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EXISTENCE AND STABILITY OF SOLUTIONS OF GENERAL STOCHASTIC INTEGRAL EQUATIONS

K. Balachandran[1], K. Sumathy[2] and J. K. Kim[3]

[1,2] Department of Mathematics,Bharathiar University
Coimbatore 641 046, India
e-mail: balachandran-k@lycos.com

[3] Department of Mathematics Education, Kyungnam University
Masan, Kyungnam 631-701, Korea
e-mail: jongkyuk@kyungnam.ac.kr

Abstract. In this paper we establish a set of sufficient conditions for the existence of random solutions and their asymptotic behaviour of stochastic integral equations. We use the notion of measure of noncompactness in a Banach space and fixed point theorem of Darbo type.

1. Introduction

The importance of random differential and integral equations are characterized in many social, physical, biological and engineering problems. Theory of stochastic differential and integral equations may be found in several papers and monographs [3, 4, 6, 7, 8, 9, 10, 11, 14, 15, 16]. One of the most important problem examined up to now is that concerning the existence of solutions of considered equations. The basic tools used in solving this problem are mostly the method of successive approximations or the Banach fixed point principle [8, 9, 10, 11, 14, 15]. The idea used in this papers are based on the notion of the measure of noncompactness in a Banach space and the fixed point theorem of Darbo type[1, 2, 5]. We construct first the real Banach space of tempered functions and next define the measure of noncompactness on that space where we are searching for solutions of considered equations. This approach allows us to find weaker conditions than that of the papers [6, 9, 10, 11, 12, 13, 14, 15, 16]. We replace the Lipschitz type conditions by

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those involving sublinear functions. The aim of this paper is to discuss the problem of existence and stability of solutions of a class of general stochastic integral equation.

2. Preliminaries

Throughout this paper $\mathcal{H}$ will denote an infinite dimensional real Banach space with norm $\|\cdot\|$ and the zero element 0. $K(x,r)$ stands for the closed ball centered at x of radius r. Denote by $\mathcal{M}_{\mathcal{H}}$ the family of all nonempty bounded subsets of $\mathcal{H}$ and by $\mathcal{N}_{\mathcal{H}}$ the family of all relatively compact and nonempty subset of $\mathcal{H}$.

The following axioms defining a measure of noncompactness are taken from Banas and Goebel [2].

Definition 2.1. A nonempty family $P \subset \mathcal{N}_{\mathcal{H}}$ is said to be kernel (of measure of noncompactness), provided it satisfies the following conditions:

(a) $U \in P \Rightarrow \bar{U} \in P$,
(b) $U \in P, V \subset U, V \neq \phi \Rightarrow V \in P$,
(c) $U, V \in P \Rightarrow \lambda U + (1-\lambda)V \in P, \lambda \in [0,1]$,
(d) $U \in P \Rightarrow \mathrm{Conv}U \in P$,
(e) P^c (the subfamily of P consisting of all closed sets) is closed in $\mathcal{N}^c_{\mathcal{H}}$ with respect to the topology generated by Hausdorff metric.

Definition 2.2. The function $\mu : \mathcal{M}_{\mathcal{H}} \to [0, +\infty)$ is said to be a measure of noncompactness on $(\mathcal{H}, \mathcal{M}_{\mathcal{H}})$with kernel $P(\ker\mu = P)$ if it satisfies the following condition:

(b1) $\mu(U) = 0 \Leftrightarrow U \in P$;
(b2) $\mu(U) = \mu(\bar{U})$;
(b3) $\mu(\mathrm{Conv}U) = \mu(U)$;
(b4) $U \subset V \Rightarrow \mu(U) \leq \mu(V)$;
(b5) $\mu(\lambda U + (1-\lambda)V) \leq \lambda\mu(U) + (1-\lambda)\mu(V), \lambda \in [0,1]$;
(b6) if $U_n \in \mathcal{M}_{\mathcal{H}},\ \bar{U}_n = U_n$ and $U_{n+1} \subset U_n, n = 1, 2, ...,$ and if

$$\lim_{n\to\infty} \mu(U_n) = 0, \text{ then } U = \bigcap_{n=1}^{\infty} U_n \neq \phi.$$

If a measure of noncompactness μ satisfies in addition the following two conditions:

(b7) $\mu(U+V) \leq \mu(U) + \mu(V),\ U+V = \{z : z = x+y,\ x \in U, y \in V\}$;
(b8) $\mu(\lambda U) = |\lambda|\,\mu(U), \lambda \in R$;

it will be sublinear.

