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SUBHARMONIC SOLUTIONS FOR NON-AUTONOMOUS SUBLINEAR SECOND ORDER HAMILTONIAN SYSTEMS

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Abstract. The purpose of this paper is to study the existence of subharmonic solutions for the following non-autonomous second order Hamiltonian systems

$$\ddot{u}(t) + \bigtriangledown F(t, u(t)) = 0 \qquad a.\ e.\ t \in R.$$

Some existence theorems are obtained by the minimax methods in critical point theory.

1. Introduction and Preliminaries

Consider the second order Hamiltonian systems

$$\ddot{u}(t) + \bigtriangledown F(t, u(t)) = 0 \qquad a.\ e.\ t \in R \tag{1}$$

where $F : R \times R^N \to R$ is $T-$periodic $(T > 0)$ in t for all $x \in R^N$, that is

$$F(t + T, x) = F(t, x) \tag{2}$$

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for all $x \in R^N$ and a. e. $t \in R$, and satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for each $x \in R^N$ and continuously differentiable in x for a. e. $t \in [0,T]$, and there exist $a \in L^1(R^+;R^+), b \in L^1(0,T;R^+)$, such that $|F(t,x)| \le a(|x|)b(t), |\bigtriangledown F(t,x)| \le a(|x|)b(t)$ for all $x \in R^N$ and a. e. $t \in R$.

A solution of problem (1) is called to be subharmonic if it is $kT-$periodic solution for some positive integer k.

A function $G : R^N \to R$ is called to be $(\lambda,\mu)-$subconvex if

$$G(\lambda(x+y)) \le \mu(G(x)+G(y))$$

for some $\lambda,\mu > 0$ and all $x,y \in R^N$.

Let $H^1_{kT} = \{u : [0,kT] \to R^N | u$ is absolutely continuous, $u(0) = u(kT)$ and $\dot{u} \in L^2(0,kT;R^N)\}$ is a Hilbert space with the norm defined by

$$\|u\| = [\int_0^{kT} |u(t)|^2 dt + \int_0^{kT} |\dot{u}(t)|^2 dt]^{\frac{1}{2}}$$

and $\|u\|_\infty = \max_{0\le t\le kT} |u(t)|$ for $u \in H^1_{kT}$.

The corresponding functional φ_k on H^1_{kT} given by

$$\varphi_k(u) = \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - \int_0^{kT} F(t,u(t))dt$$

is continuously differentiable and weakly lower semi-continuous on H^1_{kT} (see [3]). Moreover one has

$$< \varphi_k'(u), v >= \int_0^{kT} [(\dot{u}(t),\dot{v}(t)) - (\bigtriangledown F(t,u(t)),v(t))]dt$$

for all $u,v \in H^1_{kT}$. It is well known that the $kT-$periodic solutions of problem (1) correspond to the critical points of functional φ_k.

For $u \in H^1_{kT}$, let $\overline{u} = (kT)^{-1}\int_0^{kT} u(t)dt$ and $\widetilde{u}(t) = u(t) - \overline{u}$. Then one has Sobolev's inequality

$$\|\widetilde{u}\|^2_\infty \le \frac{kT}{12}\int_0^{kT} |\dot{u}(t)|^2 dt \tag{3}$$

and Wertinger's inequality

$$\int_0^{kT} |\widetilde{u}(t)|^2 dt \le \frac{k^2T^2}{4\pi^2}\int_0^{kT} |\dot{u}(t)|^2 dt. \tag{4}$$

