsilon], \tag{4.9}$$

*with* $\left\|\widetilde{R}_N^{(1)}[B,\varepsilon]\right\|_{L^\infty(0,T)}+\left\|\bar{R}_N^{(1)}[f,\varepsilon]\right\|_{L^\infty(0,T;L^2)}\le C$, *where* $C$ *is a constant depending only on* $N$, $T$, $f$, $B_1$, $B_2$, $u_i$, $0\le i\le N$.

*Proof.* (i) In the case of $N=1$, the proof of (4.8) is easy, hence we omit the details. We only prove the case of $N\ge 2$. Let $h=u_0+\sum_{i=1}^{N}u_i\varepsilon^i\equiv u_0+h_1$. We rewrite as below

$$\begin{aligned} B[h]&=B(\|\nabla h\|^2,\left\|\nabla h'\right\|^2)\\ &=B(\|\nabla u_0+\nabla h_1\|^2,\left\|\nabla u_0'+\nabla h_1'\right\|^2)\\ &=B(\|\nabla u_0\|^2+\xi_1,\left\|\nabla u_0'\right\|^2+\xi_2), \end{aligned} \tag{4.10}$$

where $\xi_1=\|\nabla u_0+\nabla h_1\|^2-\|\nabla u_0\|^2$, $\xi_2=\|\nabla u_0'+\nabla h_1'\|^2-\|\nabla u_0'\|^2$. By using Taylor's expansion of the function $B(\|\nabla u_0\|^2+\xi_1,\|\nabla u_0'\|^2+\xi_2)$ around the point $(\|\nabla u_0\|^2, \|\nabla u_0'\|^2)$ up to order $N+1$, we obtain

$$\begin{aligned} B[h] &= B(\|\nabla u_0\|^2, \|\nabla u_0'\|^2) \\ &\quad + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma B(\|\nabla u_0\|^2, \|\nabla u_0'\|^2)\xi_1^{\gamma_1}\xi_2^{\gamma_2} + R_N[B, u_0, \xi_1, \xi_2] \\ &= B[u_0] + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma B[u_0]\xi_1^{\gamma_1}\xi_2^{\gamma_2} + R_N[B, u_0, \xi_1, \xi_2], \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} &R_N[B, u_0, \xi_1, \xi_2] \\ &= \int_0^1 \sum_{|\gamma|=N+1} \frac{N+1}{\gamma!}(1-\theta)^N D^\gamma B\left(\|\nabla u_0\|^2 + \theta\xi_1, \|\nabla u_0'\|^2 + \theta\xi_2\right)\xi_1^{\gamma_1}\xi_2^{\gamma_2} d\theta \\ &\equiv |\varepsilon|^{N+1} R_N^{(1)}[B, u_0, \xi_1, \xi_2]. \end{aligned} \tag{4.12}$$

On the other hand,

$$\begin{aligned} \xi_1 &= \|\nabla u_0 + \nabla h_1\|^2 - \|\nabla u_0\|^2 = 2\langle \nabla u_0, \nabla h_1\rangle + \|\nabla h_1\|^2 \\ &\equiv \sum_{i=1}^{2N} \sigma_i^{(1)}\varepsilon^i, \end{aligned} \tag{4.13}$$

with $\sigma_i^{(1)}$, $1 \le i \le 2N$ are defined by $(4.6)_1$. By the formula (4.2), it follows from (4.13) that

$$\xi_1^{\gamma_1} = \left(\sum_{i=1}^{2N} \sigma_i^{(1)}\varepsilon^i\right)^{\gamma_1} = \sum_{k=\gamma_1}^{2\gamma_1 N} P_{2N}^{[\gamma_1]}[\sigma^{(1)}]_k \varepsilon^k, \tag{4.14}$$

where $\sigma^{(1)} = (\sigma_1^{(1)}, \cdots, \sigma_{2N}^{(1)})$. Similarly, we have

$$\xi_2^{\gamma_2} = \left(\sum_{i=1}^{2N} \sigma_i^{(2)}\varepsilon^i\right)^{\gamma_2} = \sum_{k=\gamma_2}^{2\gamma_2 N} P_{2N}^{[\gamma_2]}[\sigma^{(2)}]_k \varepsilon^k, \tag{4.15}$$

where $\sigma^{(2)} = (\sigma_1^{(2)}, \cdots, \sigma_{2N}^{(2)})$, are defined by $(4.6)_2$. Therefore, it follows from (4.14), (4.15) that

$$\begin{aligned} \xi_1^{\gamma_1}\xi_2^{\gamma_2} &= \sum_{r=|\gamma|}^{N} \Phi_r[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2]\varepsilon^r \\ &\quad + |\varepsilon|^{N+1} R_N[N, \sigma^{(1)}, \sigma^{(2)}, \gamma_1, \gamma_2, \varepsilon], \end{aligned} \tag{4.16}$$

where

$$\begin{cases} \Phi_r[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2] = \sum\limits_{\substack{\gamma_1 \le i \le 2\gamma_1 N, \\ \gamma_2 \le r-i \le 2\gamma_2 N}} P_{2N}^{[\gamma_1]}[\sigma^{(1)}]_i P_{2N}^{[\gamma_2]}[\sigma^{(2)}]_{r-i}, \\ |\varepsilon|^{N+1} R_N[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\varepsilon] = \sum\limits_{r=N+1}^{2|\gamma|N} \Phi_r[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2]\varepsilon^r. \end{cases} \tag{4.17}$$