Let $\mathcal{M} \subset \mathcal{H}$ be a nonempty set and let μ be a measure of noncompactness on $(\mathcal{H}, \mathcal{M}_{\mathcal{H}})$.

Definition 2.3. We say that a continuous mapping $T : \mathcal{M} \to \mathcal{H}$ is a contraction with respect to μ(μ-contraction) if for any set $U \in \mathcal{M}_{\mathcal{H}}$ its image $TU \in \mathcal{M}_{\mathcal{H}}$, and there exists a constant $k \in [0, 1)$ such that $\mu(TU) \leq k\mu(U)$.

We shall use the following modified version of the fixed point theorem of Darbo type .

Theorem 2.4. Let C be a nonempty, closed, convex and bounded subset of $\mathcal{H}$ and let $T : C \to C$ be an arbitrary μ contraction on $(\mathcal{H}, \mathcal{M}_{\mathcal{H}})$. Then T has at least one fixed point in C and the set Fix $T = \{x \in C : Tx = x\}$ of all fixed points of T belongs to ker μ.

Let $p(\ \cdot\ ) \in L_1([0, +\infty)) = \ L_1([0, +\infty)A, \nu)$ be a positive function.
By $L_1^p(R_+, L_2(\Omega, A, P), p)$ (or shortly L_1^p) we mean a space of all integrable with respect to the Lebesgue measure functions $x := x(t;\ \cdot)$ on R_+ with values $X(t)$ being random variables in $L_2(\Omega, A, P)$ and with the topology defined by the norm

$$\|x\|_p = \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \|x(s)\|_{L_2} \, d\nu(t) < \infty.$$

where $\nu - \text{ess } \sup_{s\in[0,t]}\ \|x(s)\|_{L_2}$ is taken with respect to the Lebesgue measure ν.
The space L_1^p with norm $\|\ \|_p$ is a real Banach space [3].

Now let $x \in L_1^p(R_+, L_2(\Omega, A, P), p)$, $\chi \in \mathcal{M}_{L_1^p}$. Define the measure of noncompactness μ' on $(L_1^p, \mathcal{M}_{L_1^p})$ as follows

$$\begin{aligned}\mu'(\chi) \ &= \ \lim_{\epsilon\to 0} \sup_{x\in\chi} \int_0^\infty \|p(t+\epsilon)X(t+\epsilon) - p(t)X(t)\|_{L_2} \, d\nu(t) \\ &\quad + \lim_{A\to\infty} \sup_{x\in\chi} \int_A^\infty p(t) \|X(t)\|_{L_2} \, dv(t) \ + \sup\{p(t)\ m(\chi(t)) :\ t \geq 0\},\end{aligned}$$

where m is a measure of noncompactness on $(L_2(\Omega, A, P), \mathcal{M}_{L_2(\Omega,A,P)})$,

$$\chi(t) = \left\{ x \in \chi \subset K(0, r)\ : \mu{-}ess \sup_{s\in[0,t]} \|X(s)\|_{L_2}\ \leq\ \|\chi\|_p / p(t) \right\} \text{ and } t \geq 0.$$

The function μ' define S sublinear measure of noncompactness on L_1^p. It is also known that kernel P' (ker $\mu'\ = P'$) being a family of all sets $U \in \mathcal{M}_{L_1^p}$ such that functions belonging to U are uniformly integrable, i.e.

$$\lim_{A\to\infty} \sup_{x\in U} \int_A^\infty p(t) \|X(t)\|_{L_2} \, d\nu(t) \ = \ 0,$$

and

$$\lim_{\epsilon\to 0}\sup_{x\in U}\int_0^\infty \|p(t+\epsilon)X(t+\epsilon)-p(t)X(t)\|_{L_2}\,d\nu(t) \;=\; 0.$$

3. Existence Theorem

Consider the general stochastic integral equation of the form

$$\begin{aligned} X(t;w) \;&=\; h(t,X(t;w))+\sum_{i=1}^{n}\int_0^t f_i(t,s,X(s;w);w)ds \\ &\quad +\sum_{j=1}^{m}\int_0^t g_j(t,s,X(s;w);w)d\beta(s;w), \quad t\geq\ 0, \end{aligned} \tag{3.1}$$

where

(i) $w\in\Omega$, the supporting set of the complete probability measure space (Ω,A,P) with A being the σ-algebra and P probability measure,
(ii) $X(t;w)$ is the unknown random process,
(iii) $h(t,X)$ is a map from $R_+\times R$ into R,
(iv) $f_i(t,s,X;w)$, $i=1,2,...,n$, $g_j(t,s,X;w)$, $j=1,2,...,m$, are maps from $R_+\times R_+\times R$ into R,
(v) $t\in R_+$ and $\beta(t;w)$ is a martingale process.

