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A STEEPEST-DESCENT KRASNOSEL'SKII–MANN ALGORITHM FOR AN INFINITE FAMILY OF NONEXPANSIVE MAPPINGS

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Abstract. For solving a variational inequality problem over the common fixed point set of an infinite family of nonexpansive mappings on uniformly smooth Banach spaces or reflexive and strictly convex ones with a uniformly Gâteaux differentiable norm, we consider an explicit iteration method, based on the steepest-descent and Krasnosel'skii–Mann algorithms. We also show that some modifications of the last are special case of our result.

1. Introduction

Let X be a Banach space with the dual space X^*. For the sake of simplicity, the norms of X and X^* are denoted by $\|.\|$. We use $\langle x, x^*\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let $A : X \to X$ be an η-strongly accretive and γ-strictly pseudocontractive mapping. Let $\{B_k\}$ be a countably infinite family of nonexpansive mappings on X such that $C := \cap_{k=1}^{\infty} \mathrm{Fix}(B_k) \neq \emptyset$, where $\mathrm{Fix}(B_k) = \{x \in X : x = B_k x\}$, the set of fixed points for B_k.

In this paper, our aim is to find a point $u_* \in X$ such that

$$u_* \in C : \quad \langle Au_*, j(u_* - u)\rangle \leq 0, \quad \forall\, u \in C, \tag{1.1}$$

where $\{B_k\}$ satisfies the following conditions:

$$\lim_{k\to\infty} \sup_{x\in D} \|B_{k+1}x - B_k x\| = 0 \tag{1.2}$$

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for any bounded subset D of X and

$$\mathrm{Fix}(B) := \cap_{k=1}^{\infty}\mathrm{Fix}(B_k) \text{ with } By = \lim_{k\to\infty} B_k y \text{ for } y \in X.$$

Recall that an operator A with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in X is said to be accretive when for each $x_1, x_2 \in \mathcal{D}(A)$, it holds

$$\langle Ax_2 - Ax_1, j(x_2 - x_1)\rangle \geq 0,$$

where $j(x_2 - x_1) \in J(x_2 - x_1)$, and J denotes the duality map from X to its dual space X^* defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\|x^*\| \quad \text{and} \quad \|x^*\| = \|x\|\}.$$

Clearly, when $C \equiv X$ ($B_k \equiv I$ for all $k \geq 1$), (1.1) is the operator equation $Ax = 0$. In order to find a solution of an η-strongly accretive and Lipschitz continuous mapping A, whose $\mathcal{D}(A) \equiv X$, a uniformly smooth Banach space, we can use the steepest-descent method, $x^1 \in X$ any element and

$$x^{k+1} = (I - \lambda_k A)x^k, \quad k \geq 1, \tag{1.3}$$

where λ_k satisfies the conditions

(C1) $\lambda_k \in (0, 1)$, $\lim_{k\to\infty} \lambda_k = 0$, and

(C2) $\sum_{k=1}^{\infty} \lambda_k = \infty$

(see [33, 37, 38] for details). Computing the set of fixed points of nonexpansive mappings is an important issue in the theory of nonlinear analysis giving rise to numerous applications in applied areas such as in image recovery and signal processing (see, for example, [4, 10]).

Among the fundamental methods for finding a fixed point of a nonexpansive mapping F over a closed convex subset C of a Hilbert space H, we would like to refer to the Krasnosel'skii–Mann method [23, 24], $x^{k+1} = (1-\alpha_k)x^k + \alpha_k Fx^k$, and the Halpern method [13, 28], $x^{k+1} = \alpha_k u + (1-\alpha_k)Fx^k$ with any $u, x^1 \in C$ and $\alpha_k \in (0, 1)$ under some conditions on α_k. In these methods, mappings which are contractions over C play an important role.

Recently, Yao *et al.* [36] proposed a modification of the Krasnosel'skii–Mann algorithm. In their method, two sequences $\{x^k\}$ and $\{y^k\}$ are built from a starting point x^1 as follows:

$$\begin{cases} y^k = (1 - \lambda_k)x^k, \\ x^{k+1} = (1 - \alpha_k)y^k + \alpha_k Fy^k, \end{cases} \tag{1.4}$$

and proved that if $\mathrm{Fix}(F) \neq \emptyset$, where F is a nonexpansive mapping on H, the parameter λ_k satisfies conditions (**C1**), (**C2**), and α_k satisfy condition:

(C3) $\alpha_k \in [a, b] \subset (0, 1)$, then the sequence $\{x^k\}$, generated by (1.4), converges strongly to a fixed point of F. This sequence $\{x^k\}$ is strongly convergent to a fixed point of the operator F, provided that F is nonexpansive, $\mathrm{Fix}(F)$ is nonempty and further several conditions are satisfied by the sequences of parameters $\{\lambda_k\}$ and $\{\alpha_k\}$ (see also [22, 28]). Note that similar results were obtained by Buong *et al.* [9] and Shehu [26] when the Hilbert space is replaced by a uniformly convex Banach space having a uniformly Gâteaux differentiable norm.

Very recently, Shehu and Ugwunnadi [27] extended (1.4) to an infinite family of nonexpansive mappings on a uniformly convex and uniformly smooth Banach space X and proved the following result.

Theorem 1.1. *Let X be a uniformly convex real Banach space which is also uniformly smooth. For any $k = 1, 2, \cdots$, let $B_k : X \to X$ be a nonexpansive mapping such that $\cap_{k=1}^{\infty}\mathrm{Fix}(B_k) \neq \emptyset$, there holds*

$$\sum_{k=1}^{\infty} \sup_{x\in D} \|B_{k+1}x - B_k x\| < \infty, \tag{1.5}$$

for any bounded subset D of X and there exists a nonexpansive mapping B such that $By = \lim_{k\to\infty} B_k y$ for $y \in X$ with $\mathrm{Fix}(B) = \cap_{k=1}^{\infty}\mathrm{Fix}(B_k)$. Let the sequences $\{x^k\}$ and $\{y^k\}$ be generated by

$$\begin{cases} x^1 \in X, \quad \textit{any element}, \\ y^k = (1-\lambda_k)x^k, \\ x^{k+1} = (1-\alpha_k)y^k + \alpha_k B_k y^k, \end{cases} \tag{1.6}$$

*where λ_k and α_k are in $[0, 1]$ and satisfy (**C1**)-(**C3**). Then, the sequences $\{x^k\}$ and $\{y^k\}$ converge to a point in $\cap_{k=1}^{\infty}\mathrm{Fix}(B_k)$.*

On the other hand, in order to find a common fixed point of an infinite family of nonexpansive mappings B_k on a closed and convex subset C of a Banach space X, Aoyama *et al.* [3] proposed a modification of the Halpern method,

$$\begin{cases} x^1 = u \in C, \quad \text{any element}, \\ x^{k+1} = \lambda_k u + (1-\lambda_k)B_k x^k, \end{cases} \tag{1.7}$$

and proved the following result.

