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$(H(\cdot,\cdot),\eta)$-MONOTONE OPERATOR WITH AN APPLICATION TO A SYSTEM OF SET-VALUED VARIATIONAL-LIKE INCLUSIONS IN BANACH SPACES

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Abstract. In this paper, we introduce and study a new class of $(H(.,.),\eta)$-monotone operator in Banach spaces. We define the generalized η-proximal operator associated with $(H(.,.),\eta)$-monotone operator and show its Lipschitz continuity. As an application, we consider the generalized system of variational-like inclusions involving $(H(.,.),\eta)$-monotone operators in Banach spaces and by using proximal operator technique, we construct an iterative algorithm for solving this system. Also, we prove the existence of solution and discuss the convergence analysis of the iterative algorithm for the generalized system of variational-like inclusions. The theorems presented in this paper improve and generalize many known results in the literature.

1. Introduction

Variational inclusions, as the generalization of variational inequalities have been widely studied by many authors in recent years. One of the most interesting and important problem in the theory of variational inequalities is the development of an efficient and implementable iterative algorithm. Various kinds of iterative algorithms have been studied to find solutions for variational

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inequalities and variational inclusions. Among these methods, the proximal operator technique for solving variational inclusions have been widely used by many authors. For further study on approximation solvability of variational inequalities and variational inclusions, see for example [1]-[16], [18]-[20], [22], [24]-[33] and the related references cited therein.

In the recent past many researchers see for example [4]-[8], [15,16], [18]-[20], [22,25], [28]-[33] and the related references cited therein have introduced the concepts of η-subdifferential operators, maximal η-monotone operators, H-monotone operators, (H, η)-monotone operators, A-monotone operators, (A, η)-monotone operators, G-η-monotone operators, M-monotone operators, M-η-monotone operators in Hilbert spaces; H-monotone operators, H-η- monotone operators and $(H(.,.), \eta)$-monotone operators in Banach spaces, respectively. By using the proximal operator technique, many systems of variational inequalities and variational inclusions have been studied by some authors, see for example Ding and Feng [4], Kazmi *et.al.*, [15], Peng and Zhu [22], Zeng [31] and the related references cited therein.

Motivated and inspired by the research works mentioned above, in this paper, we introduce and study a new class of $(H(.,.), \eta)$-monotone operators in Banach spaces which provides a unifying framework for maximal monotone operators, η-subdifferential operators, maximal η-monotone operators, H-monotone operators, (H, η)-monotone operators, A-monotone operators, G-η-monotone operators, M-monotone operators, H-η-monotone operators and $(H(.,.), \eta)$-monotone operators. We also define a generalized η-proximal operator associated with $(H(.,.), \eta)$-monotone operator and show its Lipschitz continuity. As an application, we consider the solvability of a generalized system of variational-like inclusions involving $(H(.,.), \eta)$-monotone operators in Banach spaces. By using the technique of proximal operator, we construct an iterative algorithm for solving such generalized system of variational-like inclusions. Under some suitable conditions, we prove the convergence of iterative sequences generated by the algorithm. The results presented in this paper improve and extend many known results in the literature.

2. Preliminaries and basic results

Let X be a real Banach space with dual space $X^\star$, $\langle \cdot, \cdot \rangle$ be the dual pair between X and $X^\star$, 2^X denote the family of all the nonempty subsets of X. The normalized duality mapping $J : X \longrightarrow 2^{X^\star}$ is defined by

$$J(u) = \{f \in X^\star : \langle f, u \rangle = \|f\|\|u\|, \ \|f\| = \|u\|\}, \ \ \forall u \in X.$$

If $X \equiv \mathcal{H}$, a Hilbert space, then J is an identity mapping.

Let $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X; $\mathcal{D}(\cdot,\cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$\mathcal{D}(A,B)=\max\left\{\sup_{u\in A}\ d(u,B),\sup_{v\in B}\ d(A,v)\right\},\ \ A,B\in CB(X).$$

The following concepts and results are needed in the sequel:

Definition 2.1. ([21]) Let X be a complete metric space, $T:X\to CB(X)$ be a set-valued mapping. Then for any $\epsilon>0$ and for any $u,v\in X$, $x\in T(u)$, there exists $y\in T(v)$ such that

$$d(x,y)\le(1+\epsilon)\mathcal{D}(T(u),T(v)),$$

where $\mathcal{D}$ is the Hausdorff metric on $CB(X)$.

Definition 2.2. Let $A,B:X\to X$, $T:X\to X^\star$, $H:X\times X\to X^\star$ and $\eta:X\times X\to X$ be single-valued mappings. Then $\forall u,v,\cdot\in X$,

(i) T is monotone if

$$\left\langle Tu-Tv,u-v\right\rangle\ge 0.$$

(ii) T is strictly monotone if

$$\left\langle Tu-Tv,u-v\right\rangle\ge 0,$$

and equality holds if and only if $u=v$.

(iii) T is α-strongly monotone if there exists a constant $\alpha>0$ such that

$$\left\langle Tu-Tv,u-v\right\rangle\ge\alpha\|u-v\|^2.$$

(iv) T is γ-Lipschitz continuous if there exists a constant $\gamma>0$ such that

$$\|Tu-Tv\|\ \le\ \gamma\|u-v\|.$$

(v) T is η-monotone if

$$\left\langle Tu-Tv,\eta(u,v)\right\rangle\ge 0.$$

(vi) T is strictly η-monotone if

$$\left\langle Tu-Tv,\eta(u,v)\right\rangle\ge 0,$$

and equality holds if and only if $u=v$.

(vii) A is said to be δ-strongly accretive if there exists a constant $\delta>0$ and $j(u-v)\in J(u-v)$ such that

$$\left\langle Au-Av,j(u-v)\right\rangle\ge\delta\|u-v\|^2,$$

where J is the normalized duality mapping.

(viii) $H(A,\cdot)$ is α_1-strongly η-monotone with respect to A if there exists a constant $\alpha_1 > 0$ such that

$$\Big\langle H(Au,\cdot) - H(Av,\cdot), \eta(u,v)\Big\rangle \geq \alpha_1\|u-v\|^2.$$

(ix) $H(\cdot,B)$ is α_2-relaxed η-monotone with respect to B if there exists a constant $\alpha_2 > 0$ such that

$$\Big\langle H(\cdot,Bu) - H(\cdot,Bv), \eta(u,v)\Big\rangle \geq -\alpha_2\|u-v\|^2.$$

(x) $H(\cdot,\cdot)$ is λ-Lipschitz continuous with respect to A if there exists a constant $\lambda > 0$ such that

$$\|H(Au,\cdot) - H(Av,\cdot)\| \ \leq \ \lambda\|u-v\|.$$

(xi) η is τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(u,v)\| \ \leq \ \tau\|u-v\|.$$

Remark 2.3. If X is a Hilbert space, $\eta(u,v) = u - v$, $\forall u,v \in X$, then (viii) and (ix) of Definition 2.2 reduce to (i) and (ii) of Definition 1.2, respectively in [24].

