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OSTROWSKI TYPE k-FRACTIONAL INTEGRAL INEQUALITIES FOR MT-CONVEX AND h-CONVEX FUNCTIONS

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Abstract. In this paper, we give some inequalities of Ostrowski type for MT-convex and h-convex functions via k-fractional integrals. These results not only generalize those of [7, 8, 19] and but also give estimates of these type of fractional inequalities for k-fractional integrals.

1. Introduction and preliminaries

In 1938, Ostrowski [17] proved an inequality stated in the following result (see also [15, p.468]).

Theorem 1.1. *Let $f : I \to \mathbb{R}$, where I is interval in $\mathbb{R}$, be a mapping differentiable in I° the interior of I and $a, b \in I^\circ$, $a < b$. If $\left|f'(t)\right| \leq M$, for all $t \in [a, b]$, then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M, \quad x \in [a, b].$$

It is well known in literature as Ostrowski inequality. Since it gives bounds of integral average of a function f over an interval $[a, b]$ to its value $f(x)$ at

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point $x \in [a, b]$. In recent years, so many such type of inequalities have been obtained and generalized (see [1, 14]). In [18] MT-convex functions have been defined by Tunc as follows (see also [19, 20]).

Definition 1.2. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be MT-convex function, if it is non-negative and $\forall\, x, y \in I$ and $t \in (0, 1)$ satisfies the following inequality

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y).$$

Definition 1.3. ([21]) Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. We say $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex function, if f is non-negative and for $x, y \in I$ and $t \in [0, 1]$, one has

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

If above inequality is reversed then f is called h-concave.

Definition 1.4. ([21]) We say that $h : J \to \mathbb{R}$ is a super-multiplicative function, if for $x, y \in J$ one has

$$h(xy) \geq h(x)h(y).$$

Definition 1.5. ([2]) We say that $h : J \to \mathbb{R}$ is a super-additive function, if $\forall x, y \in J$ one has

$$h(x + y) \geq h(x) + h(y).$$

In [5], there are given definitions of k-gamma and k-beta functions as follows.

Definition 1.6. For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x) = \lim_{n \to \infty} \frac{n! k^n n k^{\frac{x}{k}-1}}{(x)_{n,k}}.$$

Its integral representation is given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{\frac{-t^k}{k}} dt.$$

For $k = 1$, it becomes integral representation of gamma function.

Definition 1.7. For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-beta function with two parameters x and y is defined as

$$\beta_k(x, y) = \frac{1}{k} \int_0^\infty t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt.$$

For $k = 1$, it becomes integral representation of beta function.

In [5], there is also given relation between k-gamma and k-beta functions.

Theorem 1.8. *When $x, y > 0$, for k-gamma and k-beta function following equality holds*

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{1.1}$$

The fractional calculus is a generalization of classical calculus concerned with operations of integration and differentiation of fractional order came into existence in the correspondence of G. W. Leibniz and Marquis de l'Hospital in 1695 where the question of meaning of the semi-derivative has been raised. This question consequently attracted the interest of many well known mathematicians, including Euler, Liouville, Laplace, Riemann, Grunwald, Letnikov and many others. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. Since the 19th century, the theory of fractional calculus developed rapidly. Fractional calculus has many useful applications in the fields of science such as electromagnetic waves, diffusion waves and mathematical biology.
Laurent in [6] gave today's definition of the Riemann-Liouville fractional integral.

Definition 1.9. ([6]) Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$. Here $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$. In case of $\alpha = 1$, the Riemann-Liouville fractional integrals reduces to the classical integral.

In [16] S. Mubeen and G. M. Habibullah defined Riemann-Liouville k-fractional integrals.

Definition 1.10. ([16]) Let $f \in L_1[a,b]$. Then Riemann-Liouville k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I^{\alpha,k}_{a+}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)\,dt, \quad x \geq a$$

and

$$I^{\alpha,k}_{b-}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)\,dt, \quad x \leq b.$$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined as

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt.$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha)$$

and

$$I_{a+}^{0,1} f(x) = I_{b-}^{0,1} f(x) = f(x).$$

For $k = 1$, Riemann-Liouville k-fractional integrals give Riemann-Liouville fractional integrals.