Theorem 1.2. *Let X be a real uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and C be a nonempty closed convex subset of X. Let $\{B_k\}$ be a sequence of nonexpansive mappings of C into itself such that $\cap_{k=1}^{\infty}\mathrm{Fix}(B_k) \neq \emptyset$ and there holds* (1.2)*. Let the parameter $\lambda_k \in [0, 1]$ satisfy* (**C1**)*,* (**C2**) *and either*

(i) $\sum_{k=1}^{\infty} |\lambda_{k+1} - \lambda_k| < \infty$ *or*
(ii) $\lambda_k \in (0, 1]$ *for all* $k \geq 1$ *and* $\lim_{k\to\infty}(\lambda_k/\lambda_{k+1}) = 1$.

Then, the sequence $\{x^k\}$*, defined by* (1.7)*, converges to a point in* $\cap_{k=1}^{\infty}\mathrm{Fix}(B_k)$.

For a given family $\{F_i\}$ of nonexpansive mapping F_i on C, in order to overcome conditions (i), (ii) in the Theorem 1.2 and condition (1.2), by using W_k-mapping, generated by $F_k, F_{k-1}, \cdots, F_1$ and real numbers $\gamma_k, \gamma_{k-1}, \cdots, \gamma_1$ as follows:

$$\begin{aligned}
&U_{k,k+1} = I,\\
&U_{k,k} = \gamma_k F_k U_{k,k+1} + (1-\gamma_k)I,\\
&U_{k,k-1} = \gamma_{k-1} F_{k-1} U_{k,k} + (1-\gamma_{k-1})I,\\
&\dots\\
&U_{k,2} = \gamma_2 F_2 U_{k,3} + (1-\gamma_2)I,\\
&W_k = U_{k,1} = \gamma_1 F_1 U_{k,2} + (1-\gamma_1)I,
\end{aligned}$$

and the Halpern method, Qin *et al.* [25] introduced the following method,

$$\begin{cases} u, x^1 \in C, \quad \text{any elements},\\ y^k = \alpha_k x^k + (1-\alpha_k)W_k x^k,\\ x^{k+1} = \lambda_k u + (1-\lambda_k)y^k, \end{cases} \tag{1.8}$$

and proved that if the parameters λ_k and α_k satisfy conditions (**C1**)-(**C3**), then the sequence $\{x^k\}$, defined by (1.8), converges strongly in a reflexive and strictly convex Banach space X, which also has a weak continuous duality mapping.

Variational inequalities over the fixed point set of nonexpansive mappings play an important role in solving practical problems such as the signal recovery problem, beamforming problem, power control problem, bandwidth allocation problem, and finance problem (see, *e.g.*, [14]-[18]). In order to solve the class of variational inequalities, in 2001, Yamada [34] introduced the hybrid steepest-descent method,

$$x^{k+1} = \big(I - \lambda_{k+1}\mu A\big)Fx^k,$$

and proved a strong convergence theorem, when the parameter λ_k satisfies (**C1**), (**C2**) and (i) in Theorem 1.2, $\mu \in (0, 2\eta/L^2)$, and the mapping A is η-strongly monotone and L-Lipschitz continuous on H. Methods for solving the class of variational inequalities are intensively investigated (see, [2, 6, 7, 8, 11, 19, 21, 29, 31, 35, 39, 40] and references therein). Very recently, in order to solve (1.1), Buong *et al.* [8] proposed the following better iterations

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k S_k(I - \lambda_k A)x^k$$

and

$$x^{k+1} = (1-\alpha_k)S_k x^k + \alpha_k(I - \lambda_k A)x^k,$$

where the mapping S_k is defined by

$$S_k = \sum_{i=1}^{k} s_i F_i/\tilde{s}_k,\ \ s_i > 0,\ \ \tilde{s}_k = \sum_{i=1}^{k} s_i \quad \text{and} \quad \sum_{i=1}^{\infty} s_i = \tilde{s} < \infty, \tag{1.9}$$

the parameters α_k, λ_k satisfy the condition

$$0 < \liminf_{k\to\infty} \alpha_k \le \limsup_{k\to\infty} \alpha_k < 1,$$

(**C1**) and (**C2**). In [5], Buong *et al.* introduced a new simple, strongly convergent, parallel and explicit iterative method,

$$\begin{cases} x_1 \in X, \quad \text{any element,} \\ x^{k+1} = (I - \lambda_k A)\tilde{S}_k(x^k), \quad k \ge 1, \end{cases} \tag{1.10}$$

where $\tilde{S}_k$ is defined by

$$\tilde{S}_k = \sum_{i=1}^{k} s_i F^i/\tilde{s}_k,\ \ F^i = \alpha_i I + (1-\alpha_i)F_i, \tag{1.11}$$

with $\alpha_i \in (0,1)$ and s_i satisfies the conditions in (1.9).

In this paper, to solve the problem (1.1)-(1.2), motivated by (1.6)-(1.10) and the result in [9], we consider the following method,

$$\begin{cases} x_1 \in X, \quad \text{any element,} \\ x^{k+1} = (I - \lambda_k A)\big(\alpha_k I + (1-\alpha_k)B_k\big)x^k, \quad k \ge 1, \end{cases} \tag{1.12}$$

that is a combination of the steepest-descent method and Krasnosel'skii–Mann method. We shall prove the strong convergence of $\{x^k\}$, generated by (1.12), to u_*, solving (1.1) with conditions (**C1**)-(**C3**). Next, we show that method (1.12) contains (1.6), (1.8) and an improvement of (1.11), as special cases.

The paper is organized as follows. In Section 2, we list some related facts that will be used latter our result. In Section 3, we prove a strong convergence result for (1.12) and show that modified Krasnosel'skii–Mann type method can be deduced from our result.