The first integral in the equation (3.1) is an ordinary Lebesgue integral with probabilistic characterization, while the second part is an Ito-Doob stochastic integral.

With respect to the random process $\beta(t;w)$ we shall assume that for each $t\in R_+$, a minimal σ -algebra A_t, $A_t\subset A$, is such that $\beta(t;w)$ is measurable with respect to A_t. In addition, we shall assume that the minimal σ -algebra A_t is an increasing family such that

(i) the random process $\{\beta(t;w),A_t : t\in R_+\}$ is a real martingale and
(ii) there is a real continuous non-decreasing function, $F(t)$, such that for $s,t\in R_+$ and $s<t$

we have $E\{|\beta(t;w)-\beta(s;w)|^2\}=E\{|\beta(t;w)-\beta(s;w)|^2|A_t\}=F(t)-F(s)P$-a.s. where E denotes the expected value of the random process.

Definition 3.1. A process $X(t;w)$ such that $\|x(t)\|_{L_2}\in L_1([0,\infty))$and satisfying (3.1) a.s. said to be a random solution of that equation.

Definition 3.2. A random solution $X(t;w)$ is said to be asymptotically stable in mean square sense if

$$\lim_{T\to\infty}\int_T^\infty \|X(t)\|_{L_2}\,d\nu(t) \;=\; 0.$$

One can find conditions [3] under which there exists a unique strong solution of equation (3.1). Our method is based on the technique of measure of noncompactness and a fixed point theorem of Darbo type.

Theorem 3.3. Suppose that the functions f_i and g_j in equation (3.1) are sublinear,i.e.

(i) $|f_i(t,s,X(s;w);w)| \leq a_{1i}(t,s;w)\ |x(s;w)| + b_{1i}(t,s;w)P-\text{a.s.}$,
and
$|g_j(t,s,X(s;w);w)| \leq a_{2j}(t,s;w)\ |x(s;w)| + b_{2j}(t,s;w)P-\text{a.s.}$,

where nonnegative functions $a_{1i},\ a_{2j},\ b_{1i},\ b_{2j}, i=1,2,...,n\ , j=1,2,...,m$ are continous for $t\in R_+$, defined for $t,s\in R_+$ and belonging to $L_\infty(\Omega,A,P)$ with

$$\||a_{1i}(t,s)\|| = P-\text{ess}\sup_{w\in\Omega}|a_{1i}(t,s;w)|,\ i=1,2,...,n$$
$$\||a_{2j}(t,s)\|| = P-\text{ess}\sup_{w\in\Omega}|a_{2j}(t,s;w)|,\ j=1,2,...,m,$$

(ii) $|h(t,X(t;w))-h(t,Y(t;w))| \leq k|X(t;w)-Y(t;w)|$ P-a.s., for $k\in[0,1)$
Let the M_1 and N_1 be defined as follows

$$\begin{aligned} M_1 &= k+\sup_{t\in[0,\infty)}\sum_{i=1}^n\int_0^t \||a_{1i}(t,s)\||ds \\ &\quad + \sqrt{2}\sup_{t\in[0,\infty)}\sum_{j=1}^m\left\{\int_0^t \||a_{2j}(t,s)\||^2 dF(s)\right\}^{1/2}, \end{aligned}$$

$$\begin{aligned} N_1 &= \int_0^\infty p(t)\nu-\text{ess}\sup_{s\in[0,t]}|h(s,0)|d\nu(t) \\ &\quad + \int_0^\infty p(t)\nu-\text{ess}\sup_{s\in[0,t]}\sum_{i=1}^n\left[\int_0^s \||b_{1i}(s,s_1)\||ds_1\right]d\nu(t) \\ &\quad + \sqrt{2}\int_0^\infty p(t)\nu-\text{ess}\sup_{s\in[0,t]}\sum_{j=1}^m\left\{\int_0^s \||b_{2j}(s,s_1)\||^2 dF(s_1)\right\}^{1/2} d\nu(t), \end{aligned}$$