We organize the paper in such a way that in Section 2 we give a version of Ostrowski inequality for MT-convex functions via k-fractional integrals. Also we prove some k-fractional Ostrowski type inequalities for MT-convex functions. In Section 3 we give a version of Ostrowski inequality for h-convex functions via k-fractional integrals. Also we prove some k-fractional Ostrowski type inequalities for h-convex functions.

2. Ostrowski type k-fractional integral inequalities for MT-convex functions

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L_1[a, b]$, then for all $x \in [a, b]$ and $\alpha, k > 0$, one has*

$$\begin{aligned} &\frac{(x-a)^{\frac{\alpha}{k}} + (b-x)^{\frac{\alpha}{k}}}{b-a} f(x) - \frac{\Gamma_k(\alpha + k)}{b-a} \left[J_{x^-}^{\alpha,k} f(a) + J_{x^+}^{\alpha,k} f(b) \right] \\ &= \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a} \int_0^1 t^{\frac{\alpha}{k}} f'(tx + (1-t)a) dt \\ &\quad - \frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a} \int_0^1 t^{\frac{\alpha}{k}} f'(tx + (1-t)b) dt. \end{aligned} \tag{2.1}$$

Proof. It is easy to see that

$$\begin{aligned} &\int_0^1 t^{\frac{\alpha}{k}} f'(tx + (1-t)a) dt \\ &= \left. \frac{t^{\frac{\alpha}{k}} f(tx + (1-t)a)}{x-a} \right|_0^1 - \frac{\alpha}{k(x-a)} \int_0^1 t^{\frac{\alpha}{k}-1} f(tx + (1-t)a) dt \\ &= \frac{f(x)}{x-a} - \frac{\alpha}{k(x-a)} \int_a^x \left(\frac{y-a}{x-a} \right)^{\frac{\alpha}{k}-1} \frac{f(y)}{x-a} dy \\ &= \frac{f(x)}{x-a} - \frac{\Gamma_k(\alpha + k)}{(x-a)^{\frac{\alpha}{k}+1}} I_{x^-}^{\alpha,k} f(a) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}} f^{'}(tx+(1-t)b)dt \\
&= \frac{t^{\frac{\alpha}{k}} f(tx+(1-t)b)}{x-b}\Bigg|_0^1 - \frac{\alpha}{k(x-b)}\int_0^1 t^{\frac{\alpha}{k}-1} f(tx+(1-t)b)dt \\
&= \frac{f(x)}{x-b} - \frac{\alpha}{k(x-b)}\int_b^x \left(\frac{y-b}{x-b}\right)^{\frac{\alpha}{k}-1}\frac{f(y)}{x-b}dy \\
&= \frac{-f(x)}{b-x} + \frac{\Gamma_k(\alpha+k)}{(b-x)^{\frac{\alpha}{k}+1}} I_{x^+}^{\alpha,k} f(b).
\end{aligned} \tag{2.3}$$

Multiplying (2.2) by $\frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}$ and (2.3) by $-\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}$, then adding resulting equations we get (2.1). □

Remark 2.2. If we put $k=1$ in (2.1), then we get [7, Lemma 1].

First we give an Ostrowski type inequality for MT-convex functions via k-fractional integrals.

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$, $a,b\geq 0$, $a<b$ be a differentiable function on (a,b) and $f^{'}\in L_1[a,b]$. If $\left|f^{'}\right|$ is MT-convex function on $[a,b]$ and $\left|f^{'}(x)\right|\leq M$, $x\in[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right| \\
&\leq M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right]\left[\frac{\Gamma_k(\alpha+\frac{k}{2})\Gamma_k(\frac{k}{2})}{2\Gamma_k(\alpha+k)}\right], \quad x\in[a,b].
\end{aligned} \tag{2.4}$$