2. Preliminaries

Let X be a real Banach space and let X^* be its dual space. For the sake of simplicity, the norms of X and X^* are denoted by $\|\cdot\|$. Let J be the normalized duality mapping of X. It is well known that if $x \neq 0$, then $J(-x) = -J(x)$ and $J(tx) = tJ(x)$ for all $t > 0$. Let $A : X \to X$ be a mapping. A is said to be η-strongly accretive and γ-strictly pseudocontractive when the following conditions are satisfied:

$$\langle Ax_1 - Ax_2, j(x_1 - x_2)\rangle \geq \eta\|x_1 - x_2\|^2, \tag{2.1}$$

$$\langle Ax_1 - Ax_2, j(x_1 - x_2)\rangle \leq \|x_1 - x_2\|^2 - \gamma\|(I - A)x_1 - (I - A)x_2\|^2, \tag{2.2}$$

for all $x_1, x_2 \in X$ and some element $j(x_1 - x_2) \in J(x_1 - x_2)$, where I denotes the identity mapping of X, η and $\gamma \in (0, 1)$ are some positive constants. Clearly, if A is γ-strictly pseudocontractive, then $\|Ax_1 - Ax_2\| \leq L\|x_1 - x_2\|$ with $L = 1 + 1/\gamma$ and, in this case, A is called to be L-Lipschitz continuous. In addition, if $L \in [0, 1)$, then A is called to be contractive.

Let $S_1(0) := \{x \in X : \|x\| = 1\}$. The space X is said to have a Gâteaux differentiable norm if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for each $x, y \in S_1(0)$. Such a space X is also called a smooth Banach space. The space X is said to have a uniformly Gâteaux differentiable norm if the limit is attained uniformly for $x \in S_1(0)$. The norm of X is called Fréchet differentiable, if for all $x \in S_1(0)$, the limit in (2.3) is attained uniformly for $y \in S_1(0)$. The norm of X is called to be uniformly Fréchet differentiable (and X is called uniformly smooth) if the limit is attained uniformly for all $x, y \in S_1(0)$. It is well known that every uniformly smooth real Banach space is reflexive and has a uniformly Gâteaux differentiable norm [12].

Recall that a Banach space X is said to be

(i) uniformly convex if for any $0 < \varepsilon \leq 2$, the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ imply that there exists a $\delta = \delta(\varepsilon) \geq 0$ such that $\|(x + y)/2\| \leq 1 - \delta$;

(ii) strictly convex if for all $x, y \in S_1(0)$ with $x \neq y$

$$\|(1 - \lambda)x + \lambda y\| < 1 \quad \text{for all } \lambda \in (0, 1).$$

It is well known that each uniformly convex Banach space X is reflexive and strictly convex. If the norm of X is uniformly Gâteaux differentiable, then J is a norm to weak star uniformly continuous mapping on each bounded subset of X, and if X is smooth, then the duality mapping is single-valued. In the sequel, we shall denote by j the single-valued normalized duality mapping.

Lemma 2.1. ([11]) *Let X be a smooth real Banach space and let $A : X \to X$ be an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta + \gamma > 1$. Then, for any $\lambda \in (0, 1)$, $I - \lambda A$ is a contraction with constant $1 - \lambda\tau$, where $\tau = 1 - \sqrt{(1 - \eta)/\gamma}$. Furthermore, $I - A$ is a contraction with constant $1 - \tau$.*

Lemma 2.2. ([11]) *Let X be a smooth real Banach space. Then, the following inequality holds*

$$\|x+y\|^2 \leq \|x\|^2 + \ 2\langle y, j(x+y)\rangle, \quad \forall\, x, y \in X.$$

Lemma 2.3. ([32]) *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the conditions*

$$a_{k+1} \leq (1-b_k)a_k + b_k c_k$$

for all k, where $\{b_k\}$ and $\{c_k\}$ are sequences of real numbers such that $b_k \in [0,1]$ for all k, $\sum_{k=1}^{\infty} b_k = \infty$, and $\limsup_{k\to\infty} c_k \leq 0$. Then, $lim_{k\to\infty} a_k = 0$.

Lemma 2.4. ([28]) *Let $\{x^k\}$ and $\{z^k\}$ be bounded sequences in a Banach space X such that*

$$x^{k+1} = \eta_k x^k + (1-\eta_k) z^k$$

for $k \geq 0$, where the sequence $\{\eta_k\}$ satisfies the condition

$$0 < \liminf_{k\to\infty} \eta_k \leq \limsup_{k\to\infty} \eta_k < 1.$$

Assume that

$$\limsup_{k\to\infty} \big(\|z^{k+1} - z^k\| - \|x^{k+1} - x^k\|\big) \leq 0.$$

Then $\lim_{k\to\infty} \|x^k - z^k\| = 0$.

Let μ be a continuous linear functional on l^∞ and let $a = (a_0, a_1, \cdots) \in l^\infty$. We write $\mu_k(a_k)$ instead of $\mu((a_0, a_1, \cdots))$. We recall that μ is a Banach limit when μ satisfies $\|\mu\| = \mu_k(1) = 1$ and $\mu_k(a_{k+1}) = \mu_k(a_k)$ for each $(a_0, a_1, \cdots) \in l^\infty$. For a Banach limit μ, we know that $\liminf_{k\to\infty} a_k \leq \mu_k(a_k) \leq \limsup_{k\to\infty} a_k$, for all $(a_0, a_1, \cdots) \in l^\infty$. If $b = (b_0, b_1, \cdots) \in l^\infty$ and $a_k \to c$ (respectively, $a_k - b_k \to 0$), as $k \to \infty$, we have $\mu_k(a_k) = \mu(a) = c$ (respectively, $\mu_k(a_k) = \mu_k(b_k)$).

Lemma 2.5. ([30]) *Let C be a closed and convex subset of a Banach space X whose norm is uniformly Gâteaux differentiable. Let $\{x^k\}$ be a bounded subset of X, let z be an element of C and let μ be a Banach limit. Then,*

$$\mu_k \|x^k - z\|^2 = \min_{x\in C} \mu_k \|x^k - u\|^2$$

if and only if $\mu_k \langle u - z, j(x^k - z)\rangle \leq 0$ for all $u \in C$.

3. Main results

First of all, we prove the following statement.

Proposition 3.1. *Let A be an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta+\gamma > 1$ in a uniformly smooth Banach space X. For a fixed nonexpansive mapping F on X and each $t \in (0,1)$, choose two numbers $\alpha_t \in (a,b) \subset (0,1)$ and $\lambda_t \in (0,1)$ arbitrarily such that $\lambda_t \to 0$ as $t \to 0$ and let $\{y^t\}$ be defined by*

$$y^t = (I - \lambda_t A)F^t y^t, \quad F^t = \alpha_t I + (1-\alpha_t)F. \tag{3.1}$$

Then, the net $\{y^t\}$ converges strongly to u^*, solving (1.1) with $C = \mathrm{Fix}(F)$, assumed to be nonempty, as $t \to 0$.

