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FIXED AND COINCIDENCE POINT THEOREMS FOR EXPANSIVE MAPPINGS

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Abstract. In this paper, we give fixed and coincidence point theorems for expansive mappings, which extend and improve the corresponding results of Hicks-Saliga, Jachymski, Kang, Kang-Rhoades, Khan-Khan-Sessa, Rhoades, Taniguchi and Wang-Li-Gao-Iseki.

1. Introduction

The fixed point theorem for expansive mappings was first proved by Machuca [8]. Afterwards, a number of authors obtained also fixed point theorems for certain expansive mappings in metric spaces ([2], [3], [5], [6], [7], [9], [10], [12], [13]).

$\mathbb{N}$, Ω and $\mathbb{R}^+$ denote the sets of positive integers, nonnegative integers and nonnegative real numbers, respectively. Let f be a selfmapping of a metric space (X, d). A point x in X is called a *periodic point* of f if there exists $k \in \mathbb{N}$ such that $f^k x = x$. The least positive integer k satisfying this condition is called the *periodic index* of x. Let $O_f(x)$ denote the orbit of f at x; i.e., $O_f(x) = \{f^n x : n \in \Omega\}$. Define

$$\mathcal{F} = \{F \mid F : X \times X \to \mathbb{R}^+ \text{ is continuous such that } F(x,y) = 0 \text{ if and only if } x = y\}.$$

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Definition 1.1. [11] A selfmapping f of a metric space (X, d) is said to be *nearly densifying* if $\alpha(fA) < \alpha(A)$ whenever $\alpha(A) > 0$, A is bounded and f-invariant, where $\alpha(A)$ denotes the measure of noncompactness in the sense of Kuratowski.

Definition 1.2. [4] Let f and g be selfmappings of a metric space (X, d). Then f and g are said to be *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some point t in X.

Clearly, commuting mappings are compatible, but the converse is not necessarily true.

Lemma 1.1. [4] *Let f and g be compatible selfmappings of a metric space (X, d). Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some point $t \in X$. Then $\lim_{n\to\infty} gfx_n = ft$ if f is continuous.*

In this paper, we give fixed and coincidence point theorems for expansive mappings, which extend and improve the corresponding results of Hicks-Saliga [2], Jachymski [3], Kang [5], Kang-Rhoades [6], Khan-Khan-Sessa [7], Rhoades [10], Taniguchi [12] and Wang-Li-Gao-Iseki [13].

2. Fixed point theorems for expansive mappings

Taniguchi [12], Jachymski [3] and Ċiriċ [1] considered the following conditions:

$$d(fx, fy) > \min\{d(x, y), d(x, fx), d(y, fy)\} \tag{2.1}$$

for all $x, y \in X$ with $x \neq y$, and

$$d(x, y) > \min\{d(fx, fy), d(x, fx), d(y, fy)\} - \min\{d(x, fy), d(y, fx)\} \tag{2.2}$$

for all $x, y \in X$ with $x \neq y$.

Inspired by the results of [1], [3] and [12], we have the following

Theorem 2.1. *Let C be a compact subset of a metric space (X, d), $f : C \to X$ be continuous and $g : C \to C$ be continuous and injective. Assume that there exists $F \in \mathcal{F}$ such that*

$$\begin{aligned} F(fx, fy) > \min\left\{F(gx, gy), F(fx, gx), F(fy, gy), \frac{F(fx, gx)F(fy, gy)}{F(gx, gy)}\right\} \\ - \min\{F(fx, gy), F(fy, gx)\} \end{aligned} \tag{2.3}$$

for all distinct $x, y \in C$. *If* $fC = X$ *or* $C \subseteq fC$, *then* gh *has a fixed point in* C, *where* $h = f^{-1}$ *restricted to* C.

