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# EXSISTENCE RESULTS FOR FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

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**Abstract.** In this paper, we prove the existence of solutions of fractional integrodifferential equations by using the resolvent operators and fixed point theorem. Application to illustrate the theory is also studied.

## 1. Introduction

Fractional differential equations have been an attraction to many mathematicians because of its numerous applications in various fields of science and engineering [17]. It is considered as an alternative model to a nonlinear differential equation [8]. The fractional order differential operator is nonlocal which is the most relevant feature making it a useful tool in applications. The abstract fractional differential equations with nonlocal conditions have been studied extensively in the literature [2], since it is shown that the nonlocal problems have better effects than the normal Cauchy problem. The nonlocal Cauchy problem for an abstract fractional evolution equations was discussed

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in [3] where as in [4, 6] the authors have studied the existence of solutions of fractional impulsive evolution equations and integrodifferential equations in Banach spaces by using fixed point techniques. Hernandez *et al.* [12] established the existence of solution for a class of abstract fractional differential equations with nonlocal condition. Balachandran *et al.* [11] investigated the recent developments in the theory of abstract fractional differential equations in which the resolvent operator [12] played a key role in proving their existence results. In [5], the authors have proved the existence of solution for fractional integrodifferential equations. In this paper, we study the nonlocal fractional integrodifferential equations governed by operator $A$ generating analytical resolvent operators and using the Krasnoselskii fixed point theorem.

## 2. Preliminaries

We need some basic definitions and properties of fractional calculus and semigroup to establish our results. Let $X$ be a Banach space with supnorm denoted by $\|\cdot\|_{C(J;X)}$ and $C(J;X)$ denote the space of all continuous functions from $J := [0,b]$ into Banach space $X$. The notation $X_A$ denotes the domain of $A$ endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$. In addition, $B_r(x,X)$ represents the closed ball with center at $x$ and radius $r$ in $X$.

**Definition 2.1.** (Riemann-Liouville Fractional Integral)
The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of function $f \in L_1\ (\mathbb{R}_+)$ is defined as

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)ds}{(t-s)^{1-\alpha}}, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

**Definition 2.2.** (Riemann-Liouville Fractional Derivative)
The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in N$, is defined as

$$D^{\alpha}_{0+}f(t) = D^{n}_{0+}I^{n-\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}f(s)ds, \tag{2.2}$$

where the function $f(t)$ has absolutely continuous derivatives upto order $(n–1)$.

**Definition 2.3.** (Caputo Fractional Derivative)
The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, is defined as

$$^{C}D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds, \tag{2.3}$$

where the function $f(t)$ has absolutely continuous derivatives upto order ($n$–1). If $0 < \alpha < 1$, then

$$ {}^{C}D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(s)ds}{(t-s)^{\alpha}}, \tag{2.4}$$

where $f'(s) = Df(s) = \dfrac{df(s)}{ds}$ and f is an abstract function with values in X.

Consider the fractional differential equation

$$\begin{cases} D^q u(t) = Au(t) + f(t), & t \in J, \\ u(0) = u_0, \end{cases} \tag{2.5}$$

where $0 < q < 1$, $A$ is a closed linear unbounded operator in $X$ and $f \in C(J;X)$. Equation (2.5) is equivalent to the following integral equation.

$$u(t) = u_0 + \frac{1}{\Gamma(q)}\int_0^t \frac{Au(s)}{(t-s)^{1-q}}ds + \frac{1}{\Gamma(q)}\int_0^t \frac{f(s)}{(t-s)^{1-q}}ds, \quad t \in J. \tag{2.6}$$

This equation can be written in the following form of integral equation

$$u(t) = h(t) + \frac{1}{\Gamma(q)}\int_0^t \frac{Au(s)}{(t-s)^{1-q}}ds, \quad t \geq 0, \tag{2.7}$$

where $h(t) = u_0 + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t \frac{f(s)ds}{(t-s)^{1-q}}$. We assume that the integral equation (2.7) has an associated resolvent operator $\{S(t)\}_{t\geq 0}$ on $X$.