Proof. Consider the mapping $B_t = (I - \lambda_t A)F^t$ for each $t \in (0,1)$. From Lemma 2.1 and the property of F, it follows that

$$\begin{aligned}\|B_t x - B_t y\| &= \|(I-\lambda_t A)F^t x - (I-\lambda_t A)F^t y\| \\ &\le (I - \lambda_t \tau)\|F^t x - F^t y\| \\ &\le (1-\lambda_t\tau)\big[\alpha_t\|x-y\| + (1-\alpha_t)\|Fx - Fy\|\big] \\ &\le (1-\lambda_t\tau)\|x-y\|, \quad \forall\, x, y \in X.\end{aligned}$$

Thus, B_t is a contraction in A. By Banach's Contraction Principle, there exists a unique element $y^t \in A$, satisfying (3.1). Next, we show that $\{y^t\}$ is bounded. Indeed, for any point $p \in \mathrm{Fix}(F)$, we have $p = F^t p$, and hence, by virtue of Lemmas 2.1 and 2.2,

$$\begin{aligned}\|y^t - p\|^2 &= \|(I-\lambda_k A)F^t y^t - p\|^2 \\ &= \|(I-\lambda_t A)F^t y^t - (I-\lambda_t A)F^t p - \lambda_t A p\|^2 \\ &\le (1-\lambda_t\tau)\|y^t - p\|^2 - 2\lambda_t\big\langle Ap, j(y^t - p)\big\rangle \\ &\le \frac{-2}{\tau}\big\langle Ap, j(y^t-p)\big\rangle.\end{aligned} \tag{3.2}$$

Therefore, $\{y^t\}$ is bounded. So, are the nets $\{Fy^t\}$ and $\{AF^t y^t\}$. Take a sequence $\{t_m\}$ in $(0,t_0)$ that converges to 0 as $m \to \infty$, we have a sequence $\{y^m\}$ and a functional $\varphi(x)$, defined by (3.1) and $\varphi(x) = \mu_m\|y^m - x\|^2$ for all $x \in X$. We see that $\varphi(x) \to \infty$ as $\|x\| \to \infty$ and φ is continuous and convex. So as X is reflexive, there exists an element $\tilde{y} \in X$ such that $\varphi(\tilde{y}) = \min_{x\in X}\varphi(x)$, *i.e.*, the set

$$C^* = \{u \in X : \varphi(u) = \min_{x \in X}\varphi(x)\} \neq \emptyset.$$

It is easy to see that C^* is a bounded, closed and convex subset in X (see, [1]). Further, from (3.1), $\lambda_m := \lambda_{t_m} \to 0$ and the boundedness of the sequence

$\{AF^m y^m\}$ with $F^m = \alpha_m I + (1-\alpha_m)F$ and $\alpha_m = \alpha_{t_m}$, it follows that

$$\|y^m - F^m y^m\| = \lambda_m \|AF^m y^m\| \to 0 \quad \text{as} \quad m \to \infty. \tag{3.3}$$

Since $\|y^m - Fy^m\| = \lambda_m \|AF^m y^m\|/(1-\alpha_m)$ and $\alpha_m \in (a,b) \subset (0,1)$, we have

$$\lim_{m\to\infty} \|y^m - Fy^m\| = 0.$$

On the other hand, from the properties of a Banach limit, we can write

$$\varphi(F\tilde{y}) = \mu_m \|y^m - F\tilde{y}\|^2 = \mu_m \|Fy^m - F\tilde{y}\|^2 \le \mu_m \|y^m - \tilde{y}\|^2 = \varphi(\tilde{y}),$$

which means that $FC^* \subseteq C^*$, or, C^* is invariant under the nonexpansive mapping F. So, we have that

$$F \text{ has a fixed point, say } \tilde{p}, \text{ in } C^*. \tag{3.4}$$

It means that $\tilde{p} \in \mathrm{Fix}(F) \cap C^*$. Now, from Lemma 2.5, we know that $\tilde{p}$ is a minimizer of $\varphi(x)$ on X, if and only if

$$\mu_m \langle x - \tilde{p}, j(y^m - \tilde{p})\rangle \le 0, \quad \forall\, x \in X. \tag{3.5}$$

Taking $x = -A\tilde{p} + \tilde{p}$ in (3.5) and replacing y^t with p in (3.2) by y^m with $\tilde{p}$, respectively, we obtain that $\mu_m \|y^m - \tilde{p}\|^2 = 0$. Thus, there exists a subsequence $\{y^{m_l}\}$ of $\{y^m\}$ which converges strongly to $\tilde{p}$ as $l \to \infty$. Again, by virtue of (3.2) and the norm to weak star continuous property of the normalized duality mapping j on bounded subsets of X, we obtain that

$$\langle Ap, j(\tilde{p} - p)\rangle \le 0, \quad \forall\, p \in \mathrm{Fix}(F). \tag{3.6}$$

Since p and $\tilde{p}$ belong to $\mathrm{Fix}(F)$, a closed and convex subset, replacing p in (3.6) by $sp + (1-s)\tilde{p}$ for $s \in (0,1)$, using the well-known property that $j(s(\tilde{p} - p)) = sj(\tilde{p} - p)$ for $s > 0$, dividing by s and taking $s \to 0$, we obtain that

$$\langle A\tilde{p}, j(\tilde{p} - p)\rangle \le 0, \quad \forall\, p \in \mathrm{Fix}(F).$$

The uniqueness of u_*, satisfying (1.1) with $C = \mathrm{Fix}(F)$, guarantees that $\tilde{p} = u_*$ and all net $\{y^t\}$ converges strongly to u_* as $t \to \infty$. This completes the proof. □