Hence, we deduce from (4.11), (4.16), (4.17) that

$$\begin{aligned} B[h] &= B[u_0] + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma B[u_0] \sum_{r=|\gamma|}^{N} \Phi_r[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2]\varepsilon^r \\ &\quad + |\varepsilon|^{N+1} \widehat{R}_N^{(1)}[B,u_0,\sigma^{(1)},\sigma^{(2)},\xi_1,\xi_2] \\ &= B[u_0] + \sum_{k=1}^{N} \left( \sum_{|\gamma|\le k} \frac{1}{\gamma!} D^\gamma B[u_0] \Phi_k[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2] \right) \varepsilon^k \\ &\quad + |\varepsilon|^{N+1} \widehat{R}_N^{(1)}[B,u_0,\sigma^{(1)},\sigma^{(2)},\xi_1,\xi_2] \\ &= \sum_{r=0}^{N} \rho_r[B,\sigma^{(1)},\sigma^{(2)}]\varepsilon^r + |\varepsilon|^{N+1} \widehat{R}_N^{(1)}[B,u_0,\sigma^{(1)},\sigma^{(2)},\xi_1,\xi_2], \end{aligned} \tag{4.18}$$

where $\rho_r[B] = \rho_r[B;\sigma^{(1)},\sigma^{(2)}]$, $0 \le r \le N$, are defined by (4.5) and

$$\begin{aligned} \widehat{R}_N^{(1)}[B,u_0,\sigma^{(1)},\sigma^{(2)},\xi_1,\xi_2] = \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma B[u_0] R_N[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\varepsilon] \\ + R_N^{(1)}[B,u_0,\xi_1,\xi_2]. \end{aligned} \tag{4.19}$$

By the boundedness of the functions $u_i$, $u_i'$, $0 \le i \le N$ in the function space $L^\infty(0,T;H_0^1 \cap H^2)$, we obtain from (4.12), (4.17), (4.19) that

$$\left\| \widehat{R}_N^{(1)}[B,u_0,\sigma^{(1)},\sigma^{(2)},\xi_1,\xi_2] \right\|_{L^\infty(0,T)} \le C,$$

where $C$ is a constant depending only on $N$, $T$, $B$, $\|\nabla u_i\|_{L^\infty(0,T;L^2)}$, $\|\nabla u_i'\|_{L^\infty(0,T;L^2)}$, $0 \le i \le N$. Hence, the part 1 of Lemma 4.2 is proved.

(ii) We only prove (4.9) with $N \ge 2$. By using Taylor's expansion of the function $f[u_0 + h_1]$ around the point $u_0$ up to order $N+1$, we obtain from

(4.2), that

$$
\begin{aligned}
& f[u_0+h_1] \\
& = f[u_0] + D_3 f[u_0] h_1 + D_4 f[u_0] \nabla h_1 + D_5 f[u_0] h_1' + D_6 f[u_0] \nabla h_1' \\
& \quad + \sum_{\substack{2 \le |m| \le N \\ m=(m_1,\dots,m_4) \in \mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] h_1^{m_1} (\nabla h_1)^{m_2} (h_1')^{m_3} \left(\nabla h_1'\right)^{m_4} + R_N^{(1)}[f,h_1] \\
& = f[u_0] + D_3 f[u_0] h_1 + D_4 f[u_0] \nabla h_1 + D_5 f[u_0] h_1' + D_6 f[u_0] \nabla h_1' \\
& \quad + \sum_{\substack{2 \le |m| \le N \\ m \in \mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{r=|m|}^{|m|N} \tilde{\Phi}_r[m,N,f,u,\nabla u,u',\nabla u'] \varepsilon^r + R_N^{(1)}[f,h_1] \\
& = f[u_0] + D_3 f[u_0] h_1 + D_4 f[u_0] \nabla h_1 + D_5 f[u_0] h_1' + D_6 f[u_0] \nabla h_1' \\
& \quad + \sum_{\substack{2 \le |m| \le N \\ m \in \mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{r=|m|}^{N} \tilde{\Phi}_r[m,N,f,u,\nabla u,u',\nabla u'] \varepsilon^r \\
& \quad + \sum_{\substack{2 \le |m| \le N \\ m \in \mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{r=N+1}^{|m|N} \tilde{\Phi}_r[m,N,f,u,\nabla u,u',\nabla u'] \varepsilon^r + R_N^{(1)}[f,h_1],
\end{aligned} \tag{4.20}
$$

where

$$
\begin{cases}
R_N^{(1)}[f,h_1] \\
= \sum\limits_{\substack{|m|=N+1 \\ m \in \mathbb{Z}_+^4}} \frac{N+1}{m!} \int_0^1 (1-\theta)^N D^m f[u_0+\theta h_1] h_1^{m_1} (\nabla h_1)^{m_2} (h_1')^{m_3} (\nabla h_1')^{m_4} \, d\theta, \\
\tilde{\Phi}_r[m,N,f,u,\nabla u,u',\nabla u'] \\
= \sum\limits_{\substack{k \in A(m,N) \\ |k|=r}} P_N^{[m_1]}[u]_{k_1} P_N^{[m_2]}[\nabla u]_{k_2} P_N^{[m_3]}[u']_{k_3} P_N^{[m_4]}[\nabla u']_{k_4}, \ |m| \le r \le |m| N, \\
A(m,N) = \{k=(k_1,\cdots,k_4) \in \mathbb{Z}_+^4 : m_i \le k_i \le m_i N, \ i=1,2,3,4\}.
\end{cases} \tag{4.21}
$$