Proof. It follows that from (2.3) that $F(fx, fy) > 0$ if $x \neq y$. That is, f is injective. Let $t = gh$. Then t maps C into itself. Note that f is a continuous bijection of a compact set C onto fC. Thus f is a homeomorphism. Consequently, t is continuous. (2.3) ensures that

$$\begin{aligned} F(w, tw) > \min & \left\{ F(tx, ty), F(tx, x), F(ty, y), \frac{F(tx, x)F(ty, y)}{F(tx, ty)} \right\} \\ & - \min\{F(x, ty), F(y, tx)\}, \end{aligned} \tag{2.4}$$

for all distinct x, y in C. From the compactness of C and continuity of F and t, it follows that there exists $w \in C$ satisfying $F(w, tw) = \min\{F(x, tx) : x \in C\}$. We claim that $w = tw$. Otherwise, $F(w, tw) > 0$. Using (2.4) we have

$$\begin{aligned} F(w, tw) > \min & \left\{ F(tw, t^2w), F(w, tw), F(tw, t^2w), \frac{F(w, tw)F(tw, t^2w)}{F(tw, t^2w)} \right\} \\ & - \min\{F(w, t^2w), F(tw, tw)\} \\ = & \min\{F(tw, t^2w), F(w, tw)\} \\ = & F(tw, t^2w). \end{aligned}$$

That is,

$$F(w, tw) > F(tw, t^2w) \geq \min\{F(x, tx) : x \in C\} = F(w, tw)$$

which is a contradiction. Hence w is a fixed point of t. This completes the proof. $\square$

Remark 2.1. *In case* $g = i_C$*-the identity mapping on* C *and* $F = d$, *Theorem 2.1 reduces to a result which generalizes Theorem 4.2 of Jachymski [3] and Theorem 8 of Hicks-Saliga [2].*

Remark 2.2. *Example 5 of Hicks-Saliga [2] demonstrates that the compactness of* C *is necessary in the above Theorem 2.1.*

Theorem 2.2. *Let* f *be a continuous and nearly densifying selfmapping of a complete metric space* (X, d). *Assume that there exists* $F \in \mathcal{F}$ *such that*

$$\begin{aligned} F(fx, fy) > \min & \left\{ F(x, y), F(x, fx), F(y, fy), \frac{F(x, fx)F(y, fy)}{F(x, y)} \right\} \\ & - \min\{F(x, fy), F(y, fx)\} \end{aligned} \tag{2.5}$$

for all distinct $x, y \in X$. Then the following statements are equivalent:

(1) *f has a fixed point;*
(2) *f has a periodic point;*
(3) *There exists $x_0 \in X$ such that $O_f(x_0)$ is compact;*
(4) *There exists $x_0 \in X$ such that $O_f(x_0)$ is bounded.*

Proof. (1)$\Longrightarrow$(2) and (3)$\Longrightarrow$(4) are clear.

(2)$\Longrightarrow$(3). Assume that f has a periodic point $x_0 \in X$ and that k is the periodic index of x_0. It is easy to see that $O_f(x_0) = \{f^i x_0 : 0 \leq i < k\}$. Hence $O_f(x_0)$ is compact.

(4)$\Longrightarrow$(1). Set $A = \overline{O_f(x_0)}$, where $\overline{O_f(x_0)}$ denotes the closure of $O_f(x_0)$. By the continuity of f and $fO_f(x_0) \subseteq O_f(x_0)$, it follows that

$$fA \subseteq \overline{fO_f(x_0)} \subseteq A.$$

That is, A is invariant under f. Suppose that $\alpha(O_f(x_0)) > 0$. Since

$$\begin{aligned}\alpha(O_f(x_0)) &= \max\{\alpha(\{x_0\}), \alpha(fO_f(x_0))\} \\ &= \alpha(fO_f(x_0))\end{aligned}$$

and f is nearly densifying, it follows that A is compact. Since F and f are continuous, so the function h defined by $hx = F(x, fx)$ for $x \in A$ is continuous and achieves its maximum value at some $w \in A$. Suppose that $w \neq fw$. It follows from (2.5) that

$$\begin{aligned}F(fw, f^2w) > \min\Bigg\{ & F(w, fw), F(w, fw), F(fw, f^2w), \\ & \qquad \frac{F(w, fw)F(fw, f^2w)}{F(w, fw)}\Bigg\} \\ & - \min\{F(w, f^2w), F(fw, fw)\} \\ = F(w, fw) &\end{aligned}$$

which implies that

$$hfw = F(fw, f^2w) > F(w, fw) = hw = \max\{hx : x \in A\} \geq hfw$$

which is impossible. Hence $w = fw$. This completes the proof. $\square$

Let Φ_1 denotes the family of all functions $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ satisfying the following conditions:

(C_1) ϕ is lower-semicontinuous in each coordinate variable;
(C_2) Let $v, w \in \mathbb{R}^+$ be such that either $v \geq \phi(v, w, w, v)$ or $v \geq \phi(w, v, w, v)$. Then $v \geq hw$, where $\phi(1, 1, 1, 1) = h > 1$.