Lemma 3.2. *Let X, A and F be as in Proposition* 3.1. *Assume that λ_k and α_k satisfy conditions* (**C1**)-(**C3**). *Then, if the sequence $\{x^k\}$, defined by* (1.12), *is bounded and $\lim_{k\to\infty} \|x^k - Fx^k\| = 0$, we have*

$$\limsup_{k\to\infty} \langle Au_*, j(u_* - x^k)\rangle \le 0. \tag{3.7}$$

Proof. Let $y^m = y^{t_m}$ be as in the proof of Proposition 3.1. Then, by using the definition of j and the non-expansivity of F^m, one can write

$$\begin{aligned}\|y^m - x^k\|^2 &= \langle (I - \lambda_m A)F^m y^m - x^k, j(y^m - x^k)\rangle \\ &= \langle F^m y^m - F^m x^k, j(y^m - x^k)\rangle \\ &\quad - \lambda_m \langle AF^m y^m, j(y^m - x^k)\rangle + \langle F^m x^k - x^k, j(y^m - x^k)\rangle \\ &\le \|y^m - x^k\|^2 - \lambda_m \langle AF^m y^m, j(y^m - x^k)\rangle + \|F^m x^k - x^k\|\tilde{M},\end{aligned}$$

where $\tilde{M} \ge \|y^m - x^k\|$. Therefore,

$$\langle AF^m y^m, j(y^m - x^k)\rangle \le \frac{\|F^m x^k - x^k\|\tilde{M}}{\lambda_m} = \frac{1-\alpha_m}{\lambda_m}\|x^k - Fx^k\|\tilde{M}.$$

By the assumption,

$$\limsup_{k\to\infty} \langle AF^m y^m, j(y^m - x^k)\rangle \le 0,$$

that together with (3.3) and Proposition 3.1 implies (3.7). Lemma is proved. □

Now, we are in position to prove our main result.

Theorem 3.3. *Let X be a uniformly smooth real Banach space, $A : X \to X$ an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta+\gamma > 1$ and, for any $k = 1, 2, \cdots$, $B_k : X \to X$ a nonexpansive mapping such that $\cap_{k=1}^{\infty}\mathrm{Fix}(B_k) \ne \emptyset$, there holds* (1.2) *for any bounded subset D of X and there exists $By = \lim_{k\to\infty} B_k y$, for $y \in X$ such that $\mathrm{Fix}(B) = \cap_{k=1}^{\infty}\mathrm{Fix}(B_k)$. Assume that λ_k and α_k satisfy conditions* (**C1**)-(**C3**)*. Then, the sequence $\{x^k\}$, defined by* (1.12)*, converges strongly to the element u_* solving* (1.1).

Proof. Put $F^k = \alpha_k I + (1 - \alpha_k)B_k$. Since $F^k p = p$ for any point $p \in \cap_{k=1}^{\infty}\mathrm{Fix}(B_k)$ and $k \ge 1$, by Lemma 2.2,

$$\begin{aligned}\|x^{k+1} - p\| &= \|(1 - \lambda_k A)F^k x^k - (1 - \lambda_k A)F^k p - \lambda_k Ap\| \\ &\le (1 - \lambda_k \tau)\|x^k - p\| + \lambda_k \tau \|Ap\|/\tau \\ &\le \max\{\|x^1 - p\|, \|Ap\|/\tau\}.\end{aligned}$$

Therefore, $\{x^k\}$ is bounded, so are the sequences $\{B_k x^k\}$, $\{F^k x^k\}$, $\{F^{k+1} x^k\}$ and $\{AF^k x^k\}$. Without any loss of generality, we assume that they are bounded by a positive constant M_1. Further, it is easy to see that

$$\begin{aligned}x^{k+1} &= \lambda_k(I - A)F^k x^k + (1 - \lambda_k)F^k x^k \\ &= \lambda_k(I - A)F^k x^k + (1 - \lambda_k)\big[\alpha_k x^k + (1 - \alpha_k)B_k x^k\big] \qquad (3.8)\\ &= \eta_k x^k + (1 - \eta_k)z^k,\end{aligned}$$

where

$$\eta_k = (1-\lambda_k)\alpha_k \quad \text{and} \quad z^k = \frac{\lambda_k (I-A)F^k x^k}{1-\eta_k} + \frac{(1-\lambda_k)(1-\alpha_k)B_k x^k}{1-\eta_k}.$$

Clearly, $0 < \liminf_{k\to\infty} \eta_k \le \limsup_{k\to\infty} \eta_k < 1$. Moreover, we have

$$\begin{aligned}
&\frac{\lambda_{k+1}(I-A)F^{k+1}x^{k+1}}{1-\eta_{k+1}} - \frac{\lambda_k (I-A)F^k x^k}{1-\eta_k} \\
&= \frac{\lambda_{k+1}}{1-\eta_{k+1}}\left[(I-A)F^{k+1}x^{k+1} - (I-A)F^{k+1}x^k\right] \\
&\quad + \frac{\lambda_{k+1}}{1-\eta_{k+1}}\left[(I-A)F^{k+1}x^k - (I-A)F^k x^k\right] \\
&\quad + \left[\frac{\lambda_{k+1}}{1-\eta_{k+1}} - \frac{\lambda_k}{1-\eta_k}\right]\times (I-A)F^k x^k
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(1-\lambda_{k+1})(1-\alpha_{k+1})B_{k+1}x^{k+1}}{1-\eta_{k+1}} - \frac{(1-\lambda_k)(1-\alpha_k)B_k x^k}{1-\eta_k} \\
&= \frac{(1-\lambda_{k+1})(1-\alpha_{k+1})}{1-\eta_{k+1}}\left[B_{k+1}x^{k+1} - B_{k+1}x^k\right] \\
&\quad + \frac{(1-\lambda_{k+1})(1-\alpha_{k+1})}{1-\eta_{k+1}}\left[B_{k+1}x^k - B_k x^k\right] \\
&\quad + \left[\frac{(1-\lambda_{k+1})(1-\alpha_{k+1})}{1-\eta_{k+1}} - \frac{(1-\lambda_k)(1-\alpha_k)}{1-\eta_k}\right] B_k x^k,
\end{aligned}$$