We note that

$$
\begin{aligned}
& f[u_0] + D_3 f[u_0] h_1 + D_4 f[u_0] \nabla h_1 + D_5 f[u_0] h_1' + D_6 f[u_0] \nabla h_1' \\
& + \sum_{\substack{2 \le |m| \le N \\ m \in \mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{r=|m|}^{N} \tilde{\Phi}_r[m,N,f,u,\nabla u,u',\nabla u'] \varepsilon^r
\end{aligned}
$$

$$= f[u_0] + \sum_{\substack{1\le|m|\le N\\ m\in\mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{r=|m|}^{N} \tilde{\Phi}_r[m,N,f,u,\nabla u,u',\nabla u']\varepsilon^r$$

$$= f[u_0] + \sum_{r=1}^{N} \sum_{\substack{1\le|m|\le r\\ m\in\mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \tilde{\Phi}_r[m,N,f,u,\nabla u,u',\nabla u']\varepsilon^r$$

$$= \sum_{r=0}^{N} \pi_r[N,f]\varepsilon^r, \tag{4.22}$$

where $\pi_r[N,f]$, $0\le r\le N$ are defined by (4.7). Similarly,

$$\sum_{\substack{2\le|m|\le N\\ m\in\mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{r=N+1}^{|m|N} \tilde{\Phi}_r[m,N,f,u,\nabla u,u',\nabla u']\varepsilon^r + R_N^{(1)}[f,h_1]$$

$$= |\varepsilon|^{N+1}\, \bar{R}_N^{(1)}[f,\varepsilon], \tag{4.23}$$

with $\left\|\bar{R}_N^{(1)}[f,\varepsilon]\right\|_{L^\infty(0,T;L^2)} \le C$, where $C$ is a constant depending only on $N$, $T$, $f$, $u_i$, $i=0,1,\cdots,N$. This completes the proof of the lemma. □

**Remark 4.3.** Lemma 4.2 is a generalization of a formula contained in [14, p.262, formula (4.38)] and it is useful to obtain the following Lemma 4.4. These Lemmas are the key to the asymptotic expansion of the weak solution $u=u_\varepsilon$ of order $N+1$ in a small parameter $\varepsilon$.

Let $u=u_\varepsilon\in W_1(M,T)$ be a unique weak solution of the problem $(P_\varepsilon)$. Then $v = u-\sum_{r=0}^{N} u_r\varepsilon^r \equiv u-h = u-u_0-h_1$ satisfies the problem

$$\begin{cases} v''-B_1[v+h]\Delta v-\lambda B_2[v+h]\Delta v'' \\ = F_\varepsilon[v+h]-F_\varepsilon[h]+\left(B_1[v+h]-B_1[h]\right)\Delta h \\ \quad +\lambda\left(B_2[v+h]-B_2[h]\right)\Delta h''+E_\varepsilon(x,t),\ 0<x<1,\ 0<t<T, \\ v(0,t)=v(1,t)=0, \\ v(x,0)=v'(x,0)=0, \\ F_\varepsilon[v]=f[v]+\varepsilon f_1[v] \\ \qquad = f(x,t,v,\nabla v,v',\nabla v')+\varepsilon f_1(x,t,v,\nabla v,v',\nabla v'), \end{cases} \tag{4.24}$$

where

$$E_\varepsilon(x,t) = f[h] - f[u_0] + \varepsilon f_1[h] + (B_1[h] - B_1[u_0])\,\Delta h + \lambda\,(B_2[h] - B_2[u_0])\,\Delta h'' - \sum_{r=1}^{N} F_r\varepsilon^r. \tag{4.25}$$

**Lemma 4.4.** *Under the assumptions* $(H_1)$, $(H_5)$, *and* $(H_6)$, *there exists a constant* $\bar{C}_*$ *such that*

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le \bar{C}_* \,|\varepsilon|^{N+1}, \tag{4.26}$$

*where* $\bar{C}_*$ *is a constant depending only on* $N$, $T$, $f$, $f_1$, $B_1$, $B_2$, $u_r$, $0 \le r \le N$.

*Proof.* In the case of $N = 1$, the proof of Lemma 4.3 is easy. The details are omitted. We only consider $N \ge 2$.

By using formulas (4.8), (4.9) for the functions $f_1[h]$, $B_1[h]$ and $B_2[h]$, we obtain

$$\begin{cases} f_1[h] = \sum\limits_{r=0}^{N-1} \pi_r[N-1, f_1]\varepsilon^r + |\varepsilon|^N\, \bar{R}^{(1)}_{N-1}[f_1,\varepsilon], \\ B_i[h] = \sum\limits_{r=0}^{N} \rho_r[B_i]\varepsilon^r + |\varepsilon|^{N+1}\, \widetilde{R}^{(1)}_{N}[B_i,\varepsilon],\ i = 1,2. \end{cases} \tag{4.27}$$

By $(4.27)_1$, we rewrite $\varepsilon f_1[h]$ as follows

$$\varepsilon f_1[h] = \sum_{r=1}^{N} \pi_{r-1}[N-1, f_1]\varepsilon^r + \varepsilon\,|\varepsilon|^N\, \bar{R}^{(1)}_{N-1}[f_1,\varepsilon]. \tag{4.28}$$

First, we deduce from (4.9) and (4.28), that

$$\begin{aligned} &f[h] - f[u_0] + \varepsilon f_1[h] \\ &= \sum_{r=1}^{N} \left(\pi_r[N,f] + \pi_{r-1}[N-1,f_1]\right)\varepsilon^r + |\varepsilon|^{N+1}\, \bar{R}^{(1)}_{N}[f,f_1,\varepsilon], \end{aligned} \tag{4.29}$$

where $\bar{R}^{(1)}_{N}[f,f_1,\varepsilon] = \bar{R}^{(1)}_{N}[f,\varepsilon] + \frac{\varepsilon}{|\varepsilon|}\bar{R}^{(1)}_{N-1}[f_1,\varepsilon]$ is bounded in $L^\infty(0,T;L^2)$ by a constant depending only on $N$, $T$, $f$, $f_1$, $u_i$, $0 \le i \le N$.