Under the conditions that there exists $h \in L^1(0,T;R^+)$ such that

$$|\bigtriangledown F(t,x)| \le h(t) \tag{5}$$

for all $x \in R^N$ and a. e. $t \in [0,T]$, and that

$$\int_0^T F(t,x)dt \to +\infty \tag{6}$$

as $|x| \to +\infty$, the existence of $T-$periodic solutions is proved in [3]. Meanwhile, [2] proves that problem has infinitely distinct subharmonic solutions under (5) and the condition that

$$F(t,x) \to +\infty \tag{7}$$

as $|x| \to +\infty$ uniformly for $t \in [0,T]$. Motivated by the results of [3, 2], a natural question is whether problem (1) has infinitely distinct subharmonic solutions under (5) and (6). In [1] a positive answer was given if in addition $F(t,x)$ is convex in x for every $t \in [0,T]$. Tang in [4] generalizes the existence result of $T-$periodic solutions in [3] to the sublinear case. The existence of $T-$periodic solutions is proved in [4] under the conditions that there exist $g, h \in L^1(0,T;R^+)$ and $\alpha \in [0,1)$ such that

$$|\bigtriangledown (F(t,x)| \le g(t)|x|^\alpha + h(t)$$

for all $x \in R^N$ and a. e. $t \in [0,T]$, and that

$$|x|^{-2\alpha} \int_0^{kT} F(t,x)dt \to +\infty$$

as $|x| \to +\infty$. Recently, Tang-Wu [5] considered the nonconvex case and generalized the existence result of subharmonic solutions to the sublinear case under a condition weaker than (6) but stronger than (7).

Inspired and motivated by the results due to Mawhin-Willem [3], F. Giannoni [2], Fonda-Ramos [1], Tang-Wu [5] and Zhao-Wu [6], in this paper, we shall continue to consider the existence of subharmonic solutions under some new conditions by using the least action principle and minimax methods. The results in this paper develop and generalize the corresponding results.

In the sequel, we set

$$e_k(t) = k(cosk^{-1}\omega t)x_0$$

for all $t \in R$ and some $x_0 \in R^N$ with $|x_0| = 1$, where $\omega = 2\pi/T$.

2. Main results and proof

Now we state and prove our main result.

Theorem 2.1. *Suppose that F satisfies assumption (A), (2) and the following conditions:*
(i) there exist $g, h \in L^1(0,T;R^+)$ and $\alpha \in [0,1)$ such that

$$|\bigtriangledown F(t,x)| \le g(t)|x|^\alpha + h(t) \tag{8}$$

for all $x \in R^N$ and a. e. $t \in [0,T]$;
(ii)

$$(\bigtriangledown F(t,u), e_k) \ge (se_k, e_k)$$

for all $u = x + se_k$ where $x \in R^N$ and $s \in (0,1)$ and a. e. $t \in [0,T]$;

(iii)

$$|x|^{-2\alpha}\int_0^T F(t,x)dt \to +\infty \qquad as \qquad |x| \to +\infty. \tag{9}$$

Then problem (1) has $kT-$periodic solutions $u_k \in H^1_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

Proof. Without loss of generality, we may assume that functions b in assumption (A) , g, h in (8) are $T-$ periodic and assumption (A) , (8) hold for all $t \in R$ by the $T-$ periodicity of $F(t,x)$ in the first variable.

First we prove that φ_k satisfies the (PS) condition. Suppose that $\{u_n\}$ is a (PS) sequence for φ_k, that is $\varphi'_k(u_n) \to 0$ as $n \to \infty$ and $\varphi_k(u_n)$ is bounded. By Wertinger's inequality, we have

$$\int_0^{kT} |\dot{u}(t)|^2 dt \le \|\widetilde{u}\|^2 \le (\frac{k^2T^2}{4\pi^2}+1)\int_0^{kT} |\dot{u}(t)|^2 dt. \tag{10}$$

In the same way in [5], we have

$$\begin{aligned} |\int_0^{kT} (\bigtriangledown F(t,u(t)),\widetilde{u}(t))dt| \le \frac{1}{4}\int_0^{kT} |\dot{u}(t)|^2 dt + C_1|\overline{u}|^{2\alpha} \\ + C_2(\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{\alpha+1}{2}} + C_3(\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{1}{2}} \end{aligned} \tag{11}$$

for all $u \in H^1_{kT}$ and some positive constants C_1, C_2 and C_3. Hence one has