and hence,

$$\begin{aligned}
&\|z^{k+1} - z^k\| \\
&\le \frac{\lambda_{k+1}}{1-\eta_{k+1}}(1-\tau_1)\left[\|x^{k+1}-x^k\| + 2M_1\right] + \left|\frac{\lambda_{k+1}}{1-\eta_{k+1}} - \frac{\lambda_k}{1-\eta_k}\right| 2M_1 \\
&\quad + \frac{(1-\lambda_{k+1})(1-\alpha_{k+1})}{1-\eta_{k+1}}\left(\|x^{k+1}-x^k\| + \sup_{x\in D_1}\|B_{k+1}x - B_k x\|\right) \\
&\quad + \left|\frac{(1-\lambda_{k+1})(1-\alpha_{k+1})}{1-\eta_{k+1}} - \frac{(1-\lambda_k)(1-\alpha_k)}{1-\eta_k}\right| M_1 \\
&\le \left[\frac{\lambda_{k+1}(1-\tau_1)}{1-\eta_{k+1}} + \frac{(1-\lambda_{k+1})(1-\alpha_{k+1})}{1-\eta_{k+1}}\right]\|x^{k+1}-x^k\| + \tilde{c}_k, \\
&= \frac{\lambda_{k+1}(1-\tau_1) + (1-\lambda_{k+1})(1-\alpha_{k+1}))}{1-\alpha_{k+1}+\alpha_{k+1}\lambda_{k+1}}\|x^{k+1}-x^k\| + \tilde{c}_k, \\
&\le \|x^{k+1}-x^k\| + \tilde{c}_k,
\end{aligned}$$

where $D_1 = D(0, M_1)$ is the ball centered at 0 and of radius M_1, and

$$\begin{aligned}\tilde{c}_k := &\frac{\lambda_{k+1}}{1-\eta_{k+1}} 2M_1(1-\tau_1) + \left|\frac{\lambda_{k+1}}{1-\eta_{k+1}} - \frac{\lambda_k}{1-\eta_k}\right| 2M_1 \\ &+ \frac{(1-\lambda_{k+1})(1-\alpha_{k+1})}{1-\eta_{k+1}} \sup_{x\in D_1} \|B_{k+1}x - B_k x\| \\ &+ \left|\frac{(1-\lambda_{k+1})(1-\alpha_{k+1})}{1-\eta_{k+1}} - \frac{(1-\lambda_k)(1-\alpha_k)}{1-\eta_k}\right| M_1.\end{aligned}$$

It is not difficult to verify that $\tilde{c}_k \to 0$ as $k \to \infty$. Therefore,

$$\limsup_{k\to\infty} \big(\|z^{k+1} - z^k\| - \|x^{k+1} - x^k\|\big) \leq 0.$$

By virtue of Lemma 2.4,

$$\lim_{k\to\infty} \|x^k - z^k\| = 0. \tag{3.9}$$

Thus, from (3.8) and (3.9) it follows that

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = \lim_{k\to\infty} (1-h_k)\|z^k - x^k\| = 0. \tag{3.10}$$

Next, from (1.12) it follows that $\|x^{k+1} - F^k x^k\| \leq \lambda_k M_1 \to 0$ as $k \to \infty$. This together with (3.10) implies

$$\lim_{k\to\infty} \|x^k - F^k x^k\| = 0.$$

Consequently, by (1.12) and the property of α_k,

$$\lim_{k\to\infty} \|x^k - B_k x^k\| = 0. \tag{3.11}$$

Let $B : X \to X$ be a mapping defined by $By = \lim_{k\to\infty} B_k y$ for any $y \in X$. Then, $\lim_{k\to\infty} \sup\{\|B_k y - By\| : x \in D(0, M_1)\} = 0$, by the assumption. Now, we shall show that $\|x^k - Bx^k\| \to 0$ as $k \to \infty$. Indeed, it can be seen from the assumption that if D is a nonempty and bounded subset of X, then for $\varepsilon > 0$, there exists $k_0 > 1$ such that for all $k > k_0$

$$\sup_{y\in D} \|B_k y - By\| \leq \varepsilon.$$

Taking $D = \{x_k : k \geq 1\}$, we obtain that

$$\|B_k x^k - Bx^k\| \leq \sup_{y\in D} \|B_k x - Bx\| \leq \varepsilon.$$

This implies that $\|B_k x^k - Bx^k\| \to 0$ as $k \to \infty$.

On the other hand, since

$$\|x^k - Bx^k\| \leq \|x^k - B_k x^k\| + \|B_k x^k - Bx^k\|,$$

leading to

$$\lim_{k\to\infty} \|x^k - Bx^k\| = 0. \tag{3.12}$$

It means that $\{x^k\}$, defined by (1.12), statisfies (3.7) with F replaced by B. Now, the value $\|x^{k+1} - u_*\|^2$ can be estimated as follows.

$$\begin{aligned} \|x^{k+1} - u_*\|^2 &= \|(I - \lambda_k A)F^k x^k - u_*\|^2 \\ &= \|(I - \lambda_k A)F^k x^k - (I - \lambda_k A)F^k u_* - \lambda_k A u_*\|^2 \\ &\le (1 - \lambda_k \tau)\|x^k - u_*\|^2 + 2\lambda_k \tau \langle Au_*, j(u_* - x^{k+1})\rangle/\tau \\ &= (1 - b_k)\|x^k - u_*\|^2 + b_k c_k, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} &b_k = \lambda_k \tau, \\ &c_k = 2\big[\langle Au_*, j(u_* - x^k)\rangle + \langle Au_*, j(u_* - x^{k+1}) - j(u_* - x^k)\rangle\big]/\tau. \end{aligned}$$

Since $\sum_{k=1}^{\infty} \lambda_k = \infty$, $\sum_{k=1}^{\infty} b_k = \infty$. So, from (3.13), Lemmas 2.3, 3.2 and the properties of j with (3.10), it follows that $\lim_{k\to\infty} \|x^{k+1} - u_*\|^2 = 0$. This completes the proof. □

If we assume a weaker assumption on X, namely, X is reflexive and has a uniformly Gâteaux differentiable norm, then Theorem 3.3 remains valid provided that the space X is also a strongly convex Banach space.

Theorem 3.4. *Let X be a reflexive and strongly convex Banach space with a uniformly Gâteaux differentiable norm. Let A, F, and C be as in Proposition* 3.1*. Assume that the parameters α_k, λ_k are as in Theorem* 3.3*. Then, the sequence $\{x^k\}$, defined by* (1.12) *converges strongly to the unique solution u_* of the variational inequality* (1.1)*.*

Proof. Since a uniformly smooth real Banach space is reflexive and has a uniformly Gâteaux differentiable norm, the proof of Proposition 3.1 is still correct under the weaker conditions that X is a reflexive Banach space with a uniformly Gâteaux differentiable norm except for the property (3.4): F has a fixed point in C^*. This property holds true in a uniformly smooth Banach space. So to obtain Proposition 3.1 it remains to prove (3.4) in a reflexive strictly convex Banach space X. Since any closed convex subset in X is a Chebyshev set (see [20]), for any fixed point of F, there exists a unique $\tilde{y} \in C^*$ such that

$$\|y - \tilde{y}\| = \inf_{x\in C^*} \|y - x\|.$$

Since $y = Fy$, $T\tilde{y} \in C^*$ and F is nonexpansive,

$$\|y - F\tilde{y}\| = \|Fy - F\tilde{y}\| \le \|y - \tilde{y}\|.$$

Consequently, $F\tilde{y} = \tilde{y}$ and Theorem 3.3 is also valid in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. This completes the proof. □

