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# THE D'ALEMBERT AND LOBACZEVSKI DIFFERENCE OPERATORS IN $X_{\lambda}$ SPACES 

Stefan Czerwik ${ }^{1}$ and Krzysztof Król ${ }^{2}$<br>${ }^{1}$ Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland e-mail: Stefan.Czerwik@polsl.pl,<br>${ }^{2}$ Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland e-mail: Krzysztof.Krol@polsl.pl


#### Abstract

In this paper, by simple calculations we find norms of the d'Alembert and Lobaczevski difference operators connected with the d'Alembert and Lobaczevski functional equations. Because of nonlinearity of these operators, we use a norm for a quadratic operator and introduce a new class of operators, which are a sum of a linear and a quadratic operator, and provide a norm for this class. As an example, we find norms of these operators in $X_{\lambda}$ spaces. Then, we study Pexider type generalizations of the d'Alembert and Lobaczevski difference operators in $X_{\lambda}^{4}$ spaces. This paper is based on the article "Cauchy and Pexider operators in some function spaces" by S. Czerwik and K. Dlutek [2] and its continuation in a certains sense. The aim of the paper is drawing a reader's attention to a problem of a boundedness of a quadratic operator.


## 1. Introduction

We will define a bounded quadratic operator analogously as a bounded linear operator. The Lobaczevski difference operator

$$
\mathcal{L}(f)(x, y)=f^{2}\left(\frac{x+y}{2}\right)-f(x) f(y)
$$

where $f$ belongs to a function space and $x, y$ are vectors from a linear space is a quadratic operator. Using introduced definitions we will show that it is

[^0]bounded and find its norm in $X_{\lambda}$ space, i.e. a space containing all functions $f: X \rightarrow Y$ such that $\|f\| \leq M_{f} e^{\lambda\|x\|}$, where $X, Y$ are normed vector spaces, $\lambda \geq 0$ and $M_{f}$ is a constant depending on $f$. Similarly, because of nonlinearity of d'Alembert difference operator
$$
A(f)(x, y)=f(x+y)+f(x-y)-2 f(x) f(y)
$$
where $f$ belongs to a function space and $x, y$ are vectors from a linear space, we will introduce a new class of operators that are a sum of a linear and a quadratic operator and provide a norm for this class. Then we will show that the d'Alembert difference operator belongs to this class. In this way, it will be possible to find a norm of the d'Alembert difference operator in some function spaces. As an example, we will find a norm in a $X_{\lambda}$ spaces. Then we will study Pexider type generalizations of the d'Alembert difference operator
$$
A_{P}((f, g, h, k))(x, y)=f(x+y)+g(x-y)-2 h(x) k(y)
$$
and Lobaczevski difference operator
$$
\mathcal{L}_{\mathcal{P}}((f, g, h, k))(x, y)=f\left(\frac{x+y}{2}\right) g\left(\frac{x+y}{2}\right)-h(x) k(y)
$$
where $f, g, h, k$ belong to a function space.
The $X_{\lambda}$ spaces were introduced by Stefan Czerwik (see [2]). Bielecki also studied similar spaces in his paper [1]. The result from [3] will be presented later in remarks.
Our aim is to prove that the norm of the d'Alembert difference operator is equal 4 and the norm of the Lobaczevski difference operator ie equal 2 (for norms defined below). In this paper we use a standard notation (especially from [2]). New symbols will be introduced in a text.
For more information concerning similar problems the reader is referred to [4], [5], [6].

## 2. Preliminaries

In this section we will recall a definition of a quadratic operator and provide a norm for it (the Lobaczevski difference operator is an example of a quadratic operator). Next we will introduce a class of operators which are a sum of a bounded linear and a bounded quadratic operator. Such an operator in $X_{\lambda}$ spaces is the d'Alembert difference operator.
Let $X, Y$ be vector spaces over a field $\mathbb{K}$.
Definition 2.1. An operator $Q: X \rightarrow Y$ is called quadratic if it fulfils the following equations:
(a) $\forall x, y \in X \quad Q(x+y)+Q(x-y)=2 Q(x)+2 Q(y)$,
(b) $\forall k \in \mathbb{K} \forall x \in X$
$Q(k x)=k^{2} Q(x)$.

Definition 2.2. A quadratic operator $Q: X \rightarrow Y$ is called bounded if

$$
\exists c>0 \forall x \in X \quad\|Q(x)\| \leq c\|x\|^{2} .
$$

A norm of a quadratic operator $Q: X \rightarrow Y$ is defined by

$$
\begin{equation*}
\|Q\|:=\inf \left\{c>0 \mid\|Q(x)\| \leq c\|x\|^{2}, x \in X\right\} \tag{2.1}
\end{equation*}
$$

If such a number c does not exists we define $\|Q\|:=\infty$.
By $\mathcal{B}_{Q}(X, Y)$ we denote a set of all quadratic operators $Q: X \rightarrow Y$ such that $\|Q\|<\infty$.

