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ON MAZUR-ULAM THEOREM AND MAPPINGS WHICH PRESERVE DISTANCES

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Abstract. Let X and Y be two real Hilbert spaces with the dimension of X greater than 1. Several cases for a mapping $f : X \to Y$ preserving two distances with a non-integer ratio are presented.

1. Introduction

Let X and Y be two normed vector spaces. An isometry from X to Y is a mapping $f : X \to Y$ such that

$$\|f(x) - f(y)\| = \|x - y\| \quad \text{for all} \quad x, y \in X.$$

Mazur and Ulam [5] proved that every isometry from one normed real vector space onto another normed real vector space is a linear mapping up to translation. The conclusion is not valid for normed complex vector spaces (just consider the complex conjugation on C, see [4]). The hypothesis of surjectivity is essential in general. Without the onto assumption, Baker [1] proved that every isometry from a normed real vector space into a strictly convex normed real vector space must be a linear isometry up to translation.

A mapping $f : X \to Y$ satisfies the distance one preserving property (DOPP) iff for all $x, y \in X$ with $\|x-y\| = 1$, it follows that $\|f(x)-f(y)\| = 1$. A mapping $f : X \to Y$ satisfies the strong distance one preserving property (SDOPP) iff for all $x, y \in X$ with $|x-y\| = 1$ it follows that $\|f(x)-f(y)\| = 1$ and conversely (see [9]).

For Euclidean spaces $X = Y = \mathbb{R}^n$, if $2 \leq n < \infty$ and $f : X \to Y$ satisfies (DOPP) then f must be a linear isometry up to translation due to Beckman

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and Quarles [2]; however if $n = 1$ or $n = \infty$, $f : X \to Y$ satisfying (DOPP) is not necessary to be an isometry (see [2, 6, 8]).

If X and Y are normed real vector spaces, Rassias and Šemrl [9] proved the following results:

Theorem 1.1. ([9]). *Let X and Y be normed real vector spaces such that one of them has dimension greater that one. Suppose that $f : X \to Y$ is a Lipschitz mapping with $k = 1$:*

$$\|f(x) - f(y)\| \leq \|x - y\| \quad \textit{for all} \;\; x, y \in X.$$

Assume also that f is a surjective mapping satisfying (SDOPP). Then f is a linear isometry up to translation.

Especially, if one of the spaces X and Y is strictly convex, it was proved that

Theorem 1.2. ([9]). *Let X and Y be normed real vector spaces such that one of them has dimension greater than one. Assume that one of the spaces is strictly convex. Suppose that $f : X \to Y$ is a surjective mapping satisfying (SDOPP). Then f is a linear isometry up to translation.*

Theorem 1.3. ([9]). *Let X and Y be normed real vector spaces,* $\dim X \geq 2$, *such that one of them is strictly convex. Suppose that $f : X \to Y$ is a homeomorphism satisfying (DOPP). Then f is a linear isometry up to translation.*

Furthermore, if Y is strictly convex without the onto assumption on f, Benz and Berens [3] got the following result:

Theorem 1.4. ([3]). *Let X and Y be normed real vector spaces. Assume that* $\dim X \geq 2$, *and Y is strictly convex. Suppose $f : X \to Y$ satisfies the properties:*

(1) *for all $x, y \in X$ with $\|x - y\| = \rho$, $\|f(x) - f(y)\| \leq \rho$;*
(2) *for all $x, y \in X$ with $\|x - y\| = \lambda\rho$, $\|f(x) - f(y)\| \geq \lambda\rho$, where λ is a positive integer greater than one.*

Then f is a linear isometry up to translation.

If f preserves two distances with a noninteger ratio, and X and Y are real normed vector spaces such that Y is strictly convex and $\dim X \geq 2$, it is an *open problem* whether or not f must be an isometry (see [7]).

In this paper, we will study some extensions of the Mazur-Ulam theorem for conservative mappings between real Hilbert spaces. We denote by $(\cdot, \cdot)$ the inner products in X and Y.

2. Main Results

Let X and Y be real Hilbert spaces with the dimension of X greater than one.

Definition 2.1. Suppose $f : X \to Y$ is a mapping. The distance r is called *contractive* by f if and only if for all $x, y \in X$ with $\|x-y\| = r$, if follows that $\|f(x) - f(y)\| \le r$; The distance r is called *extensive* by f if and only if for all $x, y \in X$ with $\|x-y\| = r$, it follows that $\|f(x) - f(y)\| \ge r$; The distance r is called *preserved* by f if and only if for all $x, y \in X$ with $\|x-y\| = r$, it follows that $\|f(x) - f(y)\| = r$.

It is obvious by the triangle inequality that if $f : X \to Y$ preserves the distance r, then the distance nr is contractive by $f, n = 1, 2, \cdots$.

Theorem 2.1. *Suppose that $f : X \to Y$ satisfies (DOPP) and the distances a, b are contractive by f, where a and b are positive numbers with $|a-b| < 1$. Then the distance $\sqrt{2a^2+2b^2-1}$ is contractive by f. Especially, if the distance $\sqrt{2a^2+2b^2-1}$ is extensive by f, then the distances a, b and $\sqrt{2a^2+2b^2-1}$ are preserved by f.*