Remark 3.5. Theorems 3.3 and Theorem 3.4 still hold true for the following iteration:

$$\begin{cases} y^1 \in X, \quad \text{any element}, \\ y^{k+1} = \big(\alpha_k I + (1-\alpha_k)B_k\big)(I - \lambda_k A)y^k, \quad k \geq 1, \end{cases} \tag{3.14}$$

with the same conditions on X, A, B_k, λ_k, and α_k. Indeed, putting $y^k = (\alpha_k I + (1-\alpha_k)B_k\big)x^k$ in (1.12) we obtain that

$$y^{k+1} = \big(\alpha_{k+1} I + (1-\alpha_{k+1})B_k\big)(I - \lambda_k A)y^k.$$

Re-denoting $\alpha_k := \alpha_{k+1}$, we obtain (3.14). Moreover, if $\lambda_k \to 0$ then $\{x^k\}$ is convergent if and only if $\{y^k\}$ does so and their limits coincide. Indeed, from (1.12), it follows that $\|x^{k+1} - y^k\| \leq \lambda_k \|Ay^k\|$. Therefore, when $\{x^k\}$ is convergent, $\{x^k\}$ is bounded, and hence $\{y^k\}$ is bounded. Consequently, $\{Ay^k\}$ is also bounded. Since $\lambda_k \to 0$ as $k \to \infty$, from the last inequality and the convegence of $\{x^k\}$ it follows that convergence of $\{y^k\}$ and their limits coincide. The case, when $\{y^k\}$ converges, is similar.

If $A = (1-a)I$ with a fixed number $a \in (0,1)$, we can write that $A = I - f$ with $f = aI$ and A is an η-strongly accretive and γ-strictly pseudocontractive mapping on X with some positive numbers η and γ such that $\eta + \gamma > 1$. Indeed,

$$\begin{aligned} \langle Ax - Ay, j(x-y)\rangle &= (1-a)\|x-y\|^2 \\ &= \|x-y\|^2 - \frac{1}{a}\|ax - ay\|^2 = \|x-y\|^2 - \frac{1}{a}\|fx - fy\|^2 \\ &= \|x-y\|^2 - \frac{1}{a}\|(I-A)x - (I-A)y\|^2 \\ &\leq \|x-y\|^2 - \gamma\|(I-A)x - (I-A)y\|^2, \end{aligned}$$

$\gamma \in [0,1)$ is a fixed number. Clearly, $\eta + \gamma > 1$ for $\eta = 1 - a$ and any fixed $\gamma \in (a,1)$. Now, replacing A by $(1-a)I$ in (1.12), we obtain the following iteration,

$$x^{k+1} = (1 - \lambda'_k)\big(\alpha_k I + (1-\alpha_k)B_k\big)x^k, \quad k \geq 1, \tag{3.15}$$

where $\lambda'_k = \lambda_k(1-a)$, and the following result.

Theorem 3.6. *Let $\{B_k\}$ be an infinite family of nonexpansive mappings on an either uniformly smooth or reflexive and strictly convex Banach space X with a uniformly Gâteaux differentiable norm such that $\cap_{k=1}^{\infty}\mathrm{Fix}(B_k) \neq \emptyset$, there holds*

(1.2) *for any bounded subset* D *of* X *and there exists* $By = \lim_{k\to\infty} B_k y$, *for* $y \in X$ *such that* $\mathrm{Fix}(B) = \cap_{k=1}^{\infty}\mathrm{Fix}(B_k)$. *Assume that* λ_k *and* α_k *satisfy conditions* (**C1**)-(**C3**). *With a fixed* $a \in (0,1)$, *the sequence* $\{x^k\}$, *generated by* (3.15) *converges strongly to a point in* $\cap_{k=1}^{\infty}\mathrm{Fix}(B_k)$.

Remark 3.7. Further, replacing y and A in (3.14) by x and $(1-a)I$, respectively, and re-denoting $\lambda_k = \lambda'_k$, we obtain algorithm (1.6). Therefore, this algorithm converges strongly in an either reflexive and strictly convex Banach space X with a uniformly Gâteaux differentiable norm or a uniformly smooth one. Meantimes, Shehu and Ugwunnadi [27] obtained this result only for uniformly smooth Banach spaces with a uniformly convex norm.

Remark 3.8. We note that the mappings S_k, $\tilde{S}_k$ and V_k defined, respectively, by (1.9), (1.11) and

$$V_k = F^1F^2\cdots F^k, \quad F^i = (1-s_i)I + s_iF_i,\ i \le k,$$

with s_i in (1.9) (see, [6]), satisfies condition (1.2).

Therefore, we obtain the following result.

Theorem 3.9. *Let* $\{F_i\}$ *be a nonexpansive mapping on a closed convex subset* C *of an either uniformly smooth Banach space* X *or reflexive and strictly convex one with a uniformly Gâteaux differentiable norm such that* $\cap_{i=1}^{\infty}\mathrm{Fix}(F_i) \neq \emptyset$. *Assume that* λ_k *and* α_k *satisfy conditions* (**C1**)-(**C3**). *Then, the sequence* $\{x^k\}$, *generated by*

$$\begin{cases} x^1 \in C, \quad \textit{any element}, \\ y^k = \alpha_k x^k + (1-\alpha_k)\tilde{V}_k x^k, \\ x^{k+1} = \lambda'_k u + \big(1-\lambda'_k\big)y^k,\ k \ge 1, \end{cases} \tag{3.16}$$

where $\tilde{V}_k$ *is one of* $\{W_k, S_k, \tilde{S}_k, V_k\}$ *defined above, converges strongly to a point in* $\cap_{i=1}^{\infty}\mathrm{Fix}(F_i)$.

References

[1] D.R. Sahu, D. O'Regan and R.P. Agarwal, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, 2009.

[2] K. Aoyama, H. Iiduka and W. Takahashi, *Weak convergence of an iterative sequence for accretive operators in Banach spaces*, Fixed Point Theory and Appl., **2006**:35390(2006). DOI: 10.1155/FPTA/2006/35390.