Here we assume that the resolvent operator $\{S(t)\}_{t\geq 0}$ is analytic [15] and there exists a function $\varphi_A$ in $L^1_{loc}([0,\infty);\mathbb{R}^+)$ such that

$$\|S'(t)x\| \leq \varphi_A(t)\|x\|_{X_A}, \quad \text{for all } \ t > 0.$$

**Definition 2.4.** A one parameter family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on X is called a resolvent operator for (2.7) if the following conditions hold:

(i) $S(\cdot)x \in C([0,\infty);X)$ and S(0) x = x for all $x \in X$,

(ii) $S(t)D(A) \subset D(A)$ and AS(t) x = S(t)Ax for all $x \in D(A)$ and every $t \geq 0$,

(iii) for every $x \in D(A)$ and $t \geq 0$,

$$S(t)x = x + \frac{1}{\Gamma(q)}\int_0^t \frac{AS(s)x}{(t-s)^{1-q}}ds. \tag{2.8}$$

**Definition 2.5.** A function $u \in C(J;X)$ is called a mild solution of the integral equation (2.7) on $J$ if $\displaystyle\int_0^t (t-s)^{q-1}u(s)ds \in D(A)$ for all $t \in J,\ h(t) \in$

$C(J;X)$ and

$$u(t) = \frac{A}{\Gamma(q)} \int_0^t \frac{u(s)}{(t-s)^{1-q}} ds + h(t), \;\; \forall\, t \in J.$$

**Lemma 2.6.** *Under the above conditions, the following properties are valid:*

(i) *If $u(\cdot)$ is a mild solution of* (2.7) *on $J$, then the function*

$$t \to \int_0^t S(t-s)h(s)ds$$

*is continuously differentiable on $J$ and*

$$u(t) = \frac{d}{dt} \int_0^t S(t-s)h(s)ds, \qquad \forall\, t \in J. \tag{2.9}$$

(ii) *If $h \in C^\beta(J;X)$ for some $\beta \in (0,1)$, then the function defined by*

$$\begin{aligned} u(t) &= S(t)(h(t) - h(0)) \\ &\quad + \int_0^t S'(t-s)[h(s) - h(t)]ds + S(t)h(0), \;\; t \in J, \end{aligned} \tag{2.10}$$

*is a mild solution of* (2.7) *on $J$.*

(iii) *If $h \in C(J;X_A)$, then the function $u : J \to X$ defined by*

$$u(t) = \int_0^t S'(t-s)h(s)ds + h(t), \;\; t \in J, \tag{2.11}$$

*is a mild solution of* (2.7) *on $J$.*

## 3. Existence and uniqueness

In this paper, we study the existence of mild solution for a class of abstract fractional integrodifferential equations for $t \in J$ of the form

$$\begin{aligned} & D^q(u(t) + e(t, u(t))) \\ &= Au(t) + f\left(t, u(t), \int_0^t k_1(t,s,u(s))ds, \cdots, \int_0^t k_n(t,s,u(s))ds\right), \end{aligned} \tag{3.1}$$

$$u(0) + g(u) = u_0, \tag{3.2}$$

where $D^q$ is the Caputo fractional derivative of order $0 < q < 1$, $A$ is closed linear unbounded operator in a Banach space $X$ with dense domain $D(A)$, $u_0 \in X$ and $f : J \times X \times X^n \to X$, $e : J \times X \to X$, $k_i : \Delta \times X \to X$, $g : C(J;X) \to X$ are continuous. Here $\Delta = \{(t,s) : 0 \le s \le t \le b\}$. For brevity, we take $K_i u(t) = \int_0^t k_i(t,s,u(s))ds$ , $i = 0, \cdots, n$.