Definition 2.4. Let $M : X \longrightarrow 2^{X^\star}$ be a multi-valued mapping, $H : X \to X^\star$ and $\eta : X \times X \to X$ be single-valued mappings. Then

(i) M is monotone if

$$\langle x-y, u-v\rangle \geq 0,\ \forall\ u,v \in X,\ x \in M(u), y \in M(v).$$

(ii) M is η-monotone if

$$\langle x-y, \eta(u,v)\rangle \geq 0,\ \forall\ u,v \in X,\ x \in M(u), y \in M(v).$$

(iii) M is strictly η-monotone if

$$\langle x-y, \eta(u,v)\rangle \geq 0,\ \forall\ u,v \in X,\ x \in M(u), y \in M(v),$$

and equality holds if and only if $u = v$.

(iv) M is λ-strongly η-monotone if there exists a constant $\lambda > 0$, such that

$$\langle x-y, \eta(u,v)\rangle \geq \lambda\|u-v\|^2,\ \forall\ u,v \in X,\ x \in M(u), y \in M(v).$$

(v) M is m-relaxed η-monotone if there exists a constant $m > 0$, such that

$$\langle x-y, \eta(u,v)\rangle \geq -m\|u-v\|^2,\ \forall\ u,v \in X,\ x \in M(u), y \in M(v).$$

(vi) M is maximal monotone if M is monotone and

$$(J + \lambda M)(X) = X^\star,\ \forall\ \lambda > 0,$$

where J is the normalized duality mapping.

(vii) M is maximal η-monotone if M is η-monotone and

$$(J+\lambda M)(X)=X^{\star},\ \forall\ \lambda>0.$$

(viii) M is H-monotone if M is monotone and

$$(H+\lambda M)(X)=X^{\star},\ \forall\ \lambda>0.$$

(ix) M is (H,η)-monotone if M is η-monotone and

$$(H+\lambda M)(X)=X^{\star},\ \forall\ \lambda>0.$$

(x) M is H-η-monotone if M is m-relaxed η-monotone and

$$(H+\lambda M)(X)=X^{\star},\ \forall\ \lambda>0.$$

Definition 2.5. For all $u,v,\cdot\in X$, a mapping $N:X\times X\longrightarrow X^{\star}$ is said to be

(i) β_1-Lipschitz continuous with respect to first argument, if there exists a constant $\beta_1>0$ such that

$$\|N(u,\cdot)-N(v,\cdot)\|\leq\beta_1\|u-v\|.$$

(ii) β_2-Lipschitz continuous with respect to second argument, if there exists a constant $\beta_2>0$ such that

$$\|N(\cdot\ ,u)-N(\cdot\ ,v)\|\leq\beta_2\|u-v\|.$$

Definition 2.6. For all $u,v,\cdot\in X$, a mapping $F:X\times X\times X\longrightarrow X^{\star}$ is said to be

(i) ϵ_1-Lipschitz continuous with respect to first argument, if there exists a constant $\epsilon_1>0$ such that

$$\|F(u,\cdot\ ,\cdot)-F(v,\cdot\ ,\cdot)\|\leq\epsilon_1\|u-v\|.$$

(ii) ϵ_2-Lipschitz continuous with respect to second argument, if there exists a constant $\epsilon_2>0$ such that

$$\|F(\cdot\ ,u,\cdot)-F(\cdot\ ,v,\cdot)\|\leq\epsilon_2\|u-v\|.$$

(iii) ϵ_3-Lipschitz continuous with respect to third argument, if there exists a constant $\epsilon_3>0$ such that

$$\|F(\cdot\ ,\cdot\ ,u)-F(\cdot\ ,\cdot\ ,v)\|\leq\epsilon_3\|u-v\|.$$

Lemma 2.7. ([23]) *Let X be a real Banach space and $J:X\rightarrow 2^{X^{\star}}$ be the normalized duality mapping. Then, for all $u,v\in X$,*

$$\|u+v\|^2\leq\|u\|^2+2\Big\langle v,j(u+v)\Big\rangle,\quad\forall j(u+v)\in J(u+v).$$

3. $(H(.,.),\eta)$-MONOTONE OPERATOR AND FORMULATION OF THE PROBLEM

Definition 3.1. Let X be a Banach space with the dual space $X^\star$. Let $H : X\times X \to X^\star$, $\eta : X\times X \to X$, $A, B : X \to X$ be single-valued mappings. Then the set-valued mapping $M : X \to 2^{X^\star}$ is said to be $(H(.,.),\eta)$-monotone with respect to A and B if M is m-relaxed-η-monotone and $(H(A,B)+\rho M)(X) = X^\star$ for all $\rho > 0$.

Remark 3.2. (i) If $H(Au, Bu) = Au$, $\forall u \in X$, then Definition 3.1 reduces to the definition of H-η-monotone operators considered in [19]. It follows that this class of operators in Banach spaces provides a unifying framework for the class of η-subdifferential operators, maximal monotone operators, maximal η-monotone operators, H-monotone operators, (H,η)-monotone operators, G-η-monotone operators, A-monotone operators, A-η-monotone operators in Hilbert spaces and H-η-monotone operators, H-monotone operators, A-monotone operators in Banach spaces . We remark that $(H(.,.),\eta)$-monotone operator in Banach spaces acts from X to $X^\star$.
(ii) If $X \equiv \mathcal{H}$, a Hilbert space, $m = 0$ and $\eta(u,v) = u - v$, $\forall u, v \in \mathcal{H}$, then Definition 3.1 reduces to M-monotone operator studied in [24].

Now we give some properties of $(H(.,.),\eta)$-monotone operator.

Theorem 3.3. *Let $A, B : X \to X$, $\eta : X \times X \to X$ and $H : X \times X \to X^\star$ be single-valued mappings and $H(A,B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : X \to 2^{X^\star}$ be $(H(.,.),\eta)$-monotone operator with respect to A and B. If*

$$\Big\langle x - y, \eta(u,v)\Big\rangle \geq 0, \quad \forall\,(v,y) \in Graph\ (M),$$

then $(u,x) \in Graph(M)$, where $Graph\ (M) = \Big\{(a,b) \in X \times X : b \in M(a)\Big\}$.