On the other hand, we deduce from (4.8) and $(4.27)_2$ that

$$\begin{aligned} (B_1[h] - B_1[u_0])\,\Delta h &= \left(\sum_{r=1}^{N} \rho_r[B_1]\varepsilon^r + |\varepsilon|^{N+1}\, \widetilde{R}^{(1)}_{N}[B_1,\varepsilon]\right)\Delta h \\ &= \sum_{r=1}^{N}\sum_{i=1}^{r} \rho_i[B_1]\Delta u_{r-i}\varepsilon^r + |\varepsilon|^{N+1}\, \widetilde{R}^{(2)}_{N}[B_1,\varepsilon], \end{aligned} \tag{4.30}$$

where

$$\widetilde{R}_N^{(2)}[B_1,\varepsilon] = \widetilde{R}_N^{(1)}[B_1,\varepsilon]\Delta h + \frac{1}{|\varepsilon|^{N+1}} \sum_{r=N+1}^{2N} \sum_{i=1}^{r} \rho_i[B_1]\Delta u_{r-i}\varepsilon^r. \tag{4.31}$$

Similarly

$$(B_2[h] - B_2[u_0])\,\Delta h'' = \sum_{r=1}^{N} \sum_{i=1}^{r} \rho_i[B_2]\Delta u''_{r-i}\varepsilon^r + |\varepsilon|^{N+1}\,\widetilde{R}_N^{(2)}[B_2,\varepsilon], \tag{4.32}$$

where

$$\widetilde{R}_N^{(2)}[B_2,\varepsilon] = \widetilde{R}_N^{(1)}[B_2,\varepsilon]\Delta h'' + \frac{1}{|\varepsilon|^{N+1}} \sum_{r=N+1}^{2N} \sum_{i=1}^{r} \rho_i[B_2]\Delta u''_{r-i}\varepsilon^r. \tag{4.33}$$

Combining (4.4), (4.5), (4.7), (4.25), (4.29), (4.30) and (4.32), we then obtain

$$E_\varepsilon(x,t) = |\varepsilon|^{N+1}\left(\bar{R}_N^{(1)}[f,f_1,\varepsilon] + \widetilde{R}_N^{(2)}[B_1,\varepsilon] + \lambda\widetilde{R}_N^{(2)}[B_2,\varepsilon]\right). \tag{4.34}$$

By the functions $u_i \in W_1(M,T)$, $0 \le i \le N$, we obtain from (4.29), (4.31), (4.33) and (4.34) that

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le \bar{C}_*\,|\varepsilon|^{N+1}, \tag{4.35}$$

where $\bar{C}_*$ is a constant depending only on $N$, $T$, $f$, $f_1$, $B_1$, $B_2$, $u_r$, $0 \le r \le N$. This completes thee proof of lemma. □

Now, we consider the sequence of functions $\{v_m\}$ defined by

$$\begin{cases} v_0 \equiv 0, \\ v''_m - B_1[v_{m-1}+h]\Delta v_m - \lambda B_2[v_{m-1}+h]\Delta v''_m \\ = F_\varepsilon[v_{m-1}+h] - F_\varepsilon[h] + (B_1[v_{m-1}+h] - B_1[h])\,\Delta h \\ \quad +\lambda\,(B_2[v_{m-1}+h]-B_2[h])\,\Delta h'' + E_\varepsilon(x,t),\ 0<x<1,\ 0<t<T, \\ v_m(0,t) = v_m(1,t) = 0, \\ v_m(x,0) = v'_m(x,0) = 0,\ m \ge 1. \end{cases} \tag{4.36}$$

With $m = 1$, we have the problem

$$\begin{cases} v''_1 - B_1[h]\Delta v_1 - \lambda B_2[h]\Delta v''_1 = E_\varepsilon(x,t),\ 0<x<1,\ 0<t<T, \\ v_1(0,t) = v_1(1,t) = 0, \\ v_1(x,0) = v'_1(x,0) = 0. \end{cases} \tag{4.37}$$

By multiplying the two sides of (4.37) by $v'_1$, we verify without difficulty from (4.26) that

$$\begin{aligned}
&||v_1'(t)||^2 + \bar{B}_{1\varepsilon}(t) \left\|\nabla v_1(t)\right\|^2 + \lambda \bar{B}_{2\varepsilon}(t) \left\|\nabla v_1'(t)\right\|^2 \\
&= \int_0^t \left( \bar{B}_{1\varepsilon}'(s) \left\|\nabla v_1(s)\right\|^2 + \lambda \bar{B}_{2\varepsilon}'(s) \left\|\nabla v_1'(s)\right\|^2 \right) ds + 2\int_0^t \langle E_\varepsilon(s), v_1'(s)\rangle ds \\
&\leq T\bar{C}_*^2 \left|\varepsilon\right|^{2N+2} + \int_0^t \left\|v_1'(s)\right\|^2 ds \\
&\quad + \int_0^t \left( \left|\bar{B}_{1\varepsilon}'(s)\right| \left\|\nabla v_1(s)\right\|^2 + \lambda \left|\bar{B}_{2\varepsilon}'(s)\right| \left\|\nabla v_1'(s)\right\|^2 \right) ds, \qquad (4.38)
\end{aligned}$$