Theorem 2.3. *Let f and g be surjective selfmappings of a complete metric space (X, d) satisfying*

$$d(fx, gy) \geq \phi\left(d(fx, x), d(gy, y), d(x, y), \frac{d(fx, x)d(gy, y)}{d(x, y)} \right) \tag{2.6}$$

for all distinct x, y in X, where $\phi \in \Phi_1$. Then f and g have a common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since f and g are surjective, we can easily construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $fx_{2n+1} = x_{2n}$, $gx_{2n+2} = x_{2n+1}$ for $n \in \Omega$. Set $d_n = d(x_n, x_{n+1})$ for $n \in \Omega$. Suppose that $x_{2n} = x_{2n+1}$ for some $n \in \Omega$. If $x_{2n+1} \neq x_{2n+2}$, then we have by (2.6)

$$\begin{aligned} d_{2n} &= d(fx_{2n+1}, gx_{2n+2}) \\ &\geq \phi[d(fx_{2n+1}, x_{2n+1}), d(gx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), \\ &\qquad \frac{d(fx_{2n+1}, x_{2n+1})d(gx_{2n+2}, x_{2n+2})}{d(x_{2n+1}, x_{2n+2})}] \\ &= \phi(d_{2n}, d_{2n+1}, d_{2n+1}, d_{2n}) \end{aligned}$$

It follows from (C_2) and the above inequalities that

$$0 = d_{2n} \geq hd_{2n+1} > 0$$

which is impossible. Hence $x_{2n+1} = x_{2n+2}$. That is, x_{2n} is a common fixed point of f and g.

Similarly, we may prove that x_{2n+1} is a common fixed point of f and g if $x_{2n+1} = x_{2n+2}$ for some $n \in \Omega$.

Now we can suppose that $x_n \neq x_{n+1}$ for all $n \in \Omega$. From (2.6) and (C_2) it follows that

$$d_n \ \leq \ rd_{n-1} \ \leq \ \cdots \ \leq \ r^n d_0$$

for $n \in \mathbb{N}$, where $r = \frac{1}{h} < 1$. It is easily seen that

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d_k \leq \sum_{k=n}^{m-1} r^k d_0 \leq \frac{r^n}{1-r} d_0 \tag{2.7}$$

for $n, m \in \mathbb{N}$ with $m > n$. Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. From the completeness of X, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Since

f and g are surjective, there exist $v, w \in X$ such that $fv = gw = u$. We claim that $w = u$. Otherwise $w \neq u$. We consider two cases:

Case 1. There exists $k \in \mathbb{N}$ such that $x_{2n+1} = w$ for all $n \geq k$. Letting $m \to \infty$ in (2.7), we have

$$d(w, u) = d(x_{2n+1}, u) \leq \frac{r^{2n+1}}{1-r} d_0$$

for $n \geq k$. It is easy to verify that $w = u$. This is a contradiction.

Case 2. There exists a subsequence $\{x_{2n_i+1}\}_{i\in\mathbb{N}}$ of $\{x_{2n+1}\}_{n\in\mathbb{N}}$ such that $x_{2n_i+1} \neq w$ for all $i \in \mathbb{N}$. Using (2.6),

$$\begin{aligned} d(x_{2n_i}, u) &= d(fx_{2n_i+1}, gw) \\ &\geq \phi\Bigg(d(fx_{2n_i+1}, x_{2n_i+1}), d(gw, w), d(x_{2n_i+1}, w), \\ &\qquad \frac{d(fx_{2n_i+1}, x_{2n_i+1}) d(gw, w)}{d(x_{2n_i+1}, w)} \Bigg). \end{aligned}$$

Letting $i \to \infty$, we obtain

$$\begin{aligned} 0 &= d(u, u) \\ &\geq \phi\left(d(u, u), d(gw, w), d(u, w), \frac{d(u, u) d(gw, w)}{d(u, w)} \right) \\ &= \phi(0, d(u, w), d(u, w), 0) \end{aligned}$$

which implies that $u = w$. This is also a contradiction. Therefore $w = u = gw$. Similarly we can prove that $v = u = fv$. That is, u is a common fixed point of f and g. This completes the proof. $\square$