satisfy the inequalities

(iii) $0\leq M_1<1,\ N_1<\infty$,

(iv) the mapping $X(\cdot;w) \to f_i(\cdot, s, X(\cdot;w);w)$ and $X(\cdot;w) \to g_j(\cdot, s, X(\cdot;w);w)$ from $L_1^p(R_+, L_2(\Omega, A, P), p)$ into $L_1^p(R_+, L_2(\Omega, A, P), p)$ are continuous in the topology generated by the norm $\|\cdot\|_p$,

(v) $\|h(t, X(t)\|_{L_2} \in L_1([0, +\infty))$,

(vi) there exist $L_{1i},\ i = 1, 2, ..., n,\ L_{2j},\ j = 1, 2, ..., m,$ and L_3 satisfying such that

$$0 \leq \sum_{i=1}^{n} L_{1i} + \sum_{j=1}^{m} L_{2j} + L_3 < 1,$$

$$m\Big(\int_0^t f_i(t, s, U(s); w)ds\Big) \leq L_{1i} m(U(t)),\ i = 1, 2, ..., n,$$

$$m\Big(\int_0^t g_j(t, s, U(s); w)ds\Big) \leq L_{2j} m(U(t)),\ j = 1, 2, ..., m,$$

$$m\Big(h(t, U(t)\Big) \leq L_3 m(U(t)),$$

$U(t) = \{X(s) \in L_2(\Omega, A, P),\ s \geq 0,\ x \in U \subset K(0, r) : \nu - \text{ess}$ $\sup_{s\in[0,t]} \|X(s)\|_{L_2} \leq \|U\|_p / p(t)\},\ t \geq 0$ and $r = N_1/(1 - M_1)$.

Then there exists at least one solution $x \in L_1^p$ of equation (3.1) such that

$$\lim_{T\to\infty} \int_T^{\infty} p(t)\nu - \text{ess} \sup_{s\in[0,t]} \|X(s)\|_{L_2} d\nu(t) = 0.$$

Proof. For a process $X \in L_1^p$ define the process GX by

$$\begin{aligned}(GX)(t;w) &= h(t, X(t;w)) + \sum_{i=1}^{n} \int_0^t f_i(t, s, X(s;w);w)ds \\ &\quad + \sum_{j=1}^{m} \int_0^t g_j(t, s, X(s;w);w)d\beta(s;w). \end{aligned} \tag{3.2}$$

The assumption concerning $\beta(t;w)$ and $X \in L_1^p$ allow us to give the following estimate

$$\left\| \int_0^t a_{2j}(t, s)X(s)d\beta(s)\right\|_{L_2} \leq \left\{ \int_0^t |||a_{2j}(t, s)|||^2 \|X(s)\|_{L_2}^2 dF(s)\right\}^{1/2}. \tag{3.3}$$

Put

$$L_1(t) = \sum_{j=1}^{m} \int_0^t g_j(t, s, X(s;w);w)d\beta(s;w).$$

Now by (i)and (3.3) we obtain

$$\begin{aligned}
&\int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \|L_1(s)\|_{L_2} d\nu(t) \\
&= \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \left\| \sum_{j=1}^m \int_0^s g_j(s,s_1,X(s_1;w);w)d\beta(s_1;w) \right\|_{L_2} d\nu(t) \\
&\leq \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \left\| \sum_{j=1}^m \int_0^s \left\{ a_{2j}(s,s_1;w)|X(s_1;w)| \right.\right. \\
&\quad \left.\left. + b_{2j}(s,s_1;w) \right\} d\beta(s_1;w) \right\|_{L_2} d\nu(t) \\
&\leq \sqrt{2} \int_0^\infty p(t)\nu \\
&\quad - \text{ess} \sup_{s\in[0,t]} \sum_{j=1}^m \left\{ \int_0^s |||a_{2j}(s,s_1)|||^2 \ \|X(s_1)\|_{L_2}^2 dF(s_1) \right\}^{1/2} d\nu(t) \\
&\quad + \sqrt{2} \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{j=1}^m \left\{ \int_0^s |||b_{2j}(s,s_1)|||^2 dF(s_1) \right\}^{1/2} d\nu(t) \\
&\leq \sqrt{2} \sup_{s\in[0,\infty)} \sum_{j=1}^m \left\{ \int_0^t |||a_{2j}(t,s)|||^2 dF(s) \right\}^{1/2} \\
&\quad \times \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \|X(s)\|_{L_2} d\nu(t) \\
&\quad + \sqrt{2} \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{j=1}^m \left\{ \int_0^s |||b_{2j}(s,s_1)|||^2 dF(s_1) \right\}^{1/2} d\nu(t) \qquad (3.4)
\end{aligned}$$