where $\bar{B}_{1\varepsilon}(t) = B_1[h](t)$, $\bar{B}_{2\varepsilon}(t) = B_2[h](t)$. By

$$\begin{aligned}
\bar{B}_{i\varepsilon}'(t) &= 2D_1 B_i[h] \langle \nabla h(t), \nabla h'(t)\rangle \\
&\quad + 2D_2 B_i[h] \langle \nabla h'(t), \nabla h''(t)\rangle, \ i = 1, 2, \qquad (4.39)
\end{aligned}$$

we have

$$\left|\bar{B}_{i\varepsilon}'(t)\right| \leq 4M_*^2 \tilde{K}_{M_*}(B_i) \equiv \zeta_i, \text{ for all } |\varepsilon| < 1, \ i = 1, 2, \qquad (4.40)$$

with $M_* = (N+1)M$. It follows from (4.38), (4.40) that

$$\begin{aligned}
&b_* \left( ||v_1'(t)||^2 + \left\|\nabla v_1(t)\right\|^2 + \left\|\nabla v_1'(t)\right\|^2 \right) \\
&\leq T\bar{C}_*^2 \left|\varepsilon\right|^{2N+2} + d_1 \int_0^t \left( \left\|v_1'(s)\right\|^2 + \left\|\nabla v_1(s)\right\|^2 + \left\|\nabla v_1'(s)\right\|^2 \right) ds, \qquad (4.41)
\end{aligned}$$

where $b_* = \min\{1, b_0, \lambda b_0\}$, $d_1 = \max\{1, \zeta_1, \lambda\zeta_2\}$. By Gronwall's lemma we obtain from (4.41) that

$$||v_1'(t)||^2 + \left\|\nabla v_1(t)\right\|^2 + \left\|\nabla v_1'(t)\right\|^2 \leq \frac{1}{b_*} T\bar{C}_*^2 \left|\varepsilon\right|^{2N+2} \exp\left(\frac{1}{b_*} d_1 T\right). \qquad (4.42)$$

Hence

$$||v_1||_{L^\infty(0,T;H_0^1)} + \left\|v_1'\right\|_{L^\infty(0,T;H_0^1)} \leq \frac{2}{\sqrt{b_*}} \sqrt{T}\bar{C}_* \left|\varepsilon\right|^{N+1} \exp\left(\frac{1}{2b_*} d_1 T\right). \qquad (4.43)$$

We shall prove that there exists a constant $C_T$, independent of $m$ and $\varepsilon$ such that

$$\left\|v_m'\right\|_{L^\infty(0,T;H_0^1)} + \left\|v_m\right\|_{L^\infty(0,T;H_0^1)} \leq C_T \left|\varepsilon\right|^{N+1} \qquad (4.44)$$

with $|\varepsilon| < 1$, for all $m$. By multiplying the two sides of (4.36) with $v_m'$ and integrating with respect to $t$, we obtain from (4.26) that

$$\begin{aligned}
&||v_m'(t)||^2 + \bar{B}_{1m\varepsilon}(t)\left\|\nabla v_m(t)\right\|^2 + \lambda\bar{B}_{2m\varepsilon}(t)\left\|\nabla v_m'(t)\right\|^2 \\
&\leq T\bar{C}_*^2\left|\varepsilon\right|^{2N+2} \\
&\quad + \int_0^t \left(\left\|v_m'(s)\right\|^2 + \left|\bar{B}_{1m\varepsilon}'(s)\right|\left\|\nabla v_1(s)\right\|^2 + \lambda\left|\bar{B}_{2m\varepsilon}'(s)\right|\left\|\nabla v_1'(s)\right\|^2\right) ds \\
&\quad + 2\int_0^t \langle F_\varepsilon[v_{m-1}+h] - F_\varepsilon[h], v_m'(s)\rangle ds \\
&\quad + 2\int_0^t \left(B_1[v_{m-1}+h] - B_1[h]\right)\langle \Delta h(s), v_m'(s)\rangle ds \\
&\quad + 2\lambda\int_0^t \left(B_2[v_{m-1}+h] - B_2[h]\right)\langle \Delta h''(s), v_m'(s)\rangle ds \\
&\equiv T\bar{C}_*^2\left|\varepsilon\right|^{2N+2} + \widehat{J}_1 + \widehat{J}_2 + \widehat{J}_3 + \widehat{J}_4 \qquad (4.45)
\end{aligned}$$

with $\bar{B}_{1m\varepsilon}(t) = B_1[v_{m-1}+h](t)$, $\bar{B}_{2m\varepsilon}(t) = B_2[v_{m-1}+h](t)$.

We now estimate the integrals on the right–hand side of (4.45) as follows.