Now we introduce the concept of mild solution for equations (3.1)-(3.2). This equation is equivalent to the following integral equation

$$u(t) = u_0 - g(u) + e(0,u_0) - e(t,u(t)) + \frac{1}{\Gamma(q)}\int_0^t \frac{Au(s)}{(t-s)^{1-q}}ds + \frac{1}{\Gamma(q)}\int_0^t \frac{f(s,u(s),K_1u(s),\cdots,K_nu(s))}{(t-s)^{1-q}}ds, \quad \forall\, t \in J. \tag{3.3}$$

**Definition 3.1.** A function $u \in C(J;X)$ is said to be a mild solution of (3.1)-(3.2) on $J$, if $\displaystyle\int_0^t \frac{u(s)}{(t-s)^{1-q}}ds \in D(A)$ for all $t \in J$ and satisfies the integral equation (3.3).

Suppose there exists a resolvent operator $\{S(t)\}_{t\geq 0}$ which is differentiable and the functions $f, g, k_i$ and $e$ are continuous in $X_A$, then we have

$$\begin{aligned} u(t) = {} & u_0 - g(u) + e(0,u_0) - e(t,u(t)) \\ & + \frac{1}{\Gamma(q)}\int_0^t \frac{f(s,u(s),K_1u(s),\cdots,K_nu(s))}{(t-s)^{1-q}}ds \\ & + \int_0^t S'(t-s)\Bigg(u_0 - g(u) + e(0,u_0) - e(s,u(s)) \\ & + \frac{1}{\Gamma(q)}\int_0^s \frac{f(\tau,u(\tau),K_1u(\tau),\cdots,K_nu(\tau))}{(s-\tau)^{1-q}}d\tau\Bigg)ds. \end{aligned}$$

Assume the following conditions:

The fractional integrodifferential equation (3.1)-(3.2) satisfies the Lipschitz condition such that for all $x, y \in X$, $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in X^n$, $t, s \in J$ where $i = 1, 2, \cdots, n$, then we have

(H1) The function $f : J \times X \times X^n \to X_A$ is completely continuous; there exists a constant $L_1 > 0$ such that

$$\|f(t,x,x_1,\cdots,x_n) - f(t,y,y_1,\cdots,y_n)\| \leq L_1\left(\|x-y\| + \sum_{i=1}^n \|x_i - y_i\|\right).$$

(H2) The function $k_i : \Delta \times X \to X_A$ is continuous and there exist constants $B_i$, $B_i' > 0$ such that

$$\left\|\int_0^t [k_i(t,s,x) - k_i(t,s,y)]ds\right\| \leq B_i\|x-y\|,$$

$$\left\|\int_0^t k_i(t,s,x)ds\right\| \leq B_i'[1 + \|x\|].$$

(H3) There exists a constant $L_2 > 0$ of function $e : J \times X \to X_A$ such that

$$\|e(t,x) - e(t,y)\| \leq L_2\|x-y\|.$$

(H4) There exists a constant $G > 0$, of the function $g : C(J;X) \to X_A$ such that

$$\|g(x) - g(y)\| \leq G\|x-y\|.$$

(H5) $2(1 + \|\varphi_A\|_{L^1})(\gamma L_1(1 + \sum_{i=1}^n B_i) + G + L_2) \leq 1$.

For our convenience, let

$$\gamma = \frac{b^q}{q\Gamma(q)}, \quad N = \max_{t\in J}\ f(t,0,\cdots,0), \quad N_1 = \max_{t\in J} e(t,0).$$

**Theorem 3.2.** *Assume $u_0 \in D(A)$, $f$, $g$, $e$, $k_i$, satisfies* (H1)-(H5). *Then there exists a mild solution of* (3.1)-(3.2) *on $J$.*