Remark 2.3. Analogously as for a bounded linear operator one can prove an alternative definition:

$$
\begin{equation*}
\|Q\|:=\sup \{\|Q(x)\| \mid x \in X,\|x\|=1\} . \tag{2.2}
\end{equation*}
$$

It is easy to prove that the $\left(\mathcal{B}_{Q}(X, Y),\|\cdot\|\right)$ space is a linear normed space. Now we are ready to define a linear-quadratic operator.
Definition 2.4. By $\mathcal{B}_{L Q}$ we denote the set

$$
\mathcal{B}_{L Q}(X, Y)=\left\{T \in Y^{X} \mid \exists L \in \mathcal{B}(X, Y) \wedge \exists Q \in \mathcal{B}_{Q}(X, Y) T=L+Q\right\} .
$$

Moreover, for all $T=L+Q \in \mathcal{B}_{L Q}(X, Y)$ we define

$$
\|T\|:=\|L\|+\|Q\| .
$$

An operator $T \in \mathcal{B}_{L Q}(X, Y)$ is called a bounded linear-quadratic operator.
Remark 2.5. A norm in the $\mathcal{B}_{L Q}(X, Y)$ space may be defined in different ways, especially by $\|T\|=\sqrt{\|L\|^{2}+\|Q\|^{2}}$ or $\|T\|=\max \{\|L\|,\|Q\|\}$. The provided definition has to satisfy well know axioms of a norm.

Let us notice that the space $\left(\mathcal{B}_{L Q},\|\cdot\|\right)$ is a linear normed space. The proof is just straightforward.

## 3. The d'Alembert and Lobaczevski difference operators

A standard symbol $\mathbb{C}$ denotes the set of complex numbers, for a set $X$ a symbol $\mathbb{C}^{X}$ denotes a set of all functions $f: X \rightarrow \mathbb{C}$.

Definition 3.1. Let $X$ be a linear normed space. The Lobaczevski difference operator $\mathcal{L}: \mathbb{C}^{X} \rightarrow \mathbb{C}^{X^{2}}$ is defined by

$$
\begin{equation*}
\mathcal{L}(f)(x, y):=f^{2}\left(\frac{x+y}{2}\right)-f(x) f(y), \quad x, y \in X \tag{3.1}
\end{equation*}
$$

Lemma 3.2. The Lobaczevski difference operator $\mathcal{L}: \mathbb{C}^{X} \rightarrow \mathbb{C}^{X^{2}}$ defined above is a quadratic operator.

Proof. Obviously, for $f \in \mathbb{C}^{X}, x, y \in X$ and $k \in \mathbb{K}$ we get

$$
\begin{aligned}
\mathcal{L}(k f)(x, y) & =(k f)^{2}\left(\frac{x+y}{2}\right)-k f(x) \cdot k f(y) \\
& =k^{2}\left(f^{2}\left(\frac{x+y}{2}\right)-f(x) f(y)\right)=k^{2} \mathcal{L}(f)(x, y) .
\end{aligned}
$$

Moreover, for $f, g \in \mathbb{C}^{X}, x, y \in X$ we obtain

$$
\begin{aligned}
\mathcal{L}(f+g)(x, y)+\mathcal{L}(f-g)(x, y) & =(f+g)^{2}\left(\frac{x+y}{2}\right)-(f+g)(x)(f+g)(y) \\
& +(f-g)^{2}\left(\frac{x+y}{2}\right)-(f-g)(x)(f-g)(y) \\
& =2 f^{2}\left(\frac{x+y}{2}\right)+2 g^{2}\left(\frac{x+y}{2}\right) \\
& -2 f(x) f(y)-2 g(x) g(y) \\
& =2 \mathcal{L}(f)(x, y)+2 \mathcal{L}(g)(x, y) .
\end{aligned}
$$

Thus the Lobaczevski difference operator is a quadratic operator, as claimed.