*Estimation of* $\widehat{J}_1$. We have

$$\begin{aligned}
\bar{B}_{im\varepsilon}'(t) &= 2D_1B_i[v_{m-1}+h](t)\langle\nabla v_{m-1} + \nabla h, \nabla v_{m-1}' + \nabla h'\rangle \\
&\quad + 2D_2B_i[v_{m-1}+h](t)\langle\nabla v_{m-1}' + \nabla h', \nabla v_{m-1}'' + \nabla h''\rangle, \qquad (4.46)
\end{aligned}$$

hence

$$\left|\bar{B}_{im\varepsilon}'(t)\right| \leq 4\bar{M}_*^2\tilde{K}_{\bar{M}_*}(B_i) \equiv \bar{\zeta}_i, \text{ for all } |\varepsilon| < 1,\ i = 1, 2, \qquad (4.47)$$

with $\bar{M}_* = (N+2)M$. It follows from (4.47), that

$$\begin{aligned}
\widehat{J}_1 &= \int_0^t \left(\left\|v_m'(s)\right\|^2 + \left|\bar{B}_{1m\varepsilon}'(s)\right|\left\|\nabla v_1(s)\right\|^2 + \lambda\left|\bar{B}_{2m\varepsilon}'(s)\right|\left\|\nabla v_1'(s)\right\|^2\right) ds \\
&\leq \bar{d}_1\int_0^t \left(\left\|v_m'(s)\right\|^2 + \left\|\nabla v_m(s)\right\|^2 + \left\|\nabla v_m'(s)\right\|^2\right) ds, \qquad (4.48)
\end{aligned}$$

where $\bar{d}_1 = \max\{1, \bar{\zeta}_1, \lambda\bar{\zeta}_2\}$.

*Estimation of* $\widehat{J}_2$. Note that

$$\left\|f[v_{m-1}+h] - f[h]\right\| \leq 2K_{\bar{M}_*}(f)\left\|v_{m-1}\right\|_{W_1(T)},$$

and

$$\left\|f_1[v_{m-1}+h] - f_1[h]\right\| \leq 2K_{\bar{M}_*}(f_1)\left\|v_{m-1}\right\|_{W_1(T)},$$

hence, we have

$$\left\|F_\varepsilon[v_{m-1}+h] - F_\varepsilon[h]\right\| \leq \bar{d}_2\left\|v_{m-1}\right\|_{W_1(T)}, \qquad (4.49)$$

where $\bar{d}_2 = 2\left(K_{\bar{M}_*}(f) + K_{\bar{M}_*}(f_1)\right)$. Therefore, we deduce from (4.49) that

$$\begin{aligned}\widehat{J}_2 &= 2\int_0^t \|F_\varepsilon[v_{m-1}+h] - F_\varepsilon[h]\| \left\|v_m'(s)\right\| ds \\ &\leq T\bar{d}_2^2 \|v_{m-1}\|_{W_1(T)}^2 + \int_0^t \left\|v_m'(s)\right\|^2 ds. \end{aligned} \tag{4.50}$$

*Estimation of* $\widehat{J}_3$. From the inequalities

$$\begin{cases} |B_i[v_{m-1}+h] - B_i[h]| \leq 4\bar{M}_*\tilde{K}_{\bar{M}_*}(B_i) \|v_{m-1}\|_{W_1(T)}, \ i = 1, 2, \\ \|\Delta h(s)\| \leq \sum_{r=0}^{N} \|\Delta u_r\| \, |\varepsilon|^r \leq (N+1)M = M_*, \end{cases} \tag{4.51}$$

it follows that

$$\begin{aligned}\widehat{J}_3 &= 2\int_0^t \left(B_1[v_{m-1}+h] - B_1[h]\right) \langle \Delta h(s), v_m'(s)\rangle ds \\ &\leq 2\int_0^t |B_1[v_{m-1}+h] - B_1[h]| \, \|\Delta h(s)\| \left\|v_m'(s)\right\| ds \\ &\leq T\bar{d}_3^2 \|v_{m-1}\|_{W_1(T)}^2 + \int_0^t \left\|v_m'(s)\right\|^2 ds, \end{aligned} \tag{4.52}$$

in which $\bar{d}_3 = 4M_*\bar{M}_*\tilde{K}_{\bar{M}_*}(B_1)$.

*Estimation of* $\widehat{J}_4$. From the inequalities $(4.51)_1$ and

$$\left\|\Delta h''(s)\right\| \leq \sum_{r=0}^{N} \left\|\Delta u_r''(s)\right\| |\varepsilon|^r \leq \sum_{r=0}^{N} \left\|\Delta u_r''\right\|_{L^\infty(0,T;L^2)} = \tilde{M}_*, \tag{4.53}$$

it follows that

$$\begin{aligned}\widehat{J}_4 &= 2\lambda\int_0^t \left(B_2[v_{m-1}+h] - B_2[h]\right) \langle \Delta h''(s), v_m'(s)\rangle ds \\ &\leq 2\lambda\int_0^t |B_2[v_{m-1}+h] - B_2[h]| \left\|\Delta h''(s)\right\| \left\|v_m'(s)\right\| ds \\ &\leq T\bar{d}_4^2 \|v_{m-1}\|_{W_1(T)}^2 + \int_0^t \left\|v_m'(s)\right\|^2 ds, \end{aligned} \tag{4.54}$$

in which $\bar{d}_4 = 4\lambda\tilde{M}_*\bar{M}_*\tilde{K}_{\bar{M}_*}(B_2)$. Combining (4.45), (4.48), (4.50), (4.52) and (4.54), we then obtain

$$
\begin{aligned}
& b_* \left( ||v_m'(t)||^2 + \|\nabla v_m(t)\|^2 + \left\|\nabla v_m'(t)\right\|^2 \right) \\
& \leq T \left( \bar{d}_2^2 + \bar{d}_3^2 + \bar{d}_4^2 \right) \|v_{m-1}\|_{W_1(T)}^2 + T \bar{C}_*^2 \left|\varepsilon\right|^{2N+2} \\
& \quad + (3 + \bar{d}_1) \int_0^t \left( \left\|v_m'(s)\right\|^2 + \|\nabla v_m(s)\|^2 + \left\|\nabla v_m'(s)\right\|^2 \right) ds. \qquad (4.55)
\end{aligned}
$$