Remark 2.3. *Theorem 2.3 extends and improves Theorem 2.1 of Kang [5] and Theorem 2 of Khan-Khan-Sessa [7]. The condition of surjectivity is necessary in Theorem 2.3. To see this, we give the following example inspired by Kang [5].*

Example 2.1. *Let $X = \mathbb{R}^+$ with the usual metric. Define f and $g : X \to X$ by $fx = h(x + 2)$ for $x \in X$ and $gx = hx$ for $x \in [0, 1]$ and $gx = h(x + 2)$ for $x > 1$, where $h > 1$.*

Consider $\phi(t_1, t_2, t_3, t_4) = ht_3$ for $t_1, t_2, t_3, t_4 \in X$. Then $\phi \in \Phi_1$. Further, it is easily seen that all the hypothesis of Theorem 2.3 are satisfied except the surjectivity of f and g. However, f and g do not have a common fixed point in X.

Theorem 2.4. *Let f and g be surjective selfmappings of a metric space (X, d) satisfying*

$$\begin{aligned} d(fx, gy) \geq & a_1 d(fx, x) + a_2 d(gy, y) + a_3 d(x, y) \\ & + a_4 \frac{d(fx, x) d(gy, y)}{d(x, y)} \end{aligned} \tag{2.8}$$

for all distinct $x, y \in X$, where $a_1, a_2, a_3, a_4 \in \mathbb{R}^+$ and $a_1 + a_4 < 1$, $a_2 + a_4 < 1$ and $\sum_{i=1}^4 a_i > 1$. Then

(i) $f = g$;
(ii) *f is injective;*
(iii) *f has a fixed point in X if (X, d) is complete.*

Proof. (i). Note that $fX = gX = X$. Then for any $x \in X$, there exists $y \in X$ such that $fx = gy$. We assert that $x = y$. Otherwise $x \neq y$. In view of (2.8), we immediately obtain $a_3 = 0$. If $a_4 = 0$, then $a_1 \neq 0$ and $a_2 \neq 0$ because $0 \leq a_1 < 1$, $0 \leq a_2 < 1$ and $a_1 + a_2 > 1$. (2.8) ensures that $d(fx, x) = d(gy, y) = 0$. That is, $x = fx = gy = y$, which is a contradiction. If $a_4 > 0$, by (2.8) we have $d(fx, x) d(fy, y) = 0$. This means that $d(gy, y) = d(x, y) > 0$ or $d(fx, x) = d(y, x) > 0$. In view of (2.8) we have $a_2 = 0$ or $a_1 = 0$. It follows that

$$1 > a_1 + a_4 = \sum_{i=1}^4 a_i > 1$$

or

$$1 > a_2 + a_4 = \sum_{i=1}^4 a_i > 1$$

which are impossible. Hence $fx = gx$ for all $x \in X$.

(ii). Suppose that there exist $x, y \in X$ with $x \neq y$ and $fx = fy$. From (2.8) and (i) we have

$$\begin{aligned} 0 &= d(fx, fy) \\ &\geq \frac{1}{2}(a_1 + a_2)[d(fx, x) + d(fy, y)] \\ &\quad + a_3 d(x, y) + a_4 \frac{d(fx, x) d(fy, y)}{d(x, y)} \end{aligned} \tag{2.9}$$

which implies that $a_3 = 0$. If $a_4 = 0$, then $a_1 + a_2 = \sum_{i=1}^4 a_i = 1$. It follows from (2.9) that $x = fx = fy = y$, which is a contradiction; If $a_4 > 0$, then, by (2.9) we have $d(fx, x) d(fy, y) = 0$. This means that

$$d(fx, x) + d(fy, y) = d(x, y) > 0.$$

Thus (2.9) ensures that $a_1 + a_2 = 0$. Consequently

$$1 < \sum_{i=1}^{4} a_i = a_4 < 1$$

which is absurd. Hence f is injective.

(iii). Let $h = \sum_{i=1}^{4} a_i$ and $\phi(t_1, t_2, t_3, t_4) = \sum_{i=1}^{4} a_i t_i$ for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$. It is easy to verify $\phi \in \Phi_1$. By Theorem 2.3 and (i) of Theorem 2.4 we immediately conclude that f has a fixed point in X. □

Remark 2.4. *Theorems 1 and 2 of Wang-Li-Gao-Iseki [13], Theorem 1 of Taniguchi [12] and Corollary 2.3 of Kang [5] are special cases of* (iii) *of Theorem 2.4.*

3. Coincidence point theorems for expansive mappings

Let Φ_2 denote the family of all real functions $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ satisfying conditions (C_1) and (C_3) :

(C_3) Let $v, w \in \mathbb{R}^+ - \{0\}$ be such that either $v \geq \phi(v, w, w, v)$ or $v \geq \phi(w, v, w, v)$. Then $v \geq hw$, where $\phi(1, 1, 1, 1) = h > 1$.