Definition 3.3. Let $X$ be a linear normed space. The d'Alembert difference operator $A: \mathbb{C}^{X} \rightarrow \mathbb{C}^{X^{2}}$ is defined by

$$
\begin{equation*}
A(f)(x, y):=f(x+y)+f(x-y)-2 f(x) f(y), \quad x, y \in X \tag{3.2}
\end{equation*}
$$

Lemma 3.4. Let $A: \mathbb{C}^{X} \rightarrow \mathbb{C}^{X^{2}}$ be the d'Alembert difference operator, then there exist a linear operator $L_{A}$ and a quadratic operator $Q_{A}$ such that

$$
\begin{equation*}
A(f)(x, y)=L_{A}(f)(x, y)+Q_{A}(f)(x, y), \quad x, y \in X \tag{3.3}
\end{equation*}
$$

Proof. From the definition we obtain

$$
\begin{aligned}
A(f)(x, y) & =f(x+y)+f(x-y)-2 f(x) f(y) \\
& =[f(x+y)+f(x-y)]+[-2 f(x) f(y)] \\
& =L_{A}(f)(x, y)+Q_{A}(f)(x, y),
\end{aligned}
$$

where $L_{A}: \mathbb{C}^{X} \rightarrow \mathbb{C}^{X^{2}}$ and $Q_{A}: \mathbb{C}^{X} \rightarrow \mathbb{C}^{X^{2}}$ are defined by:

$$
\begin{aligned}
L_{A}(f)(x, y) & :=f(x+y)+f(x-y), \\
Q_{A}(f)(x, y) & :=-2 f(x) f(y) .
\end{aligned}
$$

It is obvious that $L_{A}$ is linear and $Q_{A}$ is quadratic (a proof is simple and analogous as the proof of previous lemma). Consequently, A is a sum of a linear and a quadratic operator and the lemma holds.

Now we are ready to use previous definitions of a bounded quadratic and a bounded linear-quadratic operator. First, we will show that the Lobaczevski difference operator is a bounded quadratic operator in $X_{\lambda}$ spaces and then knowing that the d'Alembert difference operator is the sum of the linear operator $L_{A}$ and the quadratic operator $Q_{A}$ defined above, we will prove that d'Alembert difference operator is bounded and find its norm in $X_{\lambda}$ spaces.
4. The d'Alembert and the Lobaczevski difference operators in $X_{\lambda}$ SPaces

### 4.1. The $X_{\lambda}$ and $X_{\lambda}^{2}$ spaces.

Definition 4.1. Let $X$ and $Y$ be normed vector spaces, $\lambda \geq 0$. Let $X_{\lambda}$ be a set defined by

$$
X_{\lambda}:=\left\{f: X \rightarrow Y \mid\|f(x)\| \leq M_{f} e^{\lambda\|x\|}, x \in X\right\},
$$

where $M_{f}$ is a constant depending on $f$. Moreover, for all $f \in X_{\lambda}$ we define

$$
\|f\|:=\sup _{x \in X}\left\{e^{-\lambda\|x\|}\|f(x)\|\right\} .
$$

Remark 4.2. The $X_{\lambda}$ spaces with the norm defined above where considered by S. Czerwik and K. Dutek in [3] (see also [2]).

Clearly holds the following lemma.
Lemma 4.3. The $\left(X_{\lambda},\|\cdot\|\right)$ space, where $\|\cdot\|$ is the norm defined above, is a linear normed space.

Definition 4.4. Let $X$ and $Y$ be linear normed spaces, $\lambda \geq 0$. Let $X_{\lambda}^{2}$ be a set defined by

$$
X_{\lambda}^{2}:=\left\{g: X \times X \rightarrow Y \mid\|g(x, y)\| \leq M_{g} e^{\lambda(\|x\|+\|y\|)}, x, y \in X\right\}
$$

where $M_{g}$ is a constant depending on $g$. Moreover, for all $g \in X_{\lambda}^{2}$ we define

$$
\|g\|:=\sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}\|g(x, y)\|\right\} .
$$

Let us notice that $\left(X_{\lambda}^{2},\|\cdot\|\right)$ space is a linear normed space.
The next lemmas establish images of $X_{\lambda}$ space by the d'Alembert and Lobaczevski difference operators.
We assume that $Y=\mathbb{C}$.
Lemma 4.5. Let $A: \mathbb{C}^{X} \rightarrow \mathbb{C}^{X^{2}}$ be the d'Alembert difference operator. Then

$$
\forall f \in X_{\lambda} \quad A(f) \in X_{\lambda}^{2} .
$$

Proof. Let $f \in X_{\lambda}$. Then we obtain

$$
\begin{aligned}
|A(f)(x, y)| & =|f(x+y)+f(x-y)-2 f(x) f(y)| \\
& \leq|f(x+y)|+|f(x-y)|+2|f(x) f(y)| \\
& \leq M_{f} e^{\lambda\|x+y\|}+M_{f} e^{\lambda\|x-y\|}+2 M_{f}^{2} e^{\lambda(\|x\|+\|y\|)} \leq N_{f} e^{\lambda(\|x\|+\|y\|)},
\end{aligned}
$$