$$\begin{aligned} \|\widetilde{u}_n(t)\| &\ge < \varphi'_k(u_n), \widetilde{u}_n > \\ &= \int_0^{kT} |\dot{u}_n(t)|^2 dt - \int_0^{kT} (\bigtriangledown F(t,u_n(t)),\widetilde{u}_n(t))dt \\ &\ge \frac{3}{4}\int_0^{kT} |\dot{u}(t)|^2 dt - C_1|\overline{u}_n|^{2\alpha} \\ &\quad - C_2(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{\frac{\alpha+1}{2}} - C_3(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{\frac{1}{2}} \end{aligned} \tag{12}$$

for large n. By (10) and the above inequality we have

$$C|\overline{u}_n|^\alpha \ge (\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{\frac{1}{2}} - C_4 \tag{13}$$

for some constants $C > 0, C_4 > 0$ and all large n, which implies that

$$(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{\frac{1}{2}} \le C_5(|\overline{u}_n|^\alpha + 1) \tag{14}$$

and moreover that

$$\|\widetilde{u}_n\|_\infty \le C_5(|\overline{u}_n|^\alpha + 1) \tag{15}$$

for large n and some positive constants C_5 by Sobolev's inequality and (10). Then it follows from Sobolev's inequality one has

$$\begin{aligned}
&|\int_0^{kT}[F(t,u_n(t))-F(t,\overline{u}_n)]dt| \\
&\leq \int_0^{kT}\int_0^1 |\bigtriangledown F(t,\overline{u}_n+s\widetilde{u}_n)|\cdot|\widetilde{u}_n|dsdt \\
&\leq \int_0^{kT}\int_0^1 g(t)|\overline{u}_n+s\widetilde{u}_n|^\alpha\cdot|\widetilde{u}_n|dsdt+\int_0^{kT}\int_0^1 h(t)|\widetilde{u}_n|dsdt \\
&\leq \int_0^{kT}2g(t)\int_0^1(|\overline{u}_n|^\alpha+s^\alpha|\widetilde{u}_n|^\alpha)|\widetilde{u}_n|dsdt+\int_0^{kT}h(t)|\widetilde{u}_n|dt \\
&\leq 2(|\overline{u}_n|^\alpha+\frac{1}{\alpha+1}\|\widetilde{u}_n\|_\infty^\alpha)\|\widetilde{u}_n\|_\infty\int_0^{kT}g(t)dt+\|\widetilde{u}_n\|_\infty\int_0^{kT}h(t)dt \qquad (16)\\
&\leq \frac{3}{kT}\|\widetilde{u}_n\|_\infty+\frac{kT}{3}|\overline{u}_n|^{2\alpha}(\int_0^{kT}g(t)dt)^2 \\
&\quad+\frac{2}{\alpha+1}\|\widetilde{u}_n\|_\infty^{\alpha+1}\int_0^{kT}g(t)dt+\|\widetilde{u}_n\|_\infty\int_0^{kT}h(t)dt \\
&\leq \frac{1}{4}\int_0^{kT}|\dot{u}_n(t)|^2dt+C_6|\overline{u}_n|^{2\alpha}+C_7(\int_0^{kT}|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}} \\
&\quad+C_8(\int_0^{kT}|\dot{u}_n(t)|^2dt)^{\frac{1}{2}}
\end{aligned}$$

for large n. By (14) and (16) and the boundedness of $\{\varphi_k(u_n)\}$, we have

$$\begin{aligned}
C_9 &\leq \varphi_k(u_n) \\
&= \frac{1}{2}\int_0^{kT}|\dot{u}_n(t)|^2dt-\int_0^{kT}[F(t,u_n)-F(t,\overline{u}_n)]dt-\int_0^{kT}F(t,\overline{u}_n)dt \\
&\leq \frac{1}{2}\int_0^{kT}|\dot{u}_n(t)|^2dt+|\int_0^{kT}[F(t,u_n)-F(t,\overline{u}_n)]dt|-\int_0^{kT}F(t,\overline{u}_n)dt \\
&\leq \frac{3}{4}\int_0^{kT}|\dot{u}_n(t)|^2dt+C_6|\overline{u}_n|^{2\alpha}+C_7(\int_0^{kT}|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}} \\
&\quad+C_8(\int_0^{kT}|\dot{u}_n(t)|^2dt)^{\frac{1}{2}}-k\int_0^{T}F(t,\overline{u}_n)dt \\
&\leq C_5^2(|\overline{u}_n|^\alpha+1)^2+C_6|\overline{u}_n|^{2\alpha}+C_7C_5(|\overline{u}_n|^\alpha+1)^{\alpha+1} \\
&\quad+C_8C_5(|\overline{u}_n|^\alpha+1)-k\int_0^{T}F(t,\overline{u}_n)dt
\end{aligned}$$