It is easy to see that $\Phi_1 \subset \Phi_2$. Let $\phi(t_1, t_2, t_3, t_4) = h \min\{t_i : i = 1, 2, 3, 4\}$ for $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$, where $h > 1$. Then $\phi \in \Phi_2$ but $\phi \notin \Phi_1$. Therefore Φ_1 is a proper subset of Φ_2.

Theorem 3.1. *Let A, B, S and T be continuous selfmappings of a complete metric space (X, d) such that*

(a) *$AX \supseteq TX$, $BX \supseteq SX$;*
(b) *A, S and B, T are compatible.*

Suppose that there exists $\phi \in \Phi_2$ satisfying the condition

$$d(Ax, By) \geq \phi\left(d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{d(Ax, Sx)d(By, Ty)}{d(Sx, Ty)}\right) \quad (3.1)$$

for all $x, y \in X$ with $Sx \neq Ty$. Then there exists $w \in X$ such that either $Aw = Sw$ or $Bw = Tw$ or $Aw = Sw$ and $Bw = Tw$.

Proof. Let $x_0 \in X$. (a) ensures that there exist sequences $\{y_n\}_{n\in\Omega}$ and $\{x_n\}_{n\in\mathbb{N}}$ in X such that $y_{2n} = Ax_{2n+1} = Tx_{2n}$ and $y_{2n+1} = Bx_{2n+2} = Sx_{2n+1}$ for all $n \in \Omega$. If $y_n = y_{n+1}$ for some $n \in \Omega$, then A and S or B and T have a coincidence point. Suppose that $y_n \neq y_{n+1}$ for all $n \in \Omega$.

From (3.1) and (C_3) we easily conclude that $\{y_n\}_{n\in\Omega}$ is a Cauchy sequence. Thus it converges to some point w in X since (X,d) is complete. Clearly, $Ax_{2n+1}, Sx_{2n+1} \to w$ as $n \to \infty$. Since S is continuous, $SAx_{2n+1} \to Sw$ as $n \to \infty$. From (b) and Lemma 1.1 it follows that $SAx_{2n+1} \to Aw$ as $n \to \infty$. Hence $Aw = Sw$. Similarly we have $Tw = Bw$. This completes the proof. □

Remark 3.1. *In case $T = S = i_X$-the identity mapping of X, Theorem 3.1 reduces to a result which generalizes Theorem 2.6 and Corollary 2.7 of Kang [5], Theorem 4 of Khan-Khan-Sessa [7], Theorem 3 of Rhoades [10] and Theorem 1, Theorem 2 and Theorem 3 of Wang-Li-Gao-Iseki [13].*

Let Ψ denote the family of all real function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions (C_4) and (C_5) :

(C_4) ψ is upper-semicontinuous and nondecreasing;
(C_5) $\psi(t) < t$ for each $t > 0$.

Theorem 3.2. *Let A, B, S and T be continuous selfmappings of a complete metric space (X,d) satisfying* (a), (b) *and*

$$\psi(d(Ax,By)) \geq \min\left\{ d(Ax,Sx), d(By,Ty), d(Sx,Ty), \frac{d(Ax,Sx)d(By,Ty)}{d(Sx,Ty)} \right\} \tag{3.2}$$

for all $x, y \in X$ with $Sx \neq Ty$, where $\psi \in \Psi$ and $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for each $t > 0$. Then the same conclusion of Theorem 3.1 holds