where $N_{f}=\max \left\{M_{f}, 2 M_{f}^{2}\right\}$. Thus the lemma holds.
Lemma 4.6. Let $\mathcal{L}: \mathbb{C}^{X} \rightarrow \mathbb{C}^{X^{2}}$ be the Lobaczevski difference operator. Then

$$
\forall f \in X_{\lambda} \quad \mathcal{L}(f) \in X_{\lambda}^{2}
$$

Proof. Let $f \in X_{\lambda}$. Then we obtain

$$
\begin{aligned}
|\mathcal{L}(f)(x, y)| & =\left|f^{2}\left(\frac{x+y}{2}\right)-f(x) f(y)\right| \\
& \leq\left|f\left(\frac{x+y}{2}\right)\right|^{2}+|f(x) f(y)| \\
& \leq M_{f}^{2} e^{2 \lambda\left\|\frac{x+y}{2}\right\|}+M_{f}^{2} e^{\lambda(\|x\|+\|y\|)} \leq 2 M_{f}^{2} e^{\lambda(\|x\|+\|y\|)}
\end{aligned}
$$

thus $\mathcal{L}(f) \in X_{\lambda}^{2}$ as claimed.

## 5. Boundness of the D'Alembert and Lobaczevski difference OPERATORS

We will prove
Theorem 5.1. Let $Y=\mathbb{C}$. The Lobaczevski difference operator $\mathcal{L}: X_{\lambda} \rightarrow X_{\lambda}^{2}$ defined by (3.1) belongs to $\mathcal{B}_{Q}\left(X_{\lambda}, X_{\lambda}^{2}\right)$ space and for all $f \in X_{\lambda}$ we have

$$
\|\mathcal{L}(f)\| \leq 2\|f\|^{2}
$$

Proof. We have

$$
\begin{aligned}
\|\mathcal{L}(f)\| & =\sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}\left|f^{2}\left(\frac{x+y}{2}\right)-f(x) f(y)\right|\right\} \\
& \leq \sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}\left(\left|f^{2}\left(\frac{x+y}{2}\right)\right|+|f(x) f(y)|\right)\right\} \\
& \leq \sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}\left|f^{2}\left(\frac{x+y}{2}\right)\right|\right\}+\sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}|f(x) f(y)|\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \leq \sup _{x, y \in X}\left\{e^{-2 \lambda\left\|\frac{x+y}{2}\right\|}\left|f\left(\frac{x+y}{2}\right)\right|^{2}\right\}+\sup _{x, y \in X}\left\{e^{-\lambda\|x\|}|f(x)| e^{-\lambda\|y\|}|f(y)|\right\} \\
& \leq \quad\left(\sup _{x, y \in X}\left\{e^{-\lambda\left\|\frac{x+y}{2}\right\|}\left|f\left(\frac{x+y}{2}\right)\right|\right\}\right)^{2}+\sup _{x \in X}\left\{e^{-\lambda\|x\|}|f(x)|\right\} \sup _{y \in X}\left\{e^{-\lambda\|y\|}|f(y)|\right\} \\
& \leq\|f\|^{2}+\|f\|^{2}=2\|f\|^{2} .
\end{aligned}
$$

Thus $\mathcal{L} \in \mathcal{B}_{Q}\left(X_{\lambda}, X_{\lambda}^{2}\right)$ as claimed.
Theorem 5.2. Let $Y=\mathbb{C}$. The d'Alembert difference operator $A: X_{\lambda} \rightarrow X_{\lambda}^{2}$ defined by (3.2) belongs to $\mathcal{B}_{L Q}\left(X_{\lambda}, X_{\lambda}^{2}\right)$ space and for all $f \in X_{\lambda}$ we have

$$
\|A(f)\| \leq 2\|f\|+2\|f\|^{2}
$$

Proof. In view of (3.3) we get that $A=L_{A}+Q_{A}$, where the linear operator $L_{A}: X_{\lambda} \rightarrow X_{\lambda}^{2}$ and the quadratic operator $Q_{A}: X_{\lambda} \rightarrow X_{\lambda}^{2}$ are given by:

$$
\begin{aligned}
L_{A}(f)(x, y) & :=f(x+y)+f(x-y), \\
Q_{A}(f)(x, y) & :=-2 f(x) f(y) .
\end{aligned}
$$