*Proof.* First we transform the existence of solutions of (3.1)-(3.2) into a fixed point problem. For that, by considering the Lemma 2.6(iii), we introduce the map $\Phi : C(J;X) \to C(J;X)$ by

$$\begin{aligned}\Phi u(t) &= u_0 - g(u) + e(0,u_0) - e(t,u(t)) \\ &\quad + \frac{1}{\Gamma(q)}\int_0^t \frac{f(s,u(s),K_1u(s),\ldots,K_nu(s))}{(t-s)^{1-q}}ds \\ &\quad + \int_0^t S'(t-s)\bigg(u_0 - g(u) + e(0,u_0) - e(s,u(s)) \\ &\quad + \frac{1}{\Gamma(q)}\int_0^s \frac{f(\tau,u(\tau),K_1u(\tau),\ldots,K_n(\tau))}{(s-\tau)^{1-q}}d\tau\bigg)ds.\end{aligned}$$

Now we decompose $\Phi$ as $\Phi_1 + \Phi_2$ on $B_r(0;C(J;X))$ where

$$\begin{aligned}\Phi_1 u(t) &= u_0 - g(u) + e(0,u_0) - e(t,u(t)) + \int_0^t S'(t-s)(u_0 - g(u) \\ &\quad + e(0,u_0) - e(s,u(s)))ds,\end{aligned}$$

and

$$\begin{aligned}\Phi_2 u(t) &= \frac{1}{\Gamma(q)}\int_0^t \frac{f(s,u(s),K_1u(s),\ldots,K_nu(s))}{(t-s)^{1-q}}ds \\ &\quad + \int_0^t S'(t-s)\frac{1}{\Gamma(q)}\int_0^s \frac{f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))}{(s-\tau)^{1-q}}d\tau ds.\end{aligned}$$

Here,

$$h(t) = u_0 - g(u) + e(0,u_0) - e(t,u(t)) + \frac{1}{\Gamma(q)}\int_0^t \frac{f(s,u(s),K_1u(s),\dots,K_nu(s))}{(t-s)^{1-q}}ds$$

is in $C(J;X_A)$. Let $Z = C(J;X)$ and $B_r(0,Z)=\{z \in Z : \|z\| \le r\}$. Choose

$$r \ge 2(1+\|\varphi_A\|_{L^1})(\|u_0\| + \|g(0)\| + \|e(0,u_0)\| + N_1 + \gamma L_1(\sum B_i' + N)).$$