Proof. Using Lemma 2.1 and convexity of $\left|f^{'}\right|$ we proceed as follows

$$\begin{aligned}
&\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right| \\
&\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f^{'}(tx+(1-t)a)\right|dt \\
&\quad + \frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f^{'}(tx+(1-t)b)\right|dt \\
&\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[t^{\frac{\alpha}{k}}\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f^{'}(x)\right| + t^{\frac{\alpha}{k}}\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f^{'}(a)\right|\right]dt \\
&\quad + \frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[t^{\frac{\alpha}{k}}\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f^{'}(x)\right| + t^{\frac{\alpha}{k}}\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f^{'}(b)\right|\right]dt
\end{aligned}$$

$$\leq M\frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[t^{\frac{\alpha}{k}}\frac{\sqrt{t}}{2\sqrt{1-t}}+t^{\frac{\alpha}{k}}\frac{\sqrt{1-t}}{2\sqrt{t}}\right]dt$$
$$+M\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[t^{\frac{\alpha}{k}}\frac{\sqrt{t}}{2\sqrt{1-t}}+t^{\frac{\alpha}{k}}\frac{\sqrt{1-t}}{2\sqrt{t}}\right]dt$$
$$=M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{2(b-a)}\right]\int_0^1\left[t^{\frac{\alpha}{k}+\frac{1}{2}}(1-t)^{\frac{-1}{2}}+t^{\frac{\alpha}{k}-\frac{1}{2}}(1-t)^{\frac{1}{2}}\right]dt.$$
$$=M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{2(b-a)}\right]\left[\beta_k(\alpha+\frac{1}{2},\frac{1}{2})+\beta_k\big(\alpha+\frac{3}{2},\frac{3}{2})\right]$$
$$=M\Big[\frac{(x-a)^{\alpha+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\Big]\left[\frac{\Gamma_k(\alpha+\frac{k}{2})\Gamma_k(\frac{k}{2})}{2\Gamma_k(\alpha+k)}\right].$$

Here we use (1.1). The proof is completed. □

Remark 2.4. (i) If we put $k=1$ in (2.4), then we get [7, Theorem 5].
(ii) If we put $k=1$ and $\alpha=1$ in (2.4), then we get [7, Theorem 2].

In the following we give an Ostrowski type inequality for MT-convex functions via k-fractional integrals.

Theorem 2.5. *Let $f:[a,b]\to\mathbb{R}$, $a,b\geq 0$, $a<b$ be a differentiable function on (a,b) and $f^{'}\in L_1[a,b]$. If $\left|f^{'}\right|^q$, when $q>1, \frac{1}{p}+\frac{1}{q}=1$ is MT-convex function on $[a,b]$ and $\left|f^{'}(x)\right|\leq M$, $x\in[a,b]$ then the following inequality for k-fractional fractional integrals holds*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J^{\alpha,k}_{x^-}f(a)+J^{\alpha,k}_{x^+}f(b)\right]\right|$$
$$\leq\frac{M}{(1+p\dfrac{\alpha}{k})^{\frac{1}{p}}}(\frac{\pi}{2})^{\frac{1}{q}}\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right],\quad x\in[a,b]. \tag{2.5}$$

Proof. Using lemma 2.1 and then Holder's inequality, we have

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J^{\alpha,k}_{x^-}f(a)+J^{\alpha,k}_{x^+}f(b)\right]\right|$$
$$\leq\frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f^{'}(tx+(1-t)a)\right|dt$$
$$+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f^{'}(tx+(1-t)b)\right|dt$$

$$\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{p\alpha}{k}}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{'}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{p\alpha}{k}}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{'}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \tag{2.6}$$

Since $\left|f^{'}\right|^q$ is MT-convex and $\left|f^{'}(x)\right| \leq M$, we get

$$\int_0^1 \left|f^{'}(tx+(1-t)a)\right|^q dt \leq \int_0^1 \left[\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f^{'}(x)\right|^q + \frac{\sqrt{1-t}}{2\sqrt{t}}\left|f^{'}(a)\right|^q\right] dt$$
$$\leq M^q \int_0^1 \left[\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{1-t}}{2\sqrt{t}}\right] dt = \frac{\pi}{2}M^q \tag{2.7}$$

and similarly

$$\int_0^1 \left|f^{'}(tx+(1-t)b)\right|^q dt \leq \frac{\pi}{2}M^q. \tag{2.8}$$

We also have

$$\int_0^1 t^{\frac{p\alpha}{k}}dt = \frac{1}{\frac{p\alpha}{k}+1}. \tag{2.9}$$

Using (2.7), (2.8) and (2.9) in (2.6) we can get (2.5). □

Remark 2.6. (i) If we put $k=1$ in (2.5), then we get [7, Theorem 6].
(ii) If we put $k=1$ and $\alpha=1$ in (2.5), then we get [7, Theorem 3].