The operator $L_{A}$ is linear and for $f \in X_{\lambda}$ we obtain

$$
\begin{aligned}
\left\|L_{A}(f)\right\| & =\sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}|f(x+y)+f(x-y)|\right\} \\
& \leq \sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}(|f(x+y)|+|f(x-y)|)\right\} \\
& \leq \sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}|f(x+y)|\right\}+\sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}|f(x-y)|\right\} \\
& \leq \sup _{x, y \in X}\left\{e^{-\lambda\|x+y\|}|f(x+y)|\right\}+\sup _{x, y \in X}\left\{e^{-\lambda\|x-y\|}|f(x-y)|\right\}=2\|f\| .
\end{aligned}
$$

Thus $L_{A} \in \mathcal{B}\left(X_{\lambda}, X_{\lambda}^{2}\right)$. We shall show that $Q_{A}$ is bounded and $\left\|Q_{A}\right\|=2$. Let $f \in X_{\lambda}$, then

$$
\begin{aligned}
\left\|Q_{A}(f)\right\| & =\sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}|2 f(x) f(y)|\right\} \\
& =2 \sup _{x, y \in X}\left\{e^{-\lambda\|x\|}|f(x)| e^{-\lambda\|y\|}|f(y)|\right\} \\
& =2 \sup _{x \in X}\left\{e^{-\lambda\|x\|}|f(x)|\right\} \sup _{y \in X}\left\{e^{-\lambda\|y\|}|f(y)|\right\}=2\|f\|^{2} .
\end{aligned}
$$

Because $Q_{A}$ is quadratic and bounded so $Q_{A} \in \mathcal{B}_{Q}\left(X_{\lambda}, X_{\lambda}^{2}\right)$. In view of $A=L_{A}+Q_{A}$ we get that $A \in \mathcal{B}_{L Q}\left(X_{\lambda}, X_{\lambda}^{2}\right)$ and

$$
\|A(f)\|=\left\|L_{A}(f)+Q_{A}(f)\right\| \leq\left\|L_{A}(f)\right\|+\left\|Q_{A}(f)\right\| \leq 2\|f\|+2\|f\|^{2}
$$

## 6. Norms of the d'Alembert and Lobaczevski difference OPERATORS

In this part of paper we will find norms of the d'Alembert and Lobaczevski difference operators under some additional assumptions.

Theorem 6.1. Let $\mathbb{R}_{+} \subset X, Y=\mathbb{C}$ and $\|x\|=|x|$ for $x \in \mathbb{R}_{+}$. Then

$$
\|\mathcal{L}\|=2
$$

where $\mathcal{L}$ is given by (3.1).
Proof. Let $\left\{x_{n} \mid x_{n} \geq 0\right\}$ be a decreasing sequence such that

$$
\lim _{n \rightarrow \infty} x_{n}=0 .
$$

Let us define for $n \in \mathbb{N}$ a function $f_{n}$ by

$$
f_{n}(x):= \begin{cases}-e^{\lambda x_{n}}, & x=x_{n} \\ e^{2 \lambda x_{n}}, & x=2 x_{n} \\ e^{\frac{3}{2} \lambda x_{n}}, & x=\frac{3}{2} x_{n} \\ 0, & \text { otherwise }\end{cases}
$$

Clearly we have

$$
\left\|f_{n}(x)\right\| \leq e^{\lambda x_{n}} e^{\lambda\|x\|}, \quad x \in X
$$

so $f_{n} \in X_{\lambda}$ for all $n \in \mathbb{N}$. Moreover,

$$
e^{-\lambda\|x\|}\left\|f_{n}(x)\right\|= \begin{cases}1, & x \in\left\{x_{n}, \frac{3}{2} x_{n}, 2 x_{n}\right\} \\ 0, & \text { otherwise }\end{cases}
$$

so $\left\|f_{n}\right\|=1$ for all $n \in \mathbb{N}$. Then,

$$
\begin{aligned}
\left\|\mathcal{L}\left(f_{n}\right)\right\| & =\sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}\left|f_{n}^{2}\left(\frac{x+y}{2}\right)-f_{n}(x) f_{n}(y)\right|\right\} \\
& \geq e^{-3 \lambda\left\|x_{n}\right\|}\left|f_{n}^{2}\left(\frac{3}{2} x_{n}\right)-f_{n}\left(x_{n}\right) f_{n}\left(2 x_{n}\right)\right| \\
& =e^{-3 \lambda x_{n}}\left|e^{3 \lambda x_{n}}+e^{\lambda x_{n}} e^{2 \lambda x_{n}}\right|=2
\end{aligned}
$$

and

$$
\|\mathcal{L}\|:=\sup \left\{\|\mathcal{L}(f)\| \mid f \in X_{\lambda},\|f\|=1\right\} \geq\left\|\mathcal{L}\left(f_{n}\right)\right\| \geq 2
$$

In view of the previous lemma $\|\mathcal{L}\| \leq 2$, thus $\|\mathcal{L}\|=2$.
Theorem 6.2. Let $\mathbb{R}_{+} \subset X, Y=\mathbb{C}$ and $\|x\|=|x|$ for $x \in \mathbb{R}_{+}$. Then

$$
\|A\|=4
$$

where $A$ is given by (3.2).