For any $u, v \in Z$, we have

$$\begin{aligned}
&\|\Phi_1u(t) + \Phi_2v(t)\| \\
&\le \|u_0\| + \|g(u)-g(0)\| + \|g(0)\| + \|e(0,u_0)\| + \|e(t,u(t)) - e(t,0)\| + \|e(t,0)\| \\
&\quad + \frac{1}{\Gamma(q)}\int_0^t \frac{\|f(s,v(s),K_1v(s),\dots,K_nv(s)) - f(s,0,\dots,0)\| + \|f(s,0,\dots,0)\|}{(t-s)^{1-q}}ds \\
&\quad + \int_0^t \|S'(t-s)\|\Big(\|u_0\| + \|g(u)-g(0)\| + \|g(0)\| + \|e(0,u_0)\| + \|e(s,u(s)) \\
&\quad - e(s,0)\| + \|e(s,0)\| + \frac{1}{\Gamma(q)}\Big[\int_0^s \frac{\|f(\tau,v(\tau),K_1v(\tau),\dots,K_nv(\tau))\|}{(s-\tau)^{1-q}}d\tau \\
&\quad - \int_0^s \frac{\|f(\tau,0,\dots,0)\|}{(s-\tau)^{1-q}}d\tau + \int_0^s \frac{\|f(\tau,0,\dots,0)\|}{(s-\tau)^{1-q}}d\tau\Big]\Big)ds \\
&\le \|u_0\| + Gr + \|g(0)\| + \|e(0,u_0)\| + L_2\|u(t)\| + N_1 + \frac{L_1b^q}{q\Gamma(q)} \\
&\quad \times \Big(\|v(s)\| + \sum_{i=1}^n \Big\|\int_0^t k_i(t,s,v(s))ds\Big\| + N\Big) + \int_0^t \|S'(t-s)\|\Big[\|u_0\| + Gr + \|g(0)\| \\
&\quad + \|e(0,u_0)\| + L_2\|u(s)\| + N_1 + \frac{L_1b^q}{q\Gamma(q)}\Big(\|v(s)\| + \sum_{i=1}^n \Big\|\int_0^s k_i(s,\tau,v(\tau))d\tau\Big\| + N\Big)\Big]ds \\
&\le \|u_0\| + Gr + \|g(0)\| + \|e(0,u_0)\| + L_2\|u(t)\| + N_1 \\
&\quad + \gamma L_1\Big(\|v(s)\| + \sum_{i=1}^n \Big[\Big\|\int_0^t [k_i(t,s,v(s)) - k_i(t,s,0)]ds\Big\| + \Big\|\int_0^t k_i(t,s,0)ds\Big\|\Big] + N\Big) \\
&\quad + \int_0^t \|S'(t-s)\|\Big[\|u_0\| + Gr + \|g(0)\| + \|e(0,u_0)\| + L_2\|u(s)\| + N_1 + \gamma L_1\Big(\|v(s)\| \\
&\quad + \sum_{i=1}^n \Big[\Big\|\int_0^s [k_i(s,\tau,v(\tau)) - k_i(s,\tau,0)]d\tau\Big\| + \Big\|\int_0^s k_i(s,\tau,0)d\tau\Big\|\Big] + N\Big)ds \\
&\le \|u_0\| + Gr + \|g(0)\| + \|e(0,u_0)\| + L_2r + N_1 + \gamma L_1\Big(r + \sum_{i=1}^n [B_ir + B_i'] + N\Big) + \|\varphi_A\|_{L^1} \\
&\quad \times \Big(\|u_0\| + Gr + \|g(0)\| + \|e(0,u_0)\| + L_2r + N_1 + \gamma L_1\Big(r + \sum_{i=1}^n [B_ir + B_i'] + N\Big)\Big)
\end{aligned}$$

$$\leq (1+\|\varphi_A\|_{L^1})\bigg(\|u_0\|+Gr+\|g(0)\|+\|e(0,u_0)\|+L_2r+N_1+\gamma L_1\Big(r+\sum_{i=1}^{n}B_ir\Big) + \gamma L_1\Big(\sum_{i=1}^{n}B_i'+N\Big)\bigg)$$
$$\leq r.$$

Thus $\Phi$ maps $B_r(0,Z)$ into itself and so $\Phi_1 u + \Phi_2 v \in Br$. From the assumptions (H3) and (H4), we see that, for any $u \in Z$,

$$\Big\|\int_0^t S'(t-s)(u_0+ g(u)+e(0,u_0)+e(s,u(s)))ds\| \leq \|\varphi_A\|_{L^1}(\|u_0\|+Gr+\|g(0)\|+\|e(0,u_0)\|+L_2r+N_1)$$

which implies that the function $s \to S'(t-s)(u_0+g(u)+e(0,u_0)+e(s,u(s)))$ is integrable on $J$, for all $t \in J$ and $\Phi_1 u \in Z$. Moreover for $u, v \in Z$ and $t \in J$, we get

$$\begin{aligned}\|\Phi_1u(t)-\Phi_1v(t)\| &\leq \|g(u)-g(v)\|+\|e(t,u(t))-e(t,v(t))\| \\ &\quad + \int_0^t \|S'(t-s)\|(\|g(u)-g(v)\|+\|e(t,u(t))-e(t,v(t))\|)ds \\ &\leq G\|u-v\|+L_2\|u-v\|+\|\varphi_A\|_{L^1}(G\|u-v\|+L_2\|u-v\|) \\ &\leq (1+\|\varphi_A\|_{L^1})(G+L_2)(\|u-v\|).\end{aligned}$$