Theorem 2.7. *Let $f:[a,b]\to\mathbb{R}$, $a,b\geq 0$, $a<b$ be a differentiable function on (a,b) and $f^{'}\in L_1[a,b]$. If $\left|f^{'}\right|^q$ is MT-convex function on $[a,b]$ and $\left|f^{'}(x)\right|\leq M$, $x\in[a,b], q\geq 1$, then the following inequality for k-fractional integrals holds*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[J^{\alpha,k}_{x^-}f(a)+J^{\alpha,k}_{x^+}f(b)\right]\right|$$
$$\leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\left[\frac{\Gamma_k(\alpha+\frac{k}{2})\Gamma_k(\frac{k}{2})}{2\Gamma_k(\alpha+k)}\right]^{\frac{1}{q}}\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right], x\in[a,b]. \tag{2.10}$$

Proof. Using Lemma 2.1 and power mean inequality, we have

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[J^{\alpha,k}_{x^-}f(a)+J^{\alpha,k}_{x^+}f(b)\right]\right|$$
$$\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f^{'}(tx+(1-t)a)\right|dt$$

$$
\begin{aligned}
&+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f^{'}(tx+(1-t)b)\right|dt\\
&\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\frac{\alpha}{k}}\left|f^{'}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\frac{\alpha}{k}}\left|f^{'}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \qquad (2.11)
\end{aligned}
$$

Since $\left|f^{'}\right|^q$ is MT-convex on $[a,b]$ and $\left|f^{'}\right| \leq M$, we get

$$
\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}}\left|f^{'}(tx+(1-t)a)\right|^q dt\\
&\leq \int_0^1\left[t^{\frac{\alpha}{k}}\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f^{'}(x)\right|^q + t^{\frac{\alpha}{k}}\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f^{'}(a)\right|^q\right]dt\\
&\leq M^q\int_0^1\left[t^{\frac{\alpha}{k}}\frac{\sqrt{t}}{2\sqrt{1-t}} + t^{\frac{\alpha}{k}}\frac{\sqrt{1-t}}{2\sqrt{t}}\right]dt = M^q\left[\frac{\Gamma_k(\alpha+\frac{k}{2})\Gamma_k(\frac{k}{2})}{2\Gamma_k(\alpha+k)}\right] \qquad (2.12)
\end{aligned}
$$

and similarly

$$
\int_0^1 t^{\frac{\alpha}{k}}\left|f^{'}(tx+(1-t)b)\right|^q dt \leq M^q\left[\frac{\Gamma_k(\alpha+\frac{k}{2})\Gamma_k(\frac{k}{2})}{2\Gamma_k(\alpha+k)}\right]. \qquad (2.13)
$$

Using (2.12) and (2.13) in (2.11) we can attain (2.10). □

Remark 2.8. (i) If we put $k=1$ in (2.10), then we get [7, Theorem 7].
(ii) If we put $k=1$ and $\alpha=1$ in (2.10), then we get [7, Theorem 4].

3. Ostrowski type k-fractional integral inequalities for h-convex functions

First we give an Ostrowski type inequality for h-convex functions via k-fractional integrals.

Theorem 3.1. *Let $h: J \subseteq \mathbb{R} \to \mathbb{R}([0,1] \subseteq J)$ be a non-negative and super-multiplicative function, $h(t) \geq t$ for $0 \leq t \leq 1$, $f:[a,b] \to \mathbb{R}$, $a,b \geq 0$ be a differentiable mapping on (a,b) with $a<b$ such that $f^{'} \in L_1[a,b]$. If $\left|f^{'}\right|$ is h-convex on $[a,b]$ and $\left|f^{'}(x)\right| \leq M$, $x \in [a,b]$, then the following inequalities*

for k-fractional integrals hold

$$\begin{aligned}&\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right|\\&\leq M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right]\int_0^1\left[t^{\frac{\alpha}{k}}h(t)+t^{\frac{\alpha}{k}}h(1-t)\right]dt \qquad (3.1)\\&\leq M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right]\int_0^1\left[h(t^{\frac{\alpha}{k}+1})+h(t^{\frac{\alpha}{k}}(1-t))\right]dt, \qquad (3.2)\end{aligned}$$

for $x\in[a,b]$.