By using Gronwall's lemma we deduce from (4.55) that

$$
\|v_m\|_{W_1(T)} \leq \sigma_T \|v_{m-1}\|_{W_1(T)} + \delta, \quad \text{for all} \quad m \geq 1, \qquad (4.56)
$$

with

$$
\sigma_T = \eta_T \sqrt{\bar{d}_2^2 + \bar{d}_3^2 + \bar{d}_4^2}, \ \delta = \eta_T \bar{C}_* \left|\varepsilon\right|^{N+1}, \ \eta_T = 2\sqrt{\frac{T}{b_*}} \exp\left( \frac{1}{2b_*} T \left(3 + \bar{d}_1\right) \right).
$$

Suppose that

$$
\sigma_T < 1 \quad \text{with the suitable constant} \quad T > 0. \qquad (4.57)
$$

The lemma 4.5 is easily to be proved.

**Lemma 4.5.** *Let the sequence* $\{z_m\}$ *satisfy*

$$
z_m \leq \sigma z_{m-1} + \delta \quad \text{for all} \quad m \geq 1, \ z_0 = 0, \qquad (4.58)
$$

*where* $0 \leq \sigma < 1$, $\delta \geq 0$ *are the given constants. Then*

$$
z_m \leq \delta/(1 - \sigma) \quad \text{for all} \quad m \geq 1. \qquad (4.59)
$$

Applying Lemma 4.5 with

$$
z_m = \|v_m\|_{W_1(T)}, \quad \sigma = \sigma_T = \eta_T \sqrt{\bar{d}_2^2 + \bar{d}_3^2 + \bar{d}_4^2} < 1, \quad \delta = \eta_T \bar{C}_* \left|\varepsilon\right|^{N+1},
$$

it follows from (4.59), that

$$
\begin{aligned}
\left\|v_m'\right\|_{L^\infty(0,T;H_0^1)} + \|v_m\|_{L^\infty(0,T;H_0^1)} &= \|v_m\|_{W_1(T)} \\
&\leq \delta/(1 - \sigma_T) = C_T \left|\varepsilon\right|^{N+1}, \qquad (4.60)
\end{aligned}
$$

where $C_T = \dfrac{\eta_T \bar{C}_*}{1 - \eta_T \sqrt{\bar{d}_2^2 + \bar{d}_3^2 + \bar{d}_4^2}}$.

On the other hand, the linear recurrent sequence $\{v_m\}$ defined by (4.36) converges strongly in the space $W_1(T)$ to the solution $v$ of Prob. (4.24). Hence, as $m \to +\infty$ in (4.60), it gives

$$
\left\|v'\right\|_{L^\infty(0,T;H_0^1)} + \|v\|_{L^\infty(0,T;H_0^1)} \leq C_T \left|\varepsilon\right|^{N+1}
$$

or

$$\left\| u_\varepsilon - \sum_{r=0}^{N} u_r \varepsilon^r \right\|_{W_1(T)} \leq C_T \left|\varepsilon\right|^{N+1}. \tag{4.61}$$

Thus, we have the following theorem 4.6.

**Theorem 4.6.** *Let* $(H_1)$, $(H_5)$ *and* $(H_6)$ *hold. Then there exist constants* $M > 0$ *and* $T > 0$ *such that for every* $\varepsilon$ *with* $|\varepsilon| < 1$, *Prob.* $(P_\varepsilon)$ *has a unique weak solution* $u_\varepsilon \in W_1(M,T)$ *satisfying an asymptotic estimation up to order* $N+1$ *as in* (4.61), *where the functions* $u_r$, $r = 0, 1, \cdots, N$ *are weak solutions of Prob.* $(\tilde{P}_r)$, $r = 0, 1, \cdots, N$, *respectively.*

**Remark 4.7.** Typical examples about asymptotic expansion of solutions in a small parameter can be found in many papers, such as [13]–[15]. In the case of many small parameters, there is only partial results, for example, we refer to [16], [17], [33], [34] for the asymptotic expansion of solutions in two or three small parameters.

**Acknowledgments:** The authors wish to express their sincere thanks to the referees for the suggestions and valuable comments.

## References


[1] J. Albert, *On the decay of solutions of the generalized Benjamin-Bona-Mahony equations*, J. Math. Anal. Appl., **141**(2) (1989), 527–537.

[2] C.J. Amick, J.L. Bona and M.E. Schonbek, *Decay of solutions of some nonlinear wave equations*, J. Differential Equations, **81**(1) (1989), 1–49.

[3] A. Chattopadhyay, S. Gupta, A.K. Singh and S.A. Sahu, *Propagation of shear waves in an irregular magnetoelastic monoclinic layer sandwiched between two isotropic half-spaces*, International Journal of Engineering, Science and Technology, **1**(1) (2009), 228–244.

[4] P.A. Clarkson, *New similarity reductions and Painlevé analysis for the symmetric regularised long wave and modified Benjamin-Bona-Mahoney equations*, Journal of Physics A., **22**(18) (1989), 3821–3848.

[5] M.M. Cavalcanti, V.N. Domingos Cavalcanti and J.A. Soriano, *Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation*, Adv. Differential Equations, **6**(6) (2001), 701–730.

[6] M.M. Cavalcanti, V.N. Domingos Cavalcanti and J.A. Soriano, *Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping*, Electron. J. Diff. Equ., **2002**(44) (2002), 1–14.