[3] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyota, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach spaces*, Nonlinear Anal.: TMA., **67**(8) (2007), 2350–2360.

[4] F.E. Browder and W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20**(2) (1967), 197–228.

[5] Ng. Buong, Ng.S. Ha and Ng.T.T. Thuy, *A new explicit iteration method for a class of variational inequalities*, Numerical Algorithm, **72**(2) (2016), 467–481.

[6] Ng. Buong and Ng.T.H. Phuong, *Strong convergence to solutions for a class of variational inequalities in Banach spaces by implicit iteration methods*, J. Optim. Theory Appl., **159**(2) (2013), 399–411.

[7] Ng. Buong and Ng.T.H. Phuong, *Regularization methods for a class of variational inequalities in Banach spaces*, Zh. Vychisl. Mat. Mat. Fiz., **52**(11) (2012), 1487–1496.

[8] Ng. Buong, Ng.T.H. Phuong and Ng.T.T Thuy, *Explicit iteration methods for a class of variational inequalities in Banach spaces*, Russian Math., **59**(10) (2015), 16–22.

[9] Ng. Buong, V.X. Quynh and Ng.T.T. Thuy, *A steepest-descent Krasnosel'skii–Mann algorithm for a class of variational inequalities in Banach spaces*, J. Fixed Point Theory Appl., **18**(3) (2016), 519–532.

[10] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl., **20**(1) (2004), 103–120.

[11] L.C. Ceng, Q.H. Ansari and J.C. Yao, *Mann-type steepest-descent and modified hybrid steepest descent methods for variational inequalities in Banach spaces*, Numer. Func. Anal. Optim., **29**(9-10) (2008), 987–1033.

[12] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Acad. Publ., Dordrecht (1990).

[13] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., **73** (1967), 957–961.

[14] H. Iiduka, *Fixed point optimization algorithm and its application to power control in CDMA data networks*, Math. Program., **133**(1) (2012), 227–242.

[15] H. Iiduka and I. Yamada, *An egordic algorithm for the power-control games for CDMA data networks*, J. Math. Model. Algor., **2009**, 8:1(2009). DOI:10.1007/s10852-008-9099-4.

[16] I. Iiduka, *Fixed point optimization algorithm and its application to network bandwidth allocation*, J. Comput. Appl. Math., **236**(7) (2012), 1733–1742.

[17] H. Iiduka, *Iterative algorithm for triple-hierarchical constrained nonconvex optimization problem and its application to network bandwidth allocation*, SIAM J. Optim., **22**(3) (2012), 862–878.

[18] H. Iiduka, *Fixed point optimization algorithms for distributed optimization in networked systems*, SIAM J. Optim., **23** (2013), 1–26.

[19] S. Iemoto and W. Takahashi, *Strong convergence theorems by a hybrid steepest descent method for countable nonexpansive mappings in Hilbert spaces*, Sci. Math. Jpn. Online, e-2008(2008), 557–570.

[20] L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, Nauka (in Russian) (1997).

[21] M. Kikkawa and W. Takahashi, *Strong convergence theorems by the viscosity approximation method for a countable family of nonexpansive mappings*, Taiwanese J. Math., **12**(3) (2008), 583–598.

[22] T.H. Kim and H.K. Xu, *Strong convergence of modified Mann iterations*, Nonlinear Anal., **61**(1-2) (2005), 51–60.

[23] M.A. Krasnoselskii, *Two remarks on the method of successive approximations*, Uspekhi Mat. Nauk, **10**(1) (1955), 123–127.

[24] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510.

[25] X. Qin, Y.J. Cho, J.I. Kang and S.M. Kang, *Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces*, J. Comput. Appl. Math., **230**(1) (2009), 121–127.

[26] Y. Shehu, *Modified Krasnosel'skii–Mann iterative algorithm for nonexpansive mappings in Banach spaces*, Arab J. Math., **2**(2) (2013), 209–219.

[27] Y. Shehu and G.C. Ugwunnadi, *Approximation of fixed points of nonexpansive mappings by modified Krasnosel'skii–Mann iterative algorithm*, Thai Journal of Math., **13**(2) (2015), 405–419.

[28] T. Suzuki, *A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., **135**(1) (2007), 99–106.

[29] W. Takahashi, *Weak and strong convergence theorems for families of nonexpansive mappings and their applications*, Ann. Univ. Mariae Curie–Sklodowska Sect. A., **51** (1997), 277–292.

[30] W. Takahashi and Y. Ueda, *On Reich's strong convergence theorem for resolvents of accretive operators*, J. Math. Anal. Appl., **104**(2) (1984), 546–553.

[31] Ng.T.T. Thuy, P.T. Hieu and J.J. Strodiot, *Regularization methods for accretive variational inequalities over the set of common fixed points of nonexpansive semigroups*, Optimization, **65**(8) (2016), 1553–1567.

[32] H.K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl., **116**(3) (2003), 659–678.

[33] Z. Xu and G.F. Roach, *A necessary and sufficient condition for convergence of steepest descent approximation to accretive operator equations*, J. Math. Anal. Appl., **167**(2) (1992), 340–354.

[34] I. Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, in Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (D.Butnariu, Y. Censor and S. Reich, Eds). North-Holland, Amsterdam (2001), 473–504.

[35] Y. Yao, M.A. Noor and Y.C. Liou, *A new hybrid iterative algorithm for variational inequalities*, Appl. Math. Comput., **216**(3) (2010), 822–829.

[36] Y. Yao, H. Zhou and Y.C. Liou, *Strong convergence of a modified Krasnosel'ski–Mann iterative algorithm for non-expansive mappings*, J. Appl. Math. Comput., **29**:383 (2009). DOI:10.1007/s12190-008-0139-z.

[37] H. Zhou, *A characteristic condition for convergence of steepest descent approximation to accretive operator equations*, J. Math. Anal. Appl., **271**(1) (2002), 1–6.

[38] M.M. Vainberg, *Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations*, Willey, New York (1974).

[39] S. Wang, *Convergence and weaker control conditions for hybrid iterative algorithms*, Fixed Point Theory and Appl., **2011**:3 (2011). DOI:10.1186/1687-1812-2011-3.

[40] S. Wang, L. Yu and B. Guo, *An implicit iterative scheme for an infinite countable family of asymptotically nonexpansive mappings in Banach spaces*, Fixed Point Theory and Appl., **2008**:350483(2008). DOI: 10.1155/2008/350483.