Proof. Since M is $(H(.,.),\eta)$-monotone operator with respect to A and B and $(H(A,B) + \rho M)(X) = X^\star$ holds for all $\rho > 0$. Thus there exists $(u_1, x_1) \in Graph(M)$ such that

$$H(Au,Bu) + \rho x = H(Au_1,Bu_1) + \rho x_1.$$

Since $H(A,B)$ is α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$, we have

$$\begin{aligned} 0 \leq \rho\langle x - x_1, \eta(u,u_1)\rangle &\leq -\langle H(Au,Bu) - H(Au_1,Bu_1), \eta(u,u_1)\rangle \\ &= -\langle H(Au,Bu) - H(Au_1,Bu), \eta(u,u_1)\rangle \end{aligned}$$

$$
\begin{aligned}
&- \langle H(Au_1, Bu) - H(Au_1, Bu_1), \eta(u, u_1)\rangle \\
&\leq -(\alpha-\beta) \parallel u - u_1 \parallel^2 \leq 0.
\end{aligned}
$$

This implies that $u = u_1$ and $x = x_1$. Thus $(u, x) = (u_1, x_1) \in Graph(M)$. This completes the proof. $\square$

Theorem 3.4. *Let $A, B : X \to X$, $\eta : X \times X \to X$ and $H : X \times X \to X^\star$ be single-valued mappings and $H(A, B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : X \to 2^{X^\star}$ be $(H(.,.), \eta)$-monotone operator with respect to A and B. Then $(H(A, B) + \rho M)^{-1}$ is a single-valued mapping for $0 < \rho < \dfrac{\alpha-\beta}{m}$.*

Proof. For any given $u^\star \in X^\star$, let $u, v \in (H(A, B) + \rho M)^{-1}(u^\star)$. It follows that

$$
-H(Au, Bu) + u^\star \in \rho M(u) \quad \text{and} \quad -H(Av, Bv) + u^\star \in \rho M(v).
$$

Since $M : X \to 2^{X^\star}$ be $(H(.,.), \eta)$-monotone operator with respect to A and B and $H(A, B)$ is α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$, we have

$$
\begin{aligned}
-m \parallel u - v \parallel^2 &\leq \frac{1}{\rho}\Big\langle \Big(-H(Au, Bu) + u^\star\Big) - \Big(-H(Av, Bv) + u^\star\Big), \eta(u, v)\Big\rangle \\
&= -\frac{1}{\rho}\Big\langle H(Au, Bu) - H(Av, Bv), \eta(u, v)\Big\rangle \\
&= -\frac{1}{\rho}\Big\langle H(Au, Bu) - H(Av, Bu), \eta(u, v)\Big\rangle \\
&\quad - \frac{1}{\rho}\Big\langle H(Av, Bu) - H(Av, Bv), \eta(u, v)\Big\rangle \\
&\leq -\frac{1}{\rho}(\alpha-\beta) \parallel u - v \parallel^2 .
\end{aligned}
$$

This shows that

$$
m\rho \parallel u - v \parallel^2 \geq (\alpha-\beta) \parallel u - v \parallel^2 .
$$

If $u \neq v$, then $\rho \geq \dfrac{\alpha-\beta}{m}$ contradicts with $0 < \rho < \dfrac{\alpha-\beta}{m}$. Thus, $u = v$, that is $(H(A, B) + \rho M)^{-1}$ is a single-valued mapping. This completes the proof. $\square$

Based on Theorem 3.4, we define the generalized η-proximal operator associated with $(H(A, B), \eta)$-monotone operator as under:

Definition 3.5. Let $A, B : X \to X$, $\eta : X \times X \to X$ and $H : X \times X \to X^\star$ be single-valued mappings and $H(.,.)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : X \times X \to$

$2^{X^\star}$ be $(H(.,.),\eta)$-monotone operator with respect to A and B. Then the generalized η-proximal operator $J^{H(.,.),\eta}_{M(\cdot,z'),\rho}: X \to X$ for fixed $z' \in X$ is defined by

$$J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(u) = \Big(H(A,B) + \rho M(\cdot,z')\Big)^{-1}(u),\ \forall u \in X.$$

Remark 3.6. The generalized η-proximal operator associated with $(H(.,.),\eta)$-monotone operator include as special cases the corresponding proximal operators associated with maximal monotone operators, η-subdifferential operators, maximal η-monotone operators, H-monotone operators, (H,η)-monotone operators, G-η-monotone operators, A-monotone operators, A-η-monotone operators.

One of the important properties of generalized η-proximal operator is its Lipschitz continuity which we prove as under:

Theorem 3.7. *Let $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping. Let $A, B : X \to X$ and $H : X \times X \to X^\star$ be single-valued mappings and $H(A,B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : X \times X \to 2^{X^\star}$ be $(H(.,.),\eta)$-monotone operator with respect to A and B. Then the generalized η-proximal operator $J^{H(.,.),\eta}_{M(\cdot,z'),\rho} : X \to X$ for fixed $z' \in X$ is k-Lipschitz continuous, where $k = \dfrac{\tau}{\alpha-\beta-m\rho}$, that is*

$$\|J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(u) - J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(v)\| \le k\|u-v\|,\ \forall u, v \in X.$$

Proof. Let $u, v \in X$. It follows that

$$\begin{cases} J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(u) &=& \Big(H(A,B) + \rho M(\cdot,z')\Big)^{-1}(u); \\ J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(v) &=& \Big(H(A,B) + \rho M(\cdot,z')\Big)^{-1}(v) \end{cases}$$

and hence

$$\begin{cases} \dfrac{1}{\rho}\Big\{u - H\Big(A(J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(u)), B(J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(u))\Big)\Big\} \in M\Big(J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(u), z'\Big); \\ \dfrac{1}{\rho}\Big\{v - H\Big(A(J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(v)), B(J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(v))\Big)\Big\} \in M\Big(J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(v), z'\Big). \end{cases}$$

For the sake of brevity, let $b_1 = J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(u)$ and $b_2 = J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(v)$. Since M is m-relaxed η-monotone, we have

$$\begin{aligned} &- m \parallel b_1 - b_2 \parallel^2 \\ &\le \frac{1}{\rho}\Big\langle \Big(u - H\Big(A(b_1), B(b_1)\Big)\Big) - \Big(v - H\Big(A(b_2), B(b_2)\Big)\Big), \eta(b_1,b_2)\Big\rangle \end{aligned}$$

$$= \frac{1}{\rho}\Big\langle u - v - \Big[H\Big(A(b_1), B(b_1)\Big) - H\Big(A(b_2), B(b_2)\Big)\Big], \eta(b_1, b_2)\Big\rangle.$$

By the given hypothesis and above inequality, we have

$$\begin{aligned}
&\tau \parallel u - v \parallel\parallel b_1 - b_2 \parallel \\
&\geq\parallel u - v \parallel\parallel \eta(b_1, b_2) \parallel\geq \langle u - v, \eta(b_1, b_2)\rangle \\
&\geq \Big\langle H\Big(A(b_1), B(b_1)\Big) - H\Big(A(b_2), B(b_2)\Big), \eta(b_1, b_2)\Big\rangle - m\rho \parallel b_1 - b_2 \parallel^2 \\
&\geq \Big\langle H\Big(A(b_1), B(b_1)\Big) - H\Big(A(b_2), B(b_1)\Big), \eta(b_1, b_2)\Big\rangle \\
&\quad + \Big\langle H\Big(A(b_2), B(b_1)\Big) - H\Big(A(b_2), B(b_2)\Big), \eta(b_1, b_2)\Big\rangle - m\rho \parallel b_1 - b_2 \parallel^2 \\
&\geq (\alpha - \beta - m\rho) \parallel b_1 - b_2 \parallel^2 .
\end{aligned}$$