Proof. Using Lemma 2.1 and that $\left|f'\right|$ is h-convex we have

$$\begin{aligned}&\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right|\\&\leq\frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|dt\\&\quad+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|dt\\&\leq\frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[t^{\frac{\alpha}{k}}h(t)\left|f'(x)\right|+t^{\frac{\alpha}{k}}h(1-t)\left|f'(a)\right|\right]dt\\&\quad+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[t^{\frac{\alpha}{k}}h(t)\left|f'(x)\right|+t^{\frac{\alpha}{k}}h(1-t)\left|f'(b)\right|\right]dt\\&\leq M\frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[t^{\frac{\alpha}{k}}h(t)+t^{\frac{\alpha}{k}}h(1-t)\right]dt\\&\quad+M\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[t^{\frac{\alpha}{k}}h(t)+t^{\frac{\alpha}{k}}h(1-t)\right]dt,\end{aligned}$$

which completes the proof of (3.1). By using the addition properties of h in assumption, we further have

$$\begin{aligned}\int_0^1\left[t^{\frac{\alpha}{k}}h(t)+t^{\frac{\alpha}{k}}h(1-t)\right]dt&\leq\int_0^1\left[h(t^{\frac{\alpha}{k}})h(t)+h(t^{\frac{\alpha}{k}})h(1-t)\right]dt\\&\leq\int_0^1\left[h(t^{\frac{\alpha}{k}+1})+h(t^{\frac{\alpha}{k}}(1-t))\right]dt.\end{aligned}$$

Hence the proof of (3.2) is completed. □

Remark 3.2. If we put $k=1$ in (3.1), then we get [8, Theorem 1].

Theorem 3.3. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}([0,1] \subseteq J)$ be a non-negative and super-multiplicative function, $h(t) \geq t$ for $0 \leq t \leq 1$, $f : [a,b] \to \mathbb{R}$, $a, b \geq 0$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L_1[a,b]$. If $|f'|^q$ is h-convex on $[a,b]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequalities for k-fractional integrals hold*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}} + (b-x)^{\frac{\alpha}{k}}}{b-a} f(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k} f(a) + J_{x^+}^{\alpha,k} f(b)\right]\right|$$

$$\leq M\left[\frac{(x-a)^{\frac{\alpha}{k}+1} + (b-x)^{\frac{\alpha}{k}+1}}{(1+\frac{p\alpha}{k})^{\frac{1}{p}}(b-a)}\right]\left(\int_0^1 [h(t) + h(1-t)]\, dt\right)^{\frac{1}{q}} \tag{3.3}$$

$$\leq M\left[\frac{(x-a)^{\frac{\alpha}{k}+1} + (b-x)^{\frac{\alpha}{k}+1}}{(1+\frac{p\alpha}{k})^{\frac{1}{p}}(b-a)}\right] h^{\frac{1}{q}}(1), \quad x \in [a,b]. \tag{3.4}$$

Proof. Using Lemma 2.1 and Holder's inequality, we have

$$\left|\frac{(x-a)^{\frac{\alpha}{k}} + (b-x)^{\frac{\alpha}{k}}}{b-a} f(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k} f(a) + J_{x^+}^{\alpha,k} f(b)\right]\right|$$

$$\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx + (1-t)a)\right| dt$$

$$+ \frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx + (1-t)b)\right| dt$$

$$\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx + (1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+ \frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx + (1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \tag{3.5}$$