$$\leq C_{10}|\overline{u}_n|^{2\alpha} + C_{11}|\overline{u}_n|^{\alpha} + C_{12} - k\int_0^T F(t,\overline{u}_n)dt$$
$$= |\overline{u}_n|^{2\alpha}[\frac{-k}{|\overline{u}_n|^{2\alpha}}\int_0^T F(t,\overline{u}_n)dt + C_{10} + \frac{C_{11}}{|\overline{u}_n|^{\alpha}} + \frac{C_{12}}{|\overline{u}_n|^{2\alpha}}]$$

for all large n and some real constants C_{10}, C_{11} and C_{12}. The above inequality and condition (9) imply that $\{|\overline{u}_n|\}$ is bounded. Hence $\{u_n\}$ is bounded from (14). Arguing then as in Proposition 4.1 in [3], we conclude that the (PS) condition is satisfied.

To complete our theorem, we now prove that φ_k satisfies the other conditions of the saddle point theorem.

Since

$$|x|^{-2\alpha}\int_0^T F(t,x)dt \to +\infty$$

as $|x| \to +\infty$, so for every $\beta > 0$ there exists $M \geq 1$ such that

$$|x|^{-2\alpha}\int_0^T F(t,x)dt \geq \beta \tag{17}$$

which implies that

$$\int_0^T F(t,x)dt \geq \beta M^{2\alpha} \tag{18}$$

for all $|x| \geq M$.

For $e_k(t) = k(cosk^{-1}\omega t)x_0$ we have

$$\dot{e}_k(t) = -\omega(sink^{-1}\omega t)x_0$$

for all $t \in R$ which implies that

$$\int_0^{kT} |\dot{e}_k(t)|^2 dt = \frac{1}{2}kT\omega^2.$$

Hence one has

$$\varphi_k(x+e_k) = \frac{1}{4}kT\omega^2 - \int_0^{kT} F(t, x + k(cosk^{-1}\omega t)x_0)dt$$

for all $x \in R^N$. So by (18) one has

$$\varphi_k(x+e_k) = \frac{1}{4}kT\omega^2 - \sum_{i=0}^{k-1}\int_0^T F(t, x + k(cosk^{-1}\omega(t+iT))x_0)dt$$
$$\leq \frac{1}{4}kT\omega^2 - k\beta M^{2\alpha}$$

for all $|x| \geq M + k$, which implies that

$$\varphi_k(x+e_k) \to -\infty \tag{19}$$

as $|x| \to +\infty$ by the arbitrariness of β.

On the other hand, we have

$$\varphi_k(u) \to +\infty \tag{20}$$

as $\|u\| \to \infty$ in $\widetilde{H}^1_{kT} = \{u \in H^1_{kT} | \overline{u} = 0\}$. In fact, in a similar way to (16) we have

$$\begin{aligned} &|\int_0^{kT} [F(t,u(t)) - F(t,0)]dt| \\ &\le C_{13}(\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{\alpha+1}{2}} + C_{14}(\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{1}{2}} \end{aligned}$$

for all $u \in \widetilde{H}^1_{kT}$ and some positive constants C_{13} and C_{14}. Hence we have

$$\begin{aligned} \varphi_k(u) &= \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - \int_0^{kT} [F(t,u(t)) - F(t,0)]dt - \int_0^{kT} F(t,0)dt \\ &\ge \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - C_{13}(\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{\alpha+1}{2}} \\ &\quad - C_{14}(\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{1}{2}} - \int_0^{kT} F(t,0)dt \end{aligned}$$

for all $u \in \widetilde{H}^1_{kT}$. By Wertinger's inequality, one has

$$\|u\| \to \infty \Leftrightarrow \|\dot{u}\|_2 \to \infty$$

on $\widetilde{H}^1_{kT}$. Hence (20) follows from the above inequality.