Hence

$$\|J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(u) - J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(v)\| \leq \frac{\tau}{\alpha - \beta - m\rho}\|u - v\|, \quad \forall\, u, v \in X$$

or

$$\|J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(u) - J^{H(.,.),\eta}_{M(\cdot,z'),\rho}(v)\| \leq k\|u - v\|, \;\; \forall\, u, v \in X.$$

□

Now we formulate our main problem:

Let X be a real Banach space. Let $S, T, G : X \to CB(X)$ be three set-valued mappings and $N, H : X \times X \to X^\star$, $\eta : X \times X \to X$, $F : X \times X \times X \to X^\star$, $A, B, p, g : X \to X$ be single-valued mappings. Let $M : X \times X \to 2^{X^\star}$ be set-valued mapping such that for fixed $z', z \in G(X)$, $M(., z')$, $M(., z) : X \times X \to 2^{X^\star}$ is an $(H(., .), \eta)$-monotone operator with respect to A and B and $\mathrm{Range}(g - p) \cap \mathrm{dom}(M(., z')) \neq \emptyset$, $\mathrm{Range}(g - p) \cap \mathrm{dom}(M(., z)) \neq \emptyset$. For any given $f \in X^\star$, we consider the following generalized system of variational-like inclusion problem (in short, GSVLIP):

Find $u, v \in X$, $x \in S(u)$, $y \in T(u)$, $z \in G(u)$, $x' \in S(v)$, $y' \in T(v)$, $z' \in G(v)$ such that

$$\begin{cases}
\theta^\star \in H\Big(A((g-p)(u)), B((g-p)(u))\Big) - H\Big(A((g-p)(v)), B((g-p)(v))\Big) \\
\qquad + \rho\Big\{N(x', y') + M\Big((g-p)(u), z'\Big) + F(v, v, z') + f\Big\} \\
\theta^\star \in H\Big(A((g-p)(v)), B((g-p)(v))\Big) - H\Big(A((g-p)(u)), B((g-p)(u))\Big) \\
\qquad + \gamma\Big\{N(x, y) + M\Big((g-p)(v), z\Big) + F(u, u, z) + f\Big\},
\end{cases} \tag{3.1}$$

where $\theta^\star$ is the zero element in $X^\star$.

We remark that if $\rho = \gamma = 1$, $u = v$, $x' = x$, $y' = y$, $z' = z$ and $f \equiv 0$, then GSVLIP (3.1) reduces to a variational inclusion of finding $u \in X$, $x \in S(u), y \in T(u), z \in G(u)$ such that

$$\theta^\star \in F(u,u,z) + N(x,y) + M\Big((g-p)(u), z\Big). \tag{3.2}$$

Variational inclusion (3.2) is an important generalization of variational inclusions considered by many researchers including [9,12,14,27,30]. For the applications of such variational inclusions, see [17].

Some More Special Cases:

If $\rho = \gamma = 1$, $u = v$, $x' = x$, $y' = y$, $z' = z$, if $F = p = f = 0$ and $X \equiv \mathcal{H}$, a Hilbert space, then GSVLIP (3.1) reduces to a generalized mixed quasi-variational-like inclusion with $(H(\cdot,\cdot),\eta)$-monotone operators in a Hilbert space:

Find $u \in X$, $x \in S(u), y \in T(u), z \in G(u)$ such that

$$\theta^\star \in N(x,y) + M(g(u), z). \tag{3.3}$$

If M is H-monotone in the first argument, then the problem (3.3) was introduced and studied by Zeng [31].

If $(g-p) \equiv I$, $F \equiv 0$, $f \equiv 0$, GSVLIP (3.1) reduces to a variational inclusion of finding $u, v \in X$, $x \in S(u), y \in T(u), z \in G(u), x' \in S(v), y' \in T(v), z' \in G(v)$ such that

$$\begin{cases} \theta^\star \in H\Big(A(u), B(u)\Big) - H\Big(A(v), B(v)\Big) + \rho\Big\{N(x', y') + M(u, z')\Big\} \\ \theta^\star \in H\Big(A(v), B(v)\Big) - H\Big(A(u), B(u)\Big) + \gamma\Big\{N(x, y) + M(v, z)\Big\}. \end{cases} \tag{3.4}$$

Variational inclusion (3.4) is an important generalization of variational inclusion considered by Kazmi and Bhat [12]. For applications of such variational inclusions, see [15,29].

We remark that for the suitable choices of mappings A, B, S, T, G, N, H, F, M, η, g, p and the underlying space X, GSVLIP (3.1) reduces to different classes of new and already known systems of variational inclusions/inequalities considered by many researchers including [20,25,27,33] and the related references cited therein.

4. Existence of solution, iterative algorithm and convergence analysis

First, we give the following important result:

Theorem 4.1. *Let X, A, B, S, T, G, N, H, F, M, η, g, p be same as in the GSVLIP* (3.1). *Then (u,v,x,y,z,x',y',z'), where $x\in S(u), y\in T(u), z\in G(u)$, $x'\in S(v), y'\in T(v), z'\in G(v)$ is the solution of GSVLIP* (3.1) *if and only if*

$$\begin{aligned}&(g-p)(u)\\&=J^{H(.,.),\eta}_{M(.,z'),\rho}\Big[H\Big(A((g-p)(v)),B((g-p)(v))\Big)-\rho\Big\{N(x',y')+F(v,v,z')+f\Big\}\Big],\end{aligned}$$

where

$$\begin{aligned}&(g-p)(v)\\&=J^{H(.,.),\eta}_{M(.,z),\gamma}\Big[H\Big(A((g-p)(u)),B((g-p)(u))\Big)-\gamma\Big\{N(x,y)+F(u,u,z)+f\Big\}\Big],\end{aligned}$$

$$J^{H(.,.),\eta}_{M(.,z'),\rho}(u)=\Big(H(A,B)+\rho M(.,z')\Big)^{-1}(u),$$

and

$$J^{H(.,.),\eta}_{M(.,z),\gamma}(v)=\Big(H(A,B)+\gamma M(.,z)\Big)^{-1}(v)$$

are generalized η-proximal operators and $\rho>0,\gamma>0$ are constants.

Proof. From the definition of $J^{H(.,.),\eta}_{M(.,z'),\rho}$ and $J^{H(.,.),\eta}_{M(.,z),\gamma}$, we have for

$$\begin{aligned}&(g-p)(u)\\&=J^{H(.,.),\eta}_{M(.,z'),\rho}\Big[H\Big(A((g-p)(v)),B((g-p)(v))\Big)-\rho\Big\{N(x',y')+F(v,v,z')+f\Big\}\Big]\\ \Longleftrightarrow\ &H\Big(A((g-p)(v)),B((g-p)(v))\Big)-\rho\Big\{N(x',y')+F(v,v,z')+f\Big\}\\&\in\Big(H(A,B)+\rho M(.,z')\Big)(g-p)(u)\\ \Longleftrightarrow\ &\theta^\star\in H\Big(A((g-p)(u)),B((g-p)(u))\Big)-H\Big(A((g-p)(v)),B((g-p)(v))\Big)\\&\quad+\rho\Big\{N(x',y')+M\Big((g-p)(u),z'\Big)+F(v,v,z')+f\Big\}.\end{aligned}$$

Similarly, we have the other inclusion, that is,

$$\begin{aligned}\theta^\star\in\ &H\Big(A((g-p)(v)),B((g-p)(v))\Big)-H\Big(A((g-p)(u)),B((g-p)(u))\Big)\\&+\gamma\Big\{N(x,y)+M\Big((g-p)(v),z\Big)+F(u,u,z)+f\Big\}.\end{aligned}$$

This proves the theorem. □

The above result along with Nadler's Theorem [21] allow us to suggest the following iterative algorithm for solving GSVLIP (3.1).