Proof. Because $\|A\|=\left\|L_{A}\right\|+\left\|Q_{A}\right\|$, where $L_{A}$ and $Q_{A}$ are defined above, we will find $\left\|L_{A}\right\|\left(\left\|Q_{A}\right\|=2\right.$ as proved before).
Let $\left\{x_{n}\right\}$ be a decreasing sequence of nonnegative numbers such that

$$
\lim _{n \rightarrow \infty} x_{n}=0
$$

Let us define for $n \in \mathbb{N}$ a function $f_{n}$ by

$$
f_{n}(x):= \begin{cases}e^{2 \lambda x_{n}}, & x \in\left\{0,2 x_{n}\right\} \\ 0, & \text { otherwise }\end{cases}
$$

Clearly we have

$$
\left\|f_{n}(x)\right\| \leq e^{2 \lambda x_{n}} e^{\lambda\|x\|}, \quad x \in X
$$

so $f_{n} \in X_{\lambda}$ for all $n \in \mathbb{N}$. Moreover,

$$
e^{-\lambda\|x\|}\left\|f_{n}(x)\right\|= \begin{cases}e^{2 \lambda x_{n}}, & x=0 \\ 1, & x=2 x_{n} \\ 0, & \text { otherwise }\end{cases}
$$

Because the sequence $\left\{x_{n}\right\}$ is a decreasing sequence of nonnegative numbers convergent to 0 , we obtain that $e^{2 \lambda x_{n}}>1$, so $\left\|f_{n}\right\|=e^{2 \lambda x_{n}}$ for all $n \in \mathbb{N}$. Moreover,

$$
\begin{aligned}
\left\|L_{A}\left(f_{n}\right)\right\| & =\sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}\left|f_{n}(x+y)+f_{n}(x-y)\right|\right\} \\
& \geq e^{-2 \lambda x_{n}}\left|f_{n}\left(2 x_{n}\right)+f_{n}(0)\right|=e^{-2 \lambda x_{n}} 2 e^{2 \lambda x_{n}}=2
\end{aligned}
$$

Thus $\left\|L_{A}(f)\right\| \geq 2$. Now suppose that $\left\|L_{A}\right\|<2$, then there exists $\varepsilon>0$ such that

$$
\left\|L_{A}\left(f_{n}\right)\right\| \leq(2-\varepsilon)\left\|f_{n}\right\|, \quad f_{n} \in X_{\lambda}
$$

on the other hand, for $f_{n} \in X_{\lambda}$ we have

$$
2 \leq\left\|L_{A}\left(f_{n}\right)\right\| \leq(2-\varepsilon) e^{2 \lambda x_{n}}
$$

Let us notice that if $n \rightarrow \infty$ then $x_{n} \rightarrow 0$ and $e^{2 \lambda x_{n}} \rightarrow 1$, thus $(2-\varepsilon) e^{2 \lambda x_{n}} \rightarrow 2-\varepsilon$, so we get $2 \leq 2-\varepsilon$, where $\varepsilon>0$, which is impossible. Thus we obtain that $\left\|L_{A}\right\|=2$.
Because $\|A\|=\left\|L_{A}\right\|+\left\|Q_{A}\right\|$ we get $\|A\|=4$.
Remark 6.3. In the paper [3] S. Czerwik and K. Dlutek have proved that the Cauchy difference operator $C: X_{\lambda} \rightarrow X_{\lambda}^{2}$ defined by

$$
C(f)(x, y)=f(x+y)-f(x)-f(y), \quad x, y \in X
$$

is a linear bounded operator and for all $f \in X_{\lambda}$ it satisfies the inequality

$$
\|C(f)\| \leq 3\|f\|
$$

Moreover, if $\mathbb{R}_{+} \subset X, Y=\mathbb{C}$ and $\|x\|=|x|$ for $x \in \mathbb{R}_{+}$then $\|C\|=3$.