By (H5), $\Phi_1$ is a contraction on $B_r(0,Z)$. Now we show that the operator $\Phi_2$ is completely continuous. Note that the function

$$s \to \int_0^t S'(t-s)\int_0^s \frac{f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))}{(s-\tau)^{1-q}}d\tau ds$$

is integrable from the assumptions on $f(\cdot)$ and $k_i(\cdot)$ as shown above. First we show that $\Phi_2$ is uniformly bounded. Now, for $t \in J$,

$$\begin{aligned}\|\Phi_2u(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t \frac{\|f(s,u(s),K_1u(s),\ldots,K_nu(s))\|}{(t-s)^{1-q}}ds \\ &\quad + \int_0^t \|S'(t-s)\|\frac{1}{\Gamma(q)}\int_0^s \frac{\|f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))\|}{(s-\tau)^{1-q}}d\tau ds \\ &\leq (1+\|\varphi_A\|_{L^1})\bigg[\gamma L_1 r\Big(1+\sum_{i=1}^{n}B_i\Big)+\gamma L_1\Big(\sum_{i=1}^{n}B_i'+N\Big)\bigg].\end{aligned}$$

This shows that $\Phi_2$ is uniformly bounded. Let $\{u_n\}$ be a sequence in $B_r(0;Z)$ such that $u_n \to u$ in $B_r(0;Z)$. Since the functions $f$ and $k_i$ are continuous,

$$f(s,u_n(s),K_1u_n(s),\ldots,K_nu_n(s)) \to f(s,u(s),K_1u(s),\ldots,K_nu(s)),$$

as $n \to \infty$.

Now for each $t \in J$, we have

$$\begin{aligned}
&\|\Phi_2 u_n(t) - \Phi_2 u(t)\| \\
&\leq \frac{1}{\Gamma(q)} \int_0^t \frac{\|f(s,u_n(s),K_1u_n(s),\ldots,K_nu_n(s)) - f(s,u(s),K_1u(s),\ldots,K_nu(s))\|}{(t-s)^{1-q}} ds \\
&\quad + \int_0^t \|S'(t-s)\| \Big[ \frac{1}{\Gamma(q)} \int_0^s \frac{\|f(\tau,u_n(\tau),K_1u_n(\tau),\ldots,K_nu_n(\tau))\|}{(s-\tau)^{1-q}} d\tau \\
&\quad - \frac{1}{\Gamma(q)} \int_0^s \frac{f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))}{(s-\tau)^{1-q}} d\tau \Big] ds \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

From the above it is clear that $\Phi_2$ is continuous. We need to prove that the set $\{\Phi_2 u(t) : u \in B_r(0;Z)\}$ is relatively compact in $X$ for all $t \in J$. Obviously $\{\Phi_2 u(0) : u \in B_r(0;Z)\}$ is compact. Fix $t \in (0,b]$ and $u \in B_r(0;Z)$; define the operator $\Phi_2^\varepsilon$ by

$$\begin{aligned}
\Phi_2^\varepsilon u(t) = &\frac{1}{\Gamma(q)} \int_0^{t-\varepsilon} \frac{f(s,u(s),K_1u(s),\ldots,K_nu(s))}{(t-s)^{1-q}} ds \\
&+ \int_0^{t-\varepsilon} S'(t-s) \frac{1}{\Gamma(q)} \int_0^s \frac{f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))}{(s-\tau)^{1-q}} d\tau ds.
\end{aligned}$$

Since, by (H1), $f(\cdot)$ is completely continuous, the set $X_\varepsilon = \{\Phi_2^\varepsilon u(t) : u \in B_r(0;Z)\}$ is precompact in $X$, for every $\varepsilon > 0$, $0 < \varepsilon < t$. Moreover, for every $u(\cdot) \in B_r(0;Z)$, we have

$$\begin{aligned}
&\|\Phi_2 u(t) - \Phi_2^\varepsilon u(t)\| \\
&\leq \frac{1}{\Gamma(q)} \int_{t-\varepsilon}^t \frac{\|f(s,u(s),K_1u(s),\ldots,K_nu(s))\|}{(t-s)^{1-q}} ds + \int_{t-\varepsilon}^t \|S'(t-s)\| \\
&\quad \times \frac{1}{\Gamma(q)} \int_0^s \frac{\|f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))\|}{(s-\tau)^{1-q}} d\tau ds.
\end{aligned}$$