So by (19), (20) and the saddle point Theorem (see Theorem 4.6 in [3]), there exists a critical point $u_k \in \widetilde{H}^1_{kT}$ for φ_k such that

$$-\infty < \inf_{\widetilde{H}^1_{kT}} \varphi_k \le \varphi_k(u_k) \le \sup_{R^N + e_k} \varphi_k.$$

Now we prove that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$. By condition (ii) we have

$$\begin{aligned} &k^{-1}\varphi_k(x + e_k) \\ &\le \frac{1}{4}T\omega^2 - k^{-1}\int_0^{kT} [F(t, x + e_k) - F(t,x)]dt - k^{-1}\int_0^{kT} F(t,x)dt \\ &= \frac{1}{4}T\omega^2 - k^{-1}\int_0^{kT}\int_0^1 (\bigtriangledown F(t, x + se_k), e_k)dsdt - \int_0^T F(t,x)dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}T\omega^2 - \int_0^{kT} cos(k^{-1}\omega t) \int_0^1 (\bigtriangledown F(t, x + se_k), x_0))dsdt - \int_0^T F(t, x)dt \\
&\leq \frac{1}{4}T\omega^2 - \int_0^{kT} cos(k^{-1}\omega t) \int_0^1 (se_k, x_0))dsdt - \int_0^T F(t, x)dt \\
&\leq \frac{1}{4}T\omega^2 - \frac{k}{2}\int_0^{kT} cos^2(k^{-1}\omega t)dt - \int_0^T F(t, x)dt \\
&= \frac{1}{4}T\omega^2 - \frac{Tk^2}{4} - \int_0^T F(t, x)dt
\end{aligned}
\tag{21}
$$

Hence for $\beta = 1$ there is some $M > 1$ by assumption (A) and condition (iii) there exists some constant C such that

$$k^{-1}\varphi_k(x + e_k) \leq C - \frac{Tk^2}{4}$$

for all $x \in R^N$ and all k. Hence one has

$$\sup_{x\in R^N} k^{-1}\varphi_k(x + e_k) \leq C - \frac{Tk^2}{4}$$

for all k, so we obtain

$$\limsup_{k\to+\infty} \sup_{x\in R^N} k^{-1}\varphi_k(x + e_k) = -\infty. \tag{22}$$

Then following the same way in [5] we complete our proof.

□

Theorem 2.2. *Suppose that F satisfies assumption (A), (2) and the following conditions:*
(i) there exists a function $\gamma \in L^1(0, T; R)$ with $\int_0^T \gamma(t)dt > 0$ and $\alpha \in [1, 2)$ such that

$$(\bigtriangledown F(t, x) - \bigtriangledown F(t, y), x - y) \leq \gamma(t)|x - y|^\alpha \tag{23}$$

for all $x, y \in R^N$ and a. e. $t \in [0, T]$;
(ii) $F(t, \cdot)$ is $(\lambda, \mu)-$subconvex, and $\bigtriangledown F(t, 0) = 0$, and there exist g, $h \in L^1(0, T; R^+)$ such that

$$F(t, x) \leq g(t)|x|^2 + h(t) \tag{24}$$

for all $x \in R^N$ and a. e. $t \in [0, T]$;
(iii)

$$(\bigtriangledown F(t, u), e_k) \geq (se_k, e_k)$$

for all $u = x + se_k$ where $x \in R^N$ and $s \in (0, 1)$ and a. e. $t \in [0, T]$;

(iv) assume that $a(t)$ is bounded and that

$$\int_0^T F(t,x)dt \to +\infty \quad as \quad |x| \to +\infty. \tag{25}$$

Then problem (1) has $kT-$periodic solutions $u_k \in H^1_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

Proof. Without loss of generality, we may assume that γ in (23) and g, h in (24) are $T-$ periodic and assumption (A) , (23) and (24) hold for all $t \in R$ by the $T-$ periodicity of $F(t,x)$ in the first variable.