Proof. Suppose that $p, q \in X$ with $\|p-q\| = \sqrt{2a^2+2b^2-1}$. We will prove that $\|f(p)-f(q)\| \le \sqrt{2a^2+2b^2-1}$. Since the dimension of X is greater than one, we can select p_1, p_2 in X and construct a parallelogram with $\|p_1 - p\| = \|p_2-q\| = a$, $\|p_2-p\| = \|q-p_1\| = b$, $\|q-p\| = \sqrt{2a^2+2b^2-1}$, $\|p_2-p_1\| = 1$:

$$(2-1)$$

Set $x = f(p_1) - f(p)$, $y = f(p_2) - f(p)$, $z = f(q) - f(p_1)$, $u = f(q) - f(p_2)$, $v = f(p_2) - f(p_1)$ and $w = f(q) - f(p)$, then $v = y - x$, $u = w - y$ and $z = w - x$. Since f satisfies (DOPP) and the distances a, b are contractive by f, then $\|x\| \le a$, $\|u\| \le a$, $\|y\| \le b$, $\|z\| \le b$ and $\|v\| = 1$. By the Cauchy-

Schwartz inequality, we have that

$$\begin{aligned} 1+(w,w) &= (x-y,x-y)+(w,w) \\ &= (x+y,x+y)+(w,w)-4(x,y) \\ &\geq 2(w,x+y)-4(x,y). \end{aligned} \tag{1}$$

Hence

$$\begin{aligned} (w,w) &\geq 2(w,x+y)-4(x,y)-1 \\ &= 1+2(w,x+y)-2(x-y,x-y)-4(x,y) \\ &= 1+2(w,x+y)-2(x,x)-2(y,y). \end{aligned} \tag{2}$$

Therefore

$$\begin{aligned} (w,w) &\leq 2(w,w)+2(x,x)+2(y,y)-2(w,x+y)-1 \\ &= (x,x)+(y,y)+(w-x,w-x)+(w-y,w-y)-1 \\ &= (x,x)+(y,y)+(z,z)+(u,u)-1 \\ &\leq \sqrt{2a^2+2b^2-1}. \end{aligned} \tag{3}$$

Hence, the distance $\sqrt{2a^2+2b^2-1}$ is contractive by f.

According to (3), if $f : X \to Y$ satisfies (DOPP), the distances a, b are contractive by f and the distance $\sqrt{2a^2+2b^2-1}$ is extensive by f, then the distances a, b and $\sqrt{2a^2+2b^2-1}$ are preserved by f. □

Note. *For the special case in Theorem 2.1, where $|a-b| = 1$, f must be a linear isometry up to translation due to [10].*

Corollary 2.2. *Suppose that $f : X \to Y$ satisfied (DOPP) and the distance a is contractive by f, where a is a positive number. Then the distance $\sqrt{4a^2-1}$ is contractive by f. Especially, if the distance $\sqrt{4a^2-1}$ is extensive by f, then the distances a and $\sqrt{4a^2-1}$ are preserved by f.*

Suppose that $f : X \to Y$ satisfies (DOPP). By Corollary 2.2, the distances $\sqrt{4k^2-1}$, $\sqrt{4(4k^2-1)-1}$, $\cdots$, $\sqrt{4^m k^2 - \frac{4^m-1}{3}}$ are contractive by f where $k = 1, 2, \cdots, m = 1, 2, \cdots$. Together with Theorem 1.4, Corollary 2.2 and S. Xiang [11], we get the following result.

Theorem 2.3. *Suppose that $f : X \to Y$ satisfies (DOPP). Assume the distance $n\sqrt{4^m k^2 - \frac{4^m-1}{3}}$ is extensive by f for some positive integers n, k and m. Then f must be a linear isometry up to translation.*

Proof. (1) In case the distance $\sqrt{4^m k^2 - \frac{4^m-1}{3}}$ is extensive by f for some positive integers k and m: by induction on m and Corollary 2.2, the distances $\sqrt{4k^2-1}$ and k are preserved by f. If $k \geq 2$, by Theorem 1.4, it follows that the mapping f is a linear isometry up to translation; if $k = 1$, then $\sqrt{3}$ is preserved by f. By S. Xiang [11], f is a linear isometry up to translation.

(2) In case $n \geq 2$, for any $p, q_1 \in X$ with $\|p - q_1\| = \sqrt{4^m k^2 - \frac{4^m-1}{3}}$, set

$$q_j = p + j(p - q_1), \quad j = 1, 2, \cdots, n.$$

Then $\|q_{j+1} - q_j\| = \|q_1 - p\| = \sqrt{4^m k^2 - \frac{4^m-1}{3}}$ for $j = 1, 2, \cdots, n-1$ and $\|q_n - p\| = n\sqrt{4^m k^2 - \frac{4^m-1}{3}}$. Since f satisfies (DOPP) and $\sqrt{4^m k^2 - \frac{4^m-1}{3}}$ is contractive by f, it follows that

$$\|f(q_1) - f(p)\| \leq \sqrt{4^m k^2 - \frac{4^m-1}{3}},$$
$$\|f(q_{j+1}) - f(q_j)\| \leq \sqrt{4^m k^2 - \frac{4^m-1}{3}}, \quad j = 1, 2, \cdots, n-1$$

and

$$\begin{aligned}\|f(q_n) - f(p)\| &\leq \|f(q_1) - f(p)\| + \sum_{j=1}^{n-1} \|f(q_j + 1) - f(q_j)\| \\ &\leq n\sqrt{4^m k^2 - \frac{4^m-1}{3}}.\end{aligned}$$

Since $n\sqrt{4^m k^2 - \frac{4^m-1}{3}}$ is extensive by f, we have

$$\begin{aligned}\|f(q_1) - f(p)\| &= \|f(q_2) - f(q_1)\| \\ &= \cdots \\ &= \|f(q_n) - f(q_{n-1})\| \\ &= \sqrt{4^m k^2 - \frac{4^m-1}{3}}.\end{aligned}$$

Hence the distance $\sqrt{4^m k^2 - \frac{4^m-1}{3}}$ is also preserved by f. By step (1), f is a linear isometry up to translation. □

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