This shows that precompact set $X_\varepsilon$ are arbitrarily close to the set $\{\Phi_2 u(t) : u \in B_r(0;Z)\}$. Hence the set $\{\Phi_2 u(t) : u \in B_r(0;Z)\}$ is precompact in $X$. Next, we prove that $\Phi_2(B_r(0;Z))$ is equicontinuous. The function $\Phi_2 u$, $u \in B_r(0;Z)$ are equicontinuous at $t = 0$. For $t < t+h \leq b$, $h > 0$ we have

$$\begin{aligned}
&\|\Phi_2 u(t+h) - \Phi_2 u(t)\| \\
&\leq \frac{1}{\Gamma(q)} \Big\| \int_0^{t+h} \frac{f(s,u(s),K_1u(s),\ldots,K_nu(s))}{(t+h-s)^{1-q}} ds \\
&\quad - \int_0^t \frac{f(s,u(s),K_1u(s),\ldots,K_nu(s))}{(t-s)^{1-q}} ds \Big\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q)}\left\|\int_0^{t+h} S'(t+h-s)\int_0^s \frac{f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))}{(t+h-\tau)^{1-q}}d\tau ds\right.\\
&\left.-\int_0^t S'(t-s)\int_0^s \frac{f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))}{(t-\tau)^{1-q}}d\tau ds\right\|\\
\le{}& \frac{1}{\Gamma(q)}\int_0^t\left[\frac{1}{(t+h-s)^{1-q}}-\frac{1}{(t-s)^{1-q}}\right]\|f(s,u(s),K_1u(s),\ldots,K_nu(s))\|ds\\
&+\frac{1}{\Gamma(q)}\int_t^{t+h}\frac{\|f(s,u(s),K_1u(s),\ldots,K_nu(s))\|}{(t+h-s)^{1-q}}ds\\
&+\int_0^h\|S'(t+h-s)\|\frac{1}{\Gamma(q)}\int_0^s\frac{\|f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))\|}{(s-\tau)^{1-q}}d\tau ds\\
&+\int_0^t\|S'(t-s)\|\frac{1}{\Gamma(q)}\left\|\int_0^{s+h}\frac{f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))}{(s+h-\tau)^{1-q}}d\tau\right.\\
&\left.-\int_0^s\frac{f(\tau,u(\tau),K_1u(\tau),\ldots,K_nu(\tau))}{(s-\tau)^{1-q}}d\tau\right\|ds
\end{aligned}
$$

which tends to zero as $h\to 0$, since by (H1)$f(\cdot)$ is completely continuous and the set $\{\Phi_2 u : u\in B_r(0;Z)\}$ is equicontinuous. Thus we have proved that $\Phi_2(B_r(0;Z))$ is relatively compact for $t\in J$. By Arzela-Ascoli theorem, $\Phi_2$ is compact. Hence, by the Krasnoselskii fixed point theorem there exists a fixed point $u\in Z$ such that $\Phi u=u$ which is a mild solution to (3.1) with the nonlocal condition (3.2). □