## 7. Remarks about Pexider type generalizations

7.1. The $X_{\lambda}^{3}$ and $X_{\lambda}^{4}$ spaces.

Definition 7.1. We define

$$
\begin{gathered}
X_{\lambda}^{3}:=\left\{(f, g, h) \mid f, g, h \in X_{\lambda}\right\} \\
\|(f, g, h)\|:=\max \{\|f\|,\|g\|,\|h\|\}
\end{gathered}
$$

Analogously,

$$
\begin{gathered}
X_{\lambda}^{4}:=\left\{(f, g, h, k) \mid f, g, h, k \in X_{\lambda}\right\}, \\
\|(f, g, h, k)\|:=\max \{\|f\|,\|g\|,\|h\|,\|k\|\} .
\end{gathered}
$$

It is obvious that $X_{\lambda}^{3}$ and $X_{\lambda}^{4}$ with norms provided above are vector normed spaces.

Remark 7.2. In the paper [3] S. Czerwik and K. Dutek have proved that the Pexider difference operator $P: X_{\lambda}^{3} \rightarrow X_{\lambda}^{2}$ defined by

$$
P((f, g, h))(x, y)=f(x+y)-g(x)-h(y), \quad x, y \in X
$$

is a linear bounded operator and for all $u \in X_{\lambda}^{3}$ it satisfies the inequality

$$
\|P(u)\| \leq 3\|u\|
$$

Moreover, if $\mathbb{R}_{+} \subset X, Y=\mathbb{C}$ and $\|x\|=|x|$ for $x \in \mathbb{R}_{+}$then $\|P\|=3$.

### 7.2. Pexider type generalization of the Lobaczevski difference operator.

Definition 7.3. Let $X$ be a linear normed space. The Pexider - Lobaczevski difference operator $\mathcal{L}_{P}:\left(\mathbb{C}^{X}\right)^{4} \rightarrow \mathbb{C}^{X^{2}}$ is defined by

$$
\begin{equation*}
\mathcal{L}_{P}((f, g, h, k))(x, y):=f\left(\frac{x+y}{2}\right) g\left(\frac{x+y}{2}\right)-h(x) k(y), \quad x, y \in X \tag{7.1}
\end{equation*}
$$

This operator is not quadratic. For $f=g=h=k$ we obtain the Lobaczevski difference operator which is quadratic. We will prove the following theorem.

Theorem 7.4. Let $Y=\mathbb{C}$. For all $u \in X_{\lambda}^{4}$ the Pexider - Lobaczevski difference operator $\mathcal{L}_{P}: X_{\lambda}^{4} \rightarrow X_{\lambda}^{2}$ defined by (7.1) satisfies the inequality

$$
\left\|\mathcal{L}_{P}(u)\right\| \leq 2\|u\|^{2}
$$

Proof. It is easy to show that $\forall u \in X_{\lambda}^{4} \quad \mathcal{L}_{P}(u) \in X_{\lambda}^{2}$. Take $u=(f, g, h, k)$, then we have by the definition

$$
\begin{aligned}
& \left\|\mathcal{L}_{P}((f, g, h, k))\right\| \\
= & \sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}\left|f\left(\frac{x+y}{2}\right) g\left(\frac{x+y}{2}\right)-h(x) k(y)\right|\right\} \\
\leq & \sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}\left|f\left(\frac{x+y}{2}\right) g\left(\frac{x+y}{2}\right)\right|\right\} \\
& +\sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}|h(x) k(y)|\right\} \\
\leq & \sup _{x, y \in X}\left\{e^{-2 \lambda\left\|\frac{x+y}{2}\right\|}\left|f\left(\frac{x+y}{2}\right)\right| \cdot\left|g\left(\frac{x+y}{2}\right)\right|\right\} \\
& +\sup _{x, y \in X}\left\{e^{-\lambda\|x\|}|h(x)| e^{-\lambda\|y\|}|k(y)|\right\} \\
\leq & \|f\|\|g\|+\|h\|\|k\|=2(\max \{\|f\|,\|g\|,\|h\|,\|k\|\})^{2}=2\|u\|^{2} .
\end{aligned}
$$

Corollary 7.5. Let $\mathbb{R}_{+} \subset X, Y=\mathbb{C}$ and $\|x\|=|x|$ for $x \in \mathbb{R}_{+}$. Then

$$
\inf \left\{c>0 \mid\left\|\mathcal{L}_{P}(u)\right\| \leq c\|u\|^{2}, \quad u \in X_{\lambda}^{4}\right\}=2
$$

where $\mathcal{L}_{P}$ is given by (7.1).
Proof. Assume on the contrary that

$$
d:=\inf \left\{c>0 \mid\left\|\mathcal{L}_{P}(u)\right\| \leq c\|u\|^{2}, \quad u \in X_{\lambda}^{4}\right\}<2 .
$$

Then for $f=g=h=k$, we get

$$
\left\|\mathcal{L}_{P}(u)\right\|=\|\mathcal{L}(f)\| \leq d\|(f, f, f, f)\|^{2}=d\|f\|^{2},
$$

whence

$$
\|\mathcal{L}(f)\| \leq d\|f\|^{2}
$$

By the hypothesis, $d<2$ and therefore we infer that $\|\mathcal{L}\|<2$, which is impossible in view of the previous lemma.