Let us prove that φ_k satisfies the (PS) condition. Suppose that $\{u_n\}$ is a (PS) sequence for φ_k. As $a(t)$ is bounded function, we can assume that $a_0 = \max_{t\in R^+} |a(t)| < +\infty$. By condition (i), (ii) and Sobolev's inequality, it follows that

$$\begin{aligned}
\|\widetilde{u}_n(t)\| &\geq < \varphi_k'(u_n), \widetilde{u}_n > \\
&= \int_0^{kT} |\dot{u}_n(t)|^2 dt - \int_0^{kT} (\bigtriangledown F(t,u_n(t)), \widetilde{u}_n(t))dt \\
&= \int_0^{kT} |\dot{u}_n(t)|^2 dt - \int_0^{kT} (\bigtriangledown F(t,u_n(t)) - \bigtriangledown F(t,\overline{u}_n), \widetilde{u}_n(t))dt \\
&\quad - \int_0^{kT} (\bigtriangledown F(t,\overline{u}_n), \widetilde{u}_n(t))dt \\
&\geq \int_0^{kT} |\dot{u}_n(t)|^2 dt - \int_0^{kT} \gamma(t)|\widetilde{u}_n(t)|^\alpha dt - a_0\|\widetilde{u}_n\|_\infty \int_0^{kT} b(t)dt \\
&\geq \int_0^{kT} |\dot{u}_n(t)|^2 dt - C_1'\|\widetilde{u}_n\|_\infty^{\alpha/2} - C_2'\|\widetilde{u}_n\|_\infty
\end{aligned}$$

for large n. By (10) and above inequality we have

$$C(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{\alpha/2} \geq \int_0^{kT} |\dot{u}_n(t)|^2 dt - C_1(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{1/2},$$

that is

$$(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{1/2} - C(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{\alpha/4} \leq C_2$$

which implies

$$\int_0^{kT} |\dot{u}_n(t)|^2 dt \leq C_3 \tag{26}$$

for large n and some constant C_3 as $\alpha \in [1,2)$. Then by the boundedness of $\{\varphi_k(u_n)\}$, condition (ii) and Sobolev's inequality one has

$$\begin{aligned} C_4 \le \varphi_k(u_n) &= \frac{1}{2}\int_0^{kT} |\dot{u}_n(t)|^2 dt - \int_0^{kT} F(t, u_n) \\ &\le \frac{1}{2}\int_0^{kT} |\dot{u}_n(t)|^2 dt - \frac{1}{\mu}\int_0^{kT} F(t, \lambda\overline{u}_n) dt + \int_0^{kT} F(t, -\widetilde{u}_n(t)) dt \\ &\le \frac{1}{2}\int_0^{kT} |\dot{u}_n(t)|^2 dt - \frac{1}{\mu}\int_0^{kT} F(t, \lambda\overline{u}_n) dt + \int_0^{kT} [g(t)|\widetilde{u}_n(t)|^2 + h(t)] dt \\ &\le \frac{1}{2}\int_0^{kT} |\dot{u}_n(t)|^2 dt - \frac{1}{\mu}\int_0^{kT} F(t, \lambda\overline{u}_n) dt + C_5\int_0^{kT} |\dot{u}_n(t)|^2 dt + C_6 \end{aligned} \tag{27}$$

for all large n and some constants C_4, C_5 and C_6. Hence by (26), (27) and (25) we obtain

$$|\overline{u}_n| \le C_7$$

for all large n and some constant C_7. Hence $\{u_n\}$ is a bounded sequence, and (PS) condition is satisfied.

Then the rest of proof continue as similar as in Theorem 1. We omit the details. So we complete our proof. □

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