Proof. Let $\{y_n\}_{n\in\Omega}$ and $\{x_n\}_{n\in\mathbb{N}}$ be as in the proof of Theorem 3.1. Put $d_n = d(y_n, y_{n+1})$ for $n \in \Omega$. If $y_n = y_{n+1}$ for some $n \in \Omega$, then A and S or B and T have a coincidence point. Suppose that $y_n \neq y_{n+1}$ for all $n \in \Omega$. From (3.2), (C_4) and (C_5) we have

$$\begin{aligned}
\psi(d_{2n}) &= \psi(d(Ax_{2n+1}, Bx_{2n+2})) \\
&\geq \min\left\{ d(Ax_{2n+1}, Sx_{2n+1}), d(Bx_{2n+2}, Tx_{2n+2}), d(Sx_{2n+1}, Tx_{2n+2}), \frac{d(Ax_{2n+1}, Sx_{2n+1})d(Bx_{2n+2}, Tx_{2n+2})}{d(Sx_{2n+1}, Tx_{2n+2})} \right\} \\
&= \min\left\{ d_{2n}, d_{2n+1}, d_{2n+1}, \frac{d_{2n}d_{2n+1}}{d_{2n+1}} \right\} \\
&= \min\{d_{2n}, d_{2n+1}\} \\
&= d_{2n+1}.
\end{aligned}$$

Similarly we have $d_{2n+2} \le \psi(d_{2n+1})$. Thus, for any $n \in \mathbb{N}$ we have

$$d_n \le \psi(d_{n-1}) \le \cdots \le \psi^n(d_0).$$

For any n, $m \in \mathbb{N}$ with $n < m$, we have

$$d(y_n, y_m) \le \sum_{k=n}^{m-1} d_k \le \sum_{k=n}^{m-1} \psi^k(d_0).$$

Note that $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for each $t > 0$. Therefore $\{y_n\}_{n\in\Omega}$ is a Cauchy sequence. The remaining portion of the proof can be derived as in Theorem 3.1. This completes the proof. □

Remark 3.2. *Theorem 3.2 extends and improves Theorem 2.9 and Corollary 2.7 of Kang [5], Theorem 3 of Rhoades [10] and Theorem 1, Theorem 2 and Theorem 3 of Wang-Li-Gao-Iseki [13].*

The following example reveals that the Theorem 3.1 and Theorem 3.2 extend properly Theorem 2.6, Theorem 2.9 and Corollary 2.7 of Kang [5], Theorem 4 of Khan-Khan-Sessa [7], Theorem 3 of Rhoades [10] and Theorem 1, Theorem 2 and Theorem 3 of Wang-Li-Gao-Iseki [13].

Example 3.1. *Let $X = \{1, 2, 3, 5\}$ with the usual metric. Define mappings A and B on X by $A1 = B1 = 1$, $A3 = A5 = B2 = B3 = 2$ and $A2 = B5 = 3$. Then Theorem 2.6, Theorem 2.9 and Corollary 2.7 of Kang [5] and Theorem 4 of Khan-Khan-Sessa [7] are not applicable since A and B are not surjective. For any $h > 1$ we have*

$$d(A3, A5) = 0 \not> h = h \min\{d(3, A3), d(5, A5), d(3, 5)\}$$

and

$$d(A2, B5) = 0 \not> h = h \min\{d(2, A2), d(5, B5), d(2, 5)\}.$$

That is, Theorem 3 of Rhoades [10] and Theorem 1, Theorem 2 and Theorem 3 of Wang-Li-Gao-Iseki [13] are not applicable.

Now we take that $S = A$, $T1 = T2 = T3 = 2$ and $T5 = 3$. Let

$$\phi(t_1, t_2, t_3, t_4) = h \min\{t_i : i = 1, 2, 3, 4\} \text{ for } t_1, t_2, t_3, t_4 \in \mathbb{R}^+$$

and

$$\psi(t) = \frac{t}{h} \text{ for } t \in \mathbb{R}^+,$$

where $h > 1$. Then $AX = SX$, $BX = TX$, A and S, B and T are commuting. For any x, $y \in X$ with $Sx \neq Ty$, we have

$$\begin{aligned} d(Ax, By) &\geq 0 \\ &= hd(Ax, Sx) \\ &= \phi\left(d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{d(Ax, Sx)d(By, Ty)}{d(Sx, Ty)} \right) \end{aligned}$$

and

$$\begin{aligned} \psi(d(Ax, By)) &= \frac{1}{h} d(Ax, By) \\ &\geq 0 \\ &= d(Ax, Sx) \\ &= \min\Big\{ d(Ax, Sx), d(By, Ty), d(Sx, Ty), \\ &\qquad\qquad \frac{d(Ax, Sx)d(By, Ty)}{d(Sx, Ty)} \Big\}. \end{aligned}$$

All hypothesis of Theorem 3.1 and Theorem 3.2 are therefore satisfied.

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