Since $|f'|^q$ is h-convex and $|f'(x)| \leq M$, for all $x \in [a,b]$ we get

$$\int_0^1 \left|f'(tx + (1-t)a)\right|^q dt \leq \int_0^1 [h(t)\left|f'(x)\right|^q + h(1-t)\left|f'(a)\right|^q] dt$$

$$\leq M^q \int_0^1 [h(t) + h(1-t)] dt \tag{3.6}$$

and similarly

$$\int_0^1 \left|f'(tx + (1-t)b)\right|^q dt \leq M^q \int_0^1 [h(t) + h(1-t)] dt. \tag{3.7}$$

Also we have

$$\int_0^1 t^{\frac{p\alpha}{k}} dt = \frac{1}{\frac{p\alpha}{k} + 1}. \tag{3.8}$$

Using (3.6), (3.7) and (3.8) in (3.5) we can complete the proof of (3.3). By using the super-addition property of h in the assumption, we further have

$$\int_0^1 [h(t)+h(1-t)]dt \le \int_0^1 h(1)dt = h(1).$$

Hence the proof of (3.4) is completed. □

Remark 3.4. If we put $k=1$ in (3.3), then we get [8, Theorem 2].

Theorem 3.5. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}([0,1] \subseteq J)$ be a non-negative and supermultiplicative function, $h(t) \ge t$ for $0 \le t \le 1$, $f : [a,b] \to \mathbb{R}$, $a,b \ge 0$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L_1[a,b]$. If $\left|f'\right|^q$ with $p,q > 1$, $\frac{1}{p}+\frac{1}{q}=1$ is h-convex on $[a,b]$, and $\left|f'(x)\right| \le M$, $x \in [a,b]$, then the following inequalities for k-fractional integrals holds*

$$\begin{aligned}
&\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J^{\alpha,k}_{x^-}f(a)+J^{\alpha,k}_{x^+}f(b)\right]\right| \\
&\le \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right] \\
&\quad \times \left(\int_0^1 \left[t^{\frac{\alpha}{k}}h(t)+t^{\frac{\alpha}{k}}h(1-t)\right]dt\right)^{\frac{1}{q}} \qquad (3.9)\\
&\le \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right] \\
&\quad \times \left(\int_0^1 \left[h(t^{\frac{\alpha}{k}+1}+h(t^{\frac{\alpha}{k}}(1-t))\right]dt\right)^{\frac{1}{q}}, \quad x \in [a,b]. \qquad (3.10)
\end{aligned}$$

Proof. Using Lemma 2.1 and power mean inequality, we have

$$\begin{aligned}
&\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J^{\alpha,k}_{x^-}f(a)+J^{\alpha,k}_{x^+}f(b)\right]\right| \\
&\le \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|dt \\
&\quad + \frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|dt \\
&\le \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \quad (3.11)$$

Since $\left|f'\right|^q$ is h-convex on $[a,b]$ and $\left|f'(x)\right| \leq M$, we get

$$\begin{aligned}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|^q dt &\leq \int_0^1 [t^{\frac{\alpha}{k}}h(t)\left|f'(x)\right|^q + t^{\frac{\alpha}{k}}h(1-t)\left|f'(a)\right|^q]dt \\ &\leq M^q\int_0^1 [t^{\frac{\alpha}{k}}h(t)+t^{\frac{\alpha}{k}}h(1-t)]dt \quad (3.12)\end{aligned}$$

and similarly

$$\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|^q dt \leq M^q\int_0^1 [t^{\frac{\alpha}{k}}h(t)+t^{\frac{\alpha}{k}}h(1-t)]dt. \quad (3.13)$$

Using (3.12) and (3.13) in (3.11) we complete the proof of (3.9). By using the addition properties of h in assumption, we further have

$$\begin{aligned}\int_0^1 \left[t^{\frac{\alpha}{k}}h(t)+t^{\frac{\alpha}{k}}h(1-t)\right]dt &\leq \int_0^1 \left[h(t^{\frac{\alpha}{k}})h(t)+h(t^{\frac{\alpha}{k}})h(1-t)\right]dt \\ &\leq \int_0^1 \left[h(t^{\frac{\alpha}{k}+1})+h(t^{\frac{\alpha}{k}}(1-t))\right]dt.\end{aligned}$$

Hence the proof of (3.10) is completed. □

Remark 3.6. If we put $k=1$ in (3.9), then we get [8, Theorem 3].

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