Iterative Algorithm 4.2. *For any arbitrary chosen* $u_0, v_0 \in X$, $x_0 \in S(u_0)$, $y_0 \in T(u_0)$, $z_0 \in G(u_0)$, $x'_0 \in S(v_0)$, $y'_0 \in T(v_0)$ *and* $z'_0 \in G(v_0)$, *compute the sequences* $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{x'_n\}$, $\{y'_n\}$ *and* $\{z'_n\}$ *by the iterative schemes such that*

$$(g-p)(u_{n+1}) = J^{H(.,.),\eta}_{M(.,z'_n),\rho}\Big[H\Big(A((g-p)(v_n)), B((g-p)(v_n))\Big) - \rho\Big\{N(x'_n, y'_n) + F(v_n, v_n, z'_n) + f\Big\}\Big],$$

where

$$(g-p)(v_n) = J^{H(.,.),\eta}_{M(.,z_n),\gamma}\Big[H\Big(A((g-p)(u_n)), B((g-p)(u_n))\Big) - \gamma\Big\{N(x_n, y_n) + F(u_n, u_n, z_n) + f\Big\}\Big].$$

$$\begin{aligned}
x_n \in S(u_n): &\quad \|x_{n+1} - x_n\| \le \Big(1 + (1+n)^{-1}\Big)\mathcal{D}\Big(S(u_{n+1}), S(u_n)\Big);\\
y_n \in T(u_n): &\quad \|y_{n+1} - y_n\| \le \Big(1 + (1+n)^{-1}\Big)\mathcal{D}\Big(T(u_{n+1}), T(u_n)\Big);\\
z_n \in G(u_n): &\quad \|z_{n+1} - z_n\| \le \Big(1 + (1+n)^{-1}\Big)\mathcal{D}\Big(G(u_{n+1}), G(u_n)\Big);\\
x'_n \in S(v_n): &\quad \|x'_{n+1} - x'_n\| \le \Big(1 + (1+n)^{-1}\Big)\mathcal{D}\Big(S(v_{n+1}), S(v_n)\Big);\\
y'_n \in T(v_n): &\quad \|y'_{n+1} - y'_n\| \le \Big(1 + (1+n)^{-1}\Big)\mathcal{D}\Big(T(v_{n+1}), T(v_n)\Big);\\
z'_n \in G(v_n): &\quad \|z'_{n+1} - z'_n\| \le \Big(1 + (1+n)^{-1}\Big)\mathcal{D}\Big(G(v_{n+1}), G(v_n)\Big).
\end{aligned}$$

Now, we prove the following theorem which ensures the convergence of iterative sequences generated by the Iterative Algorithm 4.2.

Theorem 4.3. *Let* X *be a real Banach space. Let* $S, T, G : X \to CB(X)$ *be* $\alpha_1, \alpha_2, \alpha_3$-$\mathcal{D}$*-Lipschitz continuous mappings, respectively. Let* $N : X \times X \to X^\star$ *be* l_1 *and* l_2*-Lipschitz continuous with respect to first and second arguments, respectively,* $\eta : X \times X \to X$ *be* τ*-Lipschitz continuous,* $F : X \times X \times X \to X^\star$ *be* β_j*-Lipschitz continuous with respect to* j*th argument for* $j = 1, 2, 3$ *and* $p, g : X \to X$ *be single-valued mappings such that* $(g-p)$ *is* s*-Lipschitz continuous and* $(g-p-I)$ *is* λ*-strongly accretive. Let* $H : X \times X \to X^\star$ *be* α*-strongly* η*-monotone with respect to* A*,* β*-relaxed* η*-monotone with respect to* B *and* h_1 *and* h_2*-Lipschitz continuous with respect to* A *and* B*, respectively. Let* $M : X \times X \to 2^{X^\star}$ *be set-valued mapping such that for fixed* $z', z \in G(X)$*,* $M(., z'), M(., z) : X \times X \to 2^{X^\star}$ *is an* $(H(.,.), \eta)$*-monotone operator with respect to* A *and* B *and* $Range(g-p) \cap dom(M(., z')) \neq \emptyset$*,* $Range(g-p) \cap dom(M(., z)) \neq \emptyset$*. In addition, suppose there exist constants* $\mu_1, \mu_2 > 0$ *such that*

$$\Big\| J^{H(.,.),\eta}_{M(.,z'_{n+1}),\rho}(u) - J^{H(.,.),\eta}_{M(.,z'_n),\rho}(u) \Big\| \le \mu_1 \parallel z'_{n+1} - z'_n \parallel,$$

$$\left\| J^{H(.,.),\eta}_{M(.,z_{n+1}),\gamma}(u) - J^{H(.,.),\eta}_{M(.,z_n),\gamma}(u) \right\| \le \mu_2 \parallel z_{n+1} - z_n \parallel . \tag{4.1}$$

Furthermore, suppose the following condition is satisfied

$$0 < Q < 1,$$

where Q is given by,

$$\begin{cases} Q \\ = \dfrac{1}{1+2\lambda}\Bigg[k\Bigg\{\sqrt{s^2h_1^2 - 2\rho l_1\alpha_1\Big(sh_1+\rho l_1\alpha_1\Big)} + \sqrt{s^2h_2^2 - 2\rho l_2\alpha_2\Big(sh_2+\rho l_2\alpha_2\Big)} \\ \quad + \Big(\beta_1+\beta_2+\beta_3\alpha_3\Big)\Bigg\} + \mu_1\alpha_3\Bigg] \\ \quad \times \Bigg[k'\Bigg\{\sqrt{s^2h_2^2 - 2\gamma l_2\alpha_2\Big(sh_2+\gamma l_2\alpha_2\Big)} + \sqrt{s^2h_1^2 - 2\gamma l_1\alpha_1\Big(sh_1+\gamma l_1\alpha_1\Big)} \\ \quad + \Big(\beta_1+\beta_2+\beta_3\alpha_3\Big)\Bigg\} + \mu_2\alpha_3\Bigg], \text{ here } k' = \dfrac{\tau}{\alpha-\beta-m\gamma}, \end{cases} \tag{4.2}$$

then the sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{x'_n\}$, $\{y'_n\}$ and $\{z'_n\}$ generated by the Iterative Algorithm 4.2 *converge strongly to the unique solution $(u, v, x, y, z, x', y', z')$, respectively, where $u, v \in X$, $x \in S(u)$, $y \in T(u)$, $z \in G(u)$, $x' \in S(v)$, $y' \in T(v)$ and $z' \in G(v)$ is the solution of GSVLIP* (3.1).