## 4. Example

Consider the following partial integrodifferential equation with fractional temporal derivative of the form

$$
\begin{cases}
\dfrac{\partial^q}{\partial t^q}\left(u(t,x)+a_1(t)u(t,x)\right)=\dfrac{\partial^2}{\partial x^2}u(t,x)+a_2(t)\sin u(t,x)\\
\qquad +\displaystyle\int_0^t c_1(t-s)e^{-u(s,x)}ds+\cdots+\int_0^t c_n(t-s)e^{-u(s,x)}ds,\ t>0,\\
u(t,0)=u(t,\pi)=0,\quad (t,x)\in[0,b]\times[0,\pi],\\
u(0,x)+\displaystyle\sum_{i=1}^n\int_0^{t_i} b_j(\tau)u(\tau,x)d\tau=z(x),
\end{cases}
\tag{4.1}
$$

where $q\in(0,1), z\in L^2[0,\pi]$ and $a_1,a_2,c_i\in L^2(J)$ , $i=1,2,3,\cdots,n$ and $b_j\in L^2(J,\mathbb{R})$, $j=1,2$. Take $X=L^2[0,\pi]$ and let $A$ be the opearator given by $Aw=w''$ with domain

$$D(A):=\{w\in X : w''\in X, w(0)=w(\pi)=0\}.$$

Clearly $A$ has a discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$, and the corresponding normalized eigenfunctions are given by

$$w_n(x) := \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(nx).$$

In addition, $\{w_n : n \in \mathbb{N}\}$ is an orthogonal basis for $X$ and

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} \langle w, w_n \rangle w_n, \quad \forall\, w \in X \quad \text{and for every } \; t > 0.$$

From these expressions, it follows that $\{T(t)\}_{t\geq 0}$ is uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$. From [15], we know that the integral equation

$$u(t) = f(t) + \frac{1}{\Gamma(q)} \int_0^t \frac{Au(s)}{(t-s)^{1-q}} ds, \quad s \geq 0,$$

has an associated analytic resolvent operator $\{S(t)\}_{t\geq 0}$ on $X$ given by

$$S(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t}(\lambda^q - A)^{-1} d\lambda, & t > 0, \\ I, & t = 0, \end{cases} \tag{4.2}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the ray $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in \left(\pi, \frac{\pi}{2}\right)$. It is easy to see that {S(t)} is differentiable, and there exists a constant $N > 0$ such that $\|S'(t)x\| \leq N\|x\|$, for $x \in D(A)$, $t > 0$. To represent the differential system(4.1) in the abstract form (3.1)-(3.2), we introduce the functions $e : J \times X \to X$, $f : J \times X \times X^n \to X$, $g : Z \to X$, $k_i : \Delta \times X \to X$ defined by

$$e(t, w)(x) = a_1(t)w(x),$$
$$f(t, w, K_1 w, \ldots, K_n w)(x) = w(x) + a_2(t) \sin\, w(x) + K_1 w + \ldots + K_n w,$$
$$g(u(x)) = \sum_{i=1}^{n} \int_0^{t_i} b_i(\tau) u(\tau, x) d\tau,$$
$$K_i w = k_i(t, s, w(x)) = c_i(t-s)e^{-w(x)}, i = 1, 2, \cdots, n.$$

Note that $\|g(u(x)) - g(v(x))\| \leq \sum\limits_{i=1}^{n} t_i \|b_i\| \; \|u - v\|$ and $L_2 = \sup_{t\in J} \|a_1(t)\|$. Here $\|\varphi_A\|_{L^1} = N$, $L_1 = (1 + \sup_{t\in J} \|a_2(t)\| + B_i)$, $B_i = \sup_{t\in J} \|c_n(t)\|$, $G = \sum\limits_{i=1}^{n} t_i \|b_i\|$ and $t_i$ is chosen such that

$$r \geq 2(1 + \|\varphi_A\|_{L^1})(N_1 + \gamma L_1(\sum_{i=1}^{n} B_i' + N))$$

and

$$2(1 + \|\varphi_A\|_{L^1})(\gamma L_1(1 + \sum_{i=1}^{n} B_i) + G + L_2) < 1.$$

Thus the conditions (H1)-(H5) of Theorem 3.2 are satisfied. Hence there is a function $u \in C(J; L^2[0,\pi])$ which is a mild solution of (4.1) on $J$.

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