### 7.3. Pexider type generalization of the d'Alembert difference operator.

Definition 7.6. Let $X$ be a linear normed space. The Pexider - d'Alembert difference operator $A_{P}:\left(\mathbb{C}^{X}\right)^{4} \rightarrow \mathbb{C}^{X^{2}}$ is defined by

$$
\begin{equation*}
A_{P}((f, g, h, k))(x, y):=f(x+y)+g(x-y)-2 h(x) k(y), \quad x, y \in X \tag{7.2}
\end{equation*}
$$

We shall prove the following theorem

Theorem 7.7. Let $Y=\mathbb{C}$. For all $u \in X_{\lambda}^{4}$ the Pexider - d'Alembert difference operator $A_{P}: X_{\lambda}^{4} \rightarrow X_{\lambda}^{2}$ defined by (7.2) satisfies the inequality

$$
\left\|A_{P}(u)\right\| \leq 2\|u\|+2\|u\|^{2} .
$$

Proof. It is easy to show that $\forall u \in X_{\lambda}^{4} \quad A_{P}(u) \in X_{\lambda}^{2}$. Take $u=(f, g, h, k)$, then we have by the definition

$$
\begin{aligned}
& \left\|A_{P}((f, g, h, k))\right\| \\
= & \sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}|f(x+y)+g(x-y)-2 h(x) k(y)|\right\} \\
\leq & \sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}|f(x+y)|\right\}+\sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}|g(x-y)|\right\} \\
& +2 \sup _{x, y \in X}\left\{e^{-\lambda(\|x\|+\|y\|)}|h(x) k(y)|\right\} \\
\leq & \sup _{x, y \in X}\left\{e^{-\lambda\|x+y\|}|f(x+y)|\right\}+\sup _{x, y \in X}\left\{e^{-\lambda\|x-y\|}|g(x-y)|\right\} \\
& +2 \sup _{x, y \in X}\left\{e^{-\lambda\|x\|}|h(x)| e^{-\lambda\|y\|}|k(y)|\right\} \\
\leq & \|f\|+\|g\|+2\|h\|\|k\| \\
= & 2 \max \{\|f\|,\|g\|,\|h\|,\|k\|\}+2(\max \{\|f\|,\|g\|,\|h\|,\|k\|\})^{2} \\
= & 2\|u\|+2\|u\|^{2} .
\end{aligned}
$$

Definition 7.8. In $X_{\lambda}^{2}:=\left\{(f, g) \mid f, g \in X_{\lambda}\right\}$ we define

$$
\|(f, g)\|:=\max \{\|f\|,\|g\|\}
$$

Corollary 7.9. Let $Y=\mathbb{C}$. For all $u \in X_{\lambda}^{2}$ the difference operator $L_{P}: X_{\lambda}^{2} \rightarrow X_{\lambda}^{2}$ defined by

$$
\begin{equation*}
L_{P}((f, g))(x, y):=f(x+y)+g(x-y), \quad x, y \in X \tag{7.3}
\end{equation*}
$$

is linear and satisfies the inequality

$$
\left\|L_{P}(u)\right\| \leq 2\|u\| .
$$

The proof is simple and analogous as the proof of the previous theorem.
Corollary 7.10. Let $\mathbb{R}_{+} \subset X, Y=\mathbb{C}$ and $\|x\|=|x|$ for $x \in \mathbb{R}_{+}$. Then

$$
\left\|L_{P}\right\|=2 .
$$

Proof. Assume on the contrary that $\left\|L_{P}\right\|<2$. Then for $f=g$, we get

$$
\left\|L_{P}((f, g))\right\|=\left\|L_{A}(f)\right\| \leq\left\|L_{P}\right\|\|(f, f)\|=\left\|L_{P}\right\|\|f\|
$$

whence $\left\|L_{A}(f)\right\| \leq\left\|L_{P}\right\|\|f\|$. By the hypothesis, $\left\|L_{P}\right\|<2$ and therefore we infer that $\left\|L_{A}\right\|<2$, which is impossible in view of Theorem 6.2.

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