[7] M.M. Cavalcanti, V.N. Domingos Cavalcanti and J.A. Soriano, *Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation*, Commun. Contemp. Math., **6**(5) (2004), 705–731.

[8] S. Dutta, *On the propagation of Love type waves in an infinite cylinder with rigidity and density varying linearly with the radial distance*, Pure and Applied Geophysics, **98**(1) (1972), 35–39.

[9] Y. Ebihara, L.A. Medeiros and M.M. Miranda, *Local solutions for a nonlinear degenerate hyperbolic equation*, Nonlinear Anal. TMA., **10** (1986), 27–40.

[10] G.R. Kirchhoff, *Vorlesungen über Mathematische Physik*: *Mechanik*, Teuber, Leipzig, 1876.

[11] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites nonlinéaires*, Dunod; Gauthier-Villars, Paris, 1969.

[12] J.L. Lions, *On some questions in boundary value problems of mathematical physics*, in: G. de la Penha, L.A. Medeiros (Eds.), International Symposium on Continuum, Mechanics and Partial Differential Equations, Rio de Janeiro 1977, Mathematics Studies, vol. 30, North-Holland, Amsterdam, 1978, pp. 284-346.

[13] N.T. Long, *Asymptotic expansion of the solution for nonlinear wave equation with the mixed homogeneous conditions*, Nonlinear Anal. TMA., **45** (2001), 261–272.

[14] N.T. Long, *On the nonlinear wave equation* $u_{tt} - B(t, \|u\|^2, \|u_x\|^2)u_{xx} = f(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2)$ *associated with the mixed homogeneous conditions*, J. Math. Anal. Appl., **306**(1) (2005), 243–268.

[15] N.T. Long, A.P.N. Dinh and T.N. Diem, *Linear recursive schemes and asymptotic expansion associated with the Kirchhoff-Carrier operator*, J. Math. Anal. Appl., **267**(1) (2002), 116–134.

[16] N.T. Long, A.P.N. Dinh and T.N. Diem, *On a shock problem involving a nonlinear viscoelastic bar*, Bound. Value Probl., **2005**(3) (2005), 337–358.

[17] N.T. Long and L.X. Truong, *Existence and asymptotic expansion for a viscoelastic problem with a mixed homogeneous condition*, Nonlinear Anal. TMA., **67**(3) (2007), 842–864.

[18] V.G. Makhankov, *Dynamics of classical solitons in nonintegrable systems*, Physics Reports C., **35**(1) (1978), 1–128.

[19] M.K. Paul, *On propagation of love-type waves on a spherical model with rigidity and density both varying exponentially with the radial distance*, Pure and Applied Geophysics, **59**(1) (1964), 33–37.

[20] L.A. Medeiros, *On some nonlinear perturbation of Kirchhoff-Carrier operator*, Comp. Appl. Math., **13** (1994), 225–233.

[21] L.A. Medeiros, J. Limaco and S.B. Menezes, *Vibrations of elastic strings: Mathematical aspects, Part one*, J. Comput. Anal. Appl., **4**(2) (2002), 91–127.

[22] L.A. Medeiros, J. Limaco and S.B. Menezes, *Vibrations of elastic strings: Mathematical aspects, Part two*, J. Comput. Anal. Appl., **4**(3) (2002), 211–263.

[23] Miranda, M Milla; Jutuca, L P San Gil, *Existence and boundary stabilization of solutions for the Kirchhoff equation*, Comm. Partial Differential Equations, **24**(9-10) (1999), 1759–1800.

[24] G.P. Menzala, *On global classical solutions of a nonlinear wave equation*, Appl. Anal., **10** (1980), 179–195.

[25] T. Ogino and S. Takeda, *Computer simulation and analysis for the spherical and cylindrical ion-acoustic solitons*, Journal of the Physical Society of Japan, **41**(1) (1976), 257–264.

[26] S.I. Pohozaev, *On a class of quasilinear hyperbolic equation*, Math. USSR. Sb., **25** (1975), 145–158.

[27] J.Y. Park, J.J. Bae and I.H. Jung, *Uniform decay of solution for wave equation of Kirchhoff type with nonlinear boundary damping and memory term*, Nonlinear Anal. TMA., **50** (2002), 871–884.

[28] J.Y. Park and J.J. Bae, *On coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory term*, Applied Math. Comput., **129** (2002), 87–105.

[29] V.Radochová, *Remark to the comparison of solution properties of Love's equation with those of wave equation*, Applications of Mathematics, **23**(3) (1978), 199–207.

[30] T.N. Rabello, M.C.C. Vieira, C.L. Frota and L.A. Medeiros, *Small vertical vibrations of strings with moving ends*, Rev. Mat. Complutent, **16** (2003), 179–206.

[31] M.L. Santos, J. Ferreira, D.C. Pereira and C.A. Raposo, *Global existence and stability for wave equation of Kirchhoff type with memory condition at the boundary*, Nonlinear Anal. TMA., **54** (2003), 959–976.

[32] C.E. Seyler and D.L. Fenstermacher, *A symmetric regularized-long-wave equation*, Physics of Fluids, **27**(1) (1984), 4–7.

[33] N.A. Triet, L.T.P. Ngoc and N.T. Long, *On a nonlinear Kirchhoff-Carrier wave equation associated with Robin conditions*, Nonlinear Anal. RWA., **11**(5) (2010), 3363–3388.

[34] N.A. Triet, L.T.P. Ngoc and N.T. Long, *A mixed Dirichlet-Robin problem for a nonlinear Kirchhoff-Carrier wave equation*, Nonlinear Anal. RWA., **13**(2) (2012), 817–839.