Proof. From Iterative Algorithm 4.2 and Theorem 3.7, we have

$$\begin{aligned}
&\|(g-p)u_{n+2} - (g-p)u_{n+1}\| \\
&= \Big\| J^{H(.,.),\eta}_{M(.,z'_{n+1}),\rho}\Big[H\Big(A((g-p)(v_{n+1})), B((g-p)(v_{n+1}))\Big) \\
&\quad - \rho\Big\{N(x'_{n+1}, y'_{n+1}) + F(v_{n+1}, v_{n+1}, z'_{n+1}) + f\Big\}\Big] \\
&\quad - J^{H(.,.),\eta}_{M(.,z'_n),\rho}\Big[H\Big(A((g-p)(v_n)), B((g-p)(v_n))\Big) \\
&\quad - \rho\Big\{N(x'_n, y'_n) + F(v_n, v_n, z'_n) + f\Big\}\Big]\Big\| \\
&\le \Big\| J^{H(.,.),\eta}_{M(.,z'_{n+1}),\rho}\Big[H\Big(A((g-p)(v_{n+1})), B((g-p)(v_{n+1}))\Big) \\
&\quad - \rho\Big\{N(x'_{n+1}, y'_{n+1}) + F(v_{n+1}, v_{n+1}, z'_{n+1}) + f\Big\}\Big] \\
&\quad - J^{H(.,.),\eta}_{M(.,z'_{n+1}),\rho}\Big[H\Big(A((g-p)(v_n)), B((g-p)(v_n))\Big) \\
&\quad - \rho\Big\{N(x'_n, y'_n) + F(v_n, v_n, z'_n) + f\Big\}\Big]\Big\| \\
&\quad + \Big\| J^{H(.,.),\eta}_{M(.,z'_{n+1}),\rho}\Big[H\Big(A((g-p)(v_n)), B((g-p)(v_n))\Big) \\
&\quad - \rho\Big\{N(x'_n, y'_n) + F(v_n, v_n, z'_n) + f\Big\}\Big] \\
&\quad - J^{H(.,.),\eta}_{M(.,z'_n),\rho}\Big[H\Big(A((g-p)(v_n)), B((g-p)(v_n))\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \rho\Big\{N(x'_n, y'_n) + F(v_n, v_n, z'_n) + f\Big\}\Big]\Big\| \\
&\le k\Big\|\Big[H\Big(A((g-p)(v_{n+1})), B((g-p)(v_{n+1}))\Big) \\
&- \rho\Big\{N(x'_{n+1}, y'_{n+1}) + F(v_{n+1}, v_{n+1}, z'_{n+1})\Big\}\Big] \\
&- \Big[H\Big(A((g-p)(v_n)), B((g-p)(v_n))\Big) \\
&- \rho\Big\{N(x'_n, y'_n) + F(v_n, v_n, z'_n)\Big\}\Big]\Big\| + \mu_1\Big\|z'_{n+1} - z'_n\Big\| \\
&\le k\Big[\Big\|H\Big(A((g-p)(v_{n+1})), B((g-p)(v_{n+1}))\Big) \\
&- H\Big(A((g-p)(v_n)), B((g-p)(v_{n+1}))\Big) - \rho\Big\{N(x'_{n+1}, y'_{n+1}) - N(x'_n, y'_{n+1})\Big\}\Big\| \\
&+ \Big\|H(A\Big((g-p)(v_n)), B((g-p)(v_{n+1}))\Big) - H\Big(A((g-p)(v_n)), B((g-p)(v_n))\Big) \\
&- \rho\Big\{N(x'_n, y'_{n+1}) - N(x'_n, y'_n)\Big\}\Big\| + \Big\|F(v_{n+1}, v_{n+1}, z'_{n+1}) - F(v_n, v_n, z'_n)\Big\|\Big] \\
&+ \mu_1\Big\|z'_{n+1} - z'_n\Big\|. \qquad (4.3)
\end{aligned}
$$

Since $(g-p)$ is s-Lipschitz continuous and $H(.,.)$ is h_1-Lipschitz continuous with respect to A and from Lemma 2.7, we have

$$
\begin{aligned}
&\Big\|H\Big(A((g-p)(v_{n+1})), B((g-p)(v_{n+1}))\Big) - H\Big(A((g-p)(v_n)), B((g-p)(v_{n+1}))\Big) \\
&- \rho\left\{N(x'_{n+1}, y'_{n+1}) - N(x'_n, y'_{n+1})\right\}\Big\|^2 \\
&\le \Big\|H\Big(A((g-p)(v_{n+1})), B((g-p)(v_{n+1}))\Big) - H\Big(A((g-p)(v_n)), B((g-p)(v_{n+1}))\Big)\Big\|^2 \\
&\quad - 2\rho\Big\langle N(x'_{n+1}, y'_{n+1}) - N(x'_n, y'_{n+1}), j\Big(H\Big(A((g-p)(v_{n+1})), B((g-p)(v_{n+1}))\Big) \\
&\quad - H\Big(A((g-p)(v_n)), B((g-p)(v_{n+1}))\Big) - \rho\Big\{N(x'_{n+1}, y'_{n+1}) - N(x'_n, y'_{n+1})\Big\}\Big)\Big\rangle \\
&\le \Big\|H\Big(A((g-p)(v_{n+1})), B((g-p)(v_{n+1}))\Big) - H\Big(A((g-p)(v_n)), B((g-p)(v_{n+1}))\Big)\Big\|^2 \\
&\quad - 2\rho\Big\|N(x'_{n+1}, y'_{n+1}) - N(x'_n, y'_{n+1})\Big\| \times \Big[\Big\|H\Big(A((g-p)(v_{n+1})), B((g-p)(v_{n+1}))\Big) \\
&\quad - H\Big(A((g-p)(v_n)), B((g-p)(v_{n+1}))\Big)\Big\| + \rho\Big\|N(x'_{n+1}, y'_{n+1}) - N(x'_n, y'_{n+1})\Big\|\Big] \\
&\le s^2h_1^2\|v_{n+1} - v_n\|^2 - 2\rho\|N(x'_{n+1}, y'_{n+1}) - N(x'_n, y'_{n+1})\| \\
&\quad \times \Big[sh_1\|v_{n+1} - v_n\| + \rho\|N(x'_{n+1}, y'_{n+1}) - N(x'_n, y'_{n+1})\|\Big]. \qquad (4.4)
\end{aligned}
$$

Since $N(.,.)$ is l_1-Lipschitz continuous with respect to first argument and l_2-Lipschitz continuous with respect to second argument and S is α_1-$\mathcal{D}$-Lipschitz continuous and T is α_2-$\mathcal{D}$-Lipschitz continuous, we have

$$
\Big\|N(x'_{n+1}, y'_{n+1}) - N(x'_n, y'_{n+1})\Big\| \le l_1\alpha_1\Big(1 + (1+n)^{-1}\Big)\|v_{n+1} - v_n\|. \qquad (4.5)
$$

$$\left\|N(x'_n,y'_{n+1})-N(x'_n,y'_n)\right\| \leq l_2\alpha_2\Big(1+(1+n)^{-1}\Big)\|v_{n+1}-v_n\|. \tag{4.6}$$

Using (4.5) in (4.4), we have

$$\begin{aligned}&\Big\|H\Big(A((g-p)(v_{n+1})),B((g-p)(v_{n+1}))\Big)-H\Big(A((g-p)(v_n)),B((g-p)(v_{n+1}))\Big)\\&-\rho\left\{N(x'_{n+1},y'_{n+1})-N(x'_n,y'_{n+1})\right\}\Big\|^2\\&\leq\Big[s^2h_1^2-2\rho l_1\alpha_1\Big(1+(1+n)^{-1}\Big)\Big\{sh_1+\rho l_1\alpha_1\Big(1+(1+n)^{-1}\Big)\Big\}\Big]\|v_{n+1}-v_n\|^2.\end{aligned} \tag{4.7}$$

Similarly, using h_2-Lipschitz continuity of $H(.,.)$ with respect to B, s-Lipschitz continuity of $(g-p)$, Lemma 2.7 and (4.6), we have the following estimate:

$$\begin{aligned}&\Big\|H\Big(A((g-p)(v_n)),B((g-p)(v_{n+1}))\Big)-H\Big(A((g-p)(v_n)),B((g-p)(v_n))\Big)\\&-\rho\left\{N(x'_n,y'_{n+1})-N(x'_n,y'_n)\right\}\Big\|^2\\&\leq\Big[s^2h_2^2-2\rho l_2\alpha_2\Big(1+(1+n)^{-1}\Big)\Big\{sh_2+\rho l_2\alpha_2\Big(1+(1+n)^{-1}\Big)\Big\}\Big]\|v_{n+1}-v_n\|^2.\end{aligned} \tag{4.8}$$

Since $F(.,.,.)$ is β_j-Lipschitz continuous in the jth argument, for $j=1,2,3$, G is α_3-$\mathcal{D}$-Lipschitz continuous and using Iterative Algorithm 4.2, we have the following estimate:

$$\begin{aligned}&\| F(v_{n+1},v_{n+1},z'_{n+1})-F(v_n,v_n,z'_n)\|\\&\leq\| F(v_{n+1},v_{n+1},z'_{n+1})-F(v_n,v_{n+1},z'_{n+1}\|\\&\quad+\| F(v_n,v_{n+1},z'_{n+1})-F(v_n,v_n,z'_{n+1})\|+\| F(v_n,v_n,z'_{n+1})-F(v_n,v_n,z'_n\|\\&\leq\Big(\beta_1+\beta_2\Big)\|v_{n+1}-v_n\|+\beta_3\Big(1+(1+n)^{-1}\Big)\mathcal{D}(G(v_{n+1}),G(v_n))\\&\leq\Big(\beta_1+\beta_2+\beta_3\alpha_3\Big(1+(1+n)^{-1}\Big)\|v_{n+1}-v_n\|.\end{aligned} \tag{4.9}$$

Using (4.7)-(4.9) in (4.3), we have

$$\begin{aligned}&\|(g-p)u_{n+2}-(g-p)u_{n+1}\|\\&\leq\Big[k\Big\{\sqrt{s^2h_1^2-2\rho l_1\alpha_1(1+(1+n)^{-1})\Big(sh_1+\rho l_1\alpha_1(1+(1+n)^{-1})\Big)}\\&\quad+\sqrt{s^2h_2^2-2\rho l_2\alpha_2(1+(1+n)^{-1})\Big(sh_2+\rho l_2\alpha_2(1+(1+n)^{-1})\Big)}\\&\quad+\Big(\beta_1+\beta_2+\beta_3\alpha_3(1+(1+n)^{-1})\Big)\Big\}\\&\quad+\mu_1\alpha_3(1+(1+n)^{-1})\Big]\|v_{n+1}-v_n\|.\end{aligned} \tag{4.10}$$

Since $(g-p-I)$ is λ-strongly accretive, by Lemma 2.7 and (4.10), we have the following estimate:

$$\begin{aligned}\|u_{n+2}-u_{n+1}\|^2 &\le \Big\|(g-p)u_{n+2}-(g-p)u_{n+1}+u_{n+2}-u_{n+1}\\ &\quad-\Big((g-p)u_{n+2}-(g-p)u_{n+1}\Big)\Big\|^2\\ &\le \|(g-p)u_{n+2}-(g-p)u_{n+1}\|^2\\ &\quad-2\Big\langle(g-p-I)u_{n+2}-(g-p-I)u_{n+1},j(u_{n+2}-u_{n+1})\Big\rangle\\ &\le \|(g-p)u_{n+2}-(g-p)u_{n+1}\|^2-2\lambda\|u_{n+2}-u_{n+1})\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\|u_{n+2}-u_{n+1}\|\\ &\le \frac{1}{\sqrt{1+2\lambda}}\ \|(g-p)u_{n+2}-(g-p)u_{n+1}\|\\ &= \frac{1}{\sqrt{1+2\lambda}}\ \Big[k\Big\{\sqrt{s^2h_1^2-2\rho l_1\alpha_1(1+(1+n)^{-1})\Big(sh_1+\rho l_1\alpha_1(1+(1+n)^{-1})\Big)}\\ &\quad+\sqrt{s^2h_2^2-2\rho l_2\alpha_2(1+(1+n)^{-1})\Big(sh_2+\rho l_2\alpha_2(1+(1+n)^{-1})\Big)}\\ &\quad+\Big(\beta_1+\beta_2+\beta_3\alpha_3(1+(1+n)^{-1})\Big)\Big\}+\mu_1\alpha_3(1+(1+n)^{-1})\Big]\|v_{n+1}-v_n\|.\end{aligned}\tag{4.11}$$

Similarly, using Iterative Algorithm 4.2, we have the following estimate:

$$\begin{aligned}&\|v_{n+1}-v_n\|\\ &\le \frac{1}{\sqrt{1+2\lambda}}\ \Big[k'\Big\{\sqrt{s^2h_2^2-2\gamma l_2\alpha_2(1+(1+n)^{-1})\Big(sh_2+\gamma l_2\alpha_2(1+(1+n)^{-1})\Big)}\\ &\quad+\sqrt{s^2h_1^2-2\gamma l_1\alpha_1(1+(1+n)^{-1})\Big(sh_1+\gamma l_1\alpha_1(1+(1+n)^{-1})\Big)}\\ &\quad+\Big(\beta_1+\beta_2+\beta_3\alpha_3(1+(1+n)^{-1})\Big)\Big\}\\ &\quad+\mu_2\alpha_3(1+(1+n)^{-1})\Big]\ \|u_{n+1}-u_n\|.\end{aligned}\tag{4.12}$$

Combining (4.11) and (4.12), we have

$$\|u_{n+2}-u_{n+1}\|\le\phi_{n+1}\ \|u_{n+1}-u_n\|,\tag{4.13}$$

where

$$\begin{aligned}\phi_{n+1}=\frac{1}{1+2\lambda}\ \Big[k\Big\{&\sqrt{s^2h_1^2-2\rho l_1\alpha_1(1+(1+n)^{-1})\Big(sh_1+\rho l_1\alpha_1(1+(1+n)^{-1})\Big)}\\ &+\sqrt{s^2h_2^2-2\rho l_2\alpha_2(1+(1+n)^{-1})\Big(sh_2+\rho l_2\alpha_2(1+(1+n)^{-1})\Big)}\end{aligned}$$

$$
\begin{aligned}
&+\Big(\beta_1+\beta_2+\beta_3\alpha_3(1+(1+n)^{-1})\Big)\Big\}+\mu_1\alpha_3(1+(1+n)^{-1})\Big]\\
&\times\Big[k'\Big\{\sqrt{s^2h_2^2-2\gamma l_2\alpha_2(1+(1+n)^{-1})\Big(sh_2+\gamma l_2\alpha_2(1+(1+n)^{-1})\Big)}\\
&+\sqrt{s^2h_1^2-2\gamma l_1\alpha_1(1+(1+n)^{-1})\Big(sh_1+\gamma l_1\alpha_1(1+(1+n)^{-1})\Big)}\\
&+\Big(\beta_1+\beta_2+\beta_3\alpha_3(1+(1+n)^{-1})\Big)\Big\}+\mu_2\alpha_3(1+(1+n)^{-1})\Big].
\end{aligned}
$$

Let

$$
\begin{aligned}
\phi=\frac{1}{1+2\lambda}\;&\Big[k\Big\{\sqrt{s^2h_1^2-2\rho l_1\alpha_1\Big(sh_1+\rho l_1\alpha_1\Big)}\\
&+\sqrt{s^2h_2^2-2\rho l_2\alpha_2\Big(sh_2+\rho l_2\alpha_2\Big)}+\Big(\beta_1+\beta_2+\beta_3\alpha_3\Big)\Big\}+\mu_1\alpha_3\Big]\\
&\times\Big[k'\Big\{\sqrt{s^2h_2^2-2\gamma l_2\alpha_2\Big(sh_2+\gamma l_2\alpha_2\Big)}\\
&+\sqrt{s^2h_1^2-2\gamma l_1\alpha_1\Big(sh_1+\gamma l_1\alpha_1\Big)}+\Big(\beta_1+\beta_2+\beta_3\alpha_3\Big)\Big\}+\mu_2\alpha_3\Big].
\end{aligned}
$$

Then we know that $\phi_n\to\phi$ as $n\to\infty$. By condition (4.2), we know that $\phi\in(0,1)$ and hence there exist $n_0>0$ and $\phi_0\in(0,1)$ such that $\phi_{n+1}\le\phi_0$ for all $n\ge n_0$. Therefore, by (4.13), we have

$$\|u_{n+2}-u_{n+1}\|\le\phi_0\|u_{n+1}-u_n\|,\ \forall\ n\ge n_0.$$

This implies

$$\|u_{n+1}-u_n\|\le\phi_0^{n-n_0}\|u_{n_0+1}-u_{n_0}\|.$$

Hence, for any $m\ge n>n_0$, we have

$$\|u_m-u_n\|\le\sum_{t=n}^{m-1}\|u_{t+1}-u_t\|\le\sum_{t=n}^{m-1}\phi_0^{t-n_0}\|u_{n_0+1}-u_{n_0}\|.$$

It follows $\|u_m-u_n\|\to 0$ as $n\to\infty$ so that $\{u_n\}$ is a Cauchy sequence in X. Hence, there exists $u\in X$ such that $u_n\to u$ as $n\to\infty$. Also from (4.12), we see that $\{v_n\}$ is a Cauchy sequence in X. Hence, there exists $v\in X$ such that $v_n\to v$ as $n\to\infty$.

Now from $\mathcal{D}$-Lipschitz continuity of S and Iterative Algorithm 4.2, we have

$$
\begin{aligned}
\|x_{n+1}-x_n\|&\le\Big(1+(1+n)^{-1}\Big)\mathcal{D}\Big(S(u_{n+1}),S(u_n)\Big)\\
&\le\Big(1+(1+n)^{-1}\Big)l_1\|u_{n+1}-u_n\|. \qquad (4.14)
\end{aligned}
$$

Since $\{u_n\}$ being Cauchy in X, (4.14) implies that $\{x_n\}$ is a Cauchy sequence in X. Similarly, we can prove that $\{y_n\}$, $\{z_n\}$, $\{x'_n\}$, $\{y'_n\}$ and $\{z'_n\}$ are also

Cauchy sequences in X. Thus, in general, there exist x, y, z, x', y', z' in X such that $x_n \to x$, $y_n \to y$, $z_n \to z$, $x'_n \to x'$, $y'_n \to y'$, $z'_n \to z'$ as $n \to \infty$.

Now, we show that $x \in S(u)$. Since $x_n \in S(u_n)$, we have

$$\begin{aligned} d(x, S(u)) &\leq \|x - x_n\| + d(x_n, S(u)) \\ &\leq \|x - x_n\| + \mathcal{D}\Big(S(u_n), S(u)\Big) \\ &\leq \|x - x_n\| + l_1\|u_n - u\| \\ &\longrightarrow 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

Since $S(u)$ is closed, it implies that $x \in S(u)$. Similarly, we can show that $y \in T(u)$, $z \in G(u)$, $x' \in S(v)$, $y' \in T(v)$, $z' \in G(v)$. By assumption (4.1), Lipschitz continuity of proximal mapping $J^{H(.,.),\eta}_{M(.,z'),\rho}$, continuity of the respective mappings and Iterative Algorithm 4.2, it follows that $u, v \in X$, $x \in S(u)$, $y \in T(u)$, $z \in G(u)$, $x' \in S(v)$, $y' \in T(v)$ and $z' \in G(v)$, where

$$\begin{aligned} J^{H(.,.),\eta}_{M(.,z'),\rho}(u) &= \Big(H(A, B) + \rho M(., z')\Big)^{-1}(u), \\ J^{H(.,.),\eta}_{M(.,z),\gamma}(v) &= \Big(H(A, B) + \gamma M(., z)\Big)^{-1}(v) \end{aligned}$$

and $\rho, \gamma > 0$ are constants. By Theorem 4.1, $(u, v, x, y, z, x', y', z')$ is the solution of the problem. This completes the proof. $\square$

Remark 4.4. Using the technique developed in this paper we can extend the results of Adly [1], Hassouni and Moudafi [9], Kazmi and Bhat [12,14], Mitrovic [20], Verma [26] and the related results cited therein for the system of variational inclusion.

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