Put $L_2(t) = h(t, X(t;w))$. By (ii), we obtain

$$\begin{aligned}
&\int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \|L_2(s)\|_{L_2} d\nu(t) \\
&\leq k \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \|X(s)\|_{L_2} d\nu(t) \\
&\quad + \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} |h(s,0)| d\nu(t). \qquad (3.5)
\end{aligned}$$

Putting

$$L_3(t) = \sum_{i=1}^{n} \int_0^t f_i(t, s, X(s;w); w) ds,$$

and using (i)and (ii) we arrive at the following estimate

$$\begin{aligned} &\int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \|L_3(s)\|_{L_2} d\nu(t) \\ &\leq \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{i=1}^{n} \int_0^s |||a_{1i}(s,s_1)||| \ \|X(s_1)\|_{L_2} d(s_1) \ d\nu(t) \\ &\quad + \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{i=1}^{n} \int_0^s |||b_{1i}(s,s_1)||| ds_1 \ d\nu(t) \\ &\leq \sup_{s\in[0,\infty)} \sum_{i=1}^{n} \int_0^t |||a_{1i}(t,s)||| ds \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \|X(s)\|_{L_2} d\nu(t) \\ &\quad + \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{i=1}^{n} \int_0^s |||b_{1i}(s,s_1)||| ds_1 \ d\nu(t). \end{aligned} \tag{3.6}$$

Now combining (3.2),(3.4),(3.5) and (3.6) we conclude that

$$\|GX\|_p \leq M_1 \|X\|_p + N_1$$

where M_1 and N_1 are the quantities given above. Thus G is a mapping of L_1^p into L_1^p, and moreover we see that G maps the ball $K(0,r)$ into $K(0,r)$, where $r = N_1/(1 - M_1)$.

We prove that the map G is continous on the ball $K(0,r)$.
Let $x, y \in K(0,r)$. By (ii) for any given $\epsilon_1 > 0$ there exists $\delta > 0$ such that

$$\int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \Big\| h(s, X(s)) - h(s, Y(s)) \Big\|_{L_2} d\nu(t) < \epsilon_1 \tag{3.7}$$

whenever $\|x - y\| < \delta$.

Furthermore, we can assume without loss of generality that there exists $T > 0$ such that $|||a_{1i}(t,s_1)||| > 1$, $|||a_{2j}(t,s_1)||| > 1$ whenever $s_1 < T$, $i = 1, 2, ..., n, j = 1, 2, ..., m$. Hence using (3.7), we have the following estimate

$$\int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \Big\| (GX)(s) - (GY)(s) \Big\|_{L_2} d\nu(t)$$

$$\begin{aligned}
\leq\ & \epsilon_1 + \int_0^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \left\| \sum_{j=1}^m \int_0^s \Big\{ g_j(s,s_1,X(s_1);w) \\
& - g_j(s,s_1,Y(s_1);w) \Big\} d\beta(s_1;w) \right\|_{L_2} d\nu(t) \\
& + \int_0^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \left\| \sum_{i=1}^n \int_0^s \Big\{ f_i(s,s_1,X(s_1);w) \\
& - f_i(s,s_1,Y(s_1);w) \Big\} ds_1 \right\|_{L_2} d\nu(t).
\end{aligned}$$

Writing

$$L^\Delta(t) = \sum_{j=1}^m \int_0^t \Big\{ g_j(t,s,X(s);w) - g_j(t,s,Y(s);w) \Big\} d\beta(s).$$

Using the inequality (3.3) we see that

$$\begin{aligned}
& \int_0^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \|L^\Delta(s)\|_{L_2} d\nu(t) \\
\leq\ & \int_0^\infty p(t)\nu \\
& -\text{ess}\sup_{s\in[0,t]} \sum_{j=1}^m \left[\int_0^s \Big\| g_j(s,s_1,X(s_1)) - g_j(s,s_1,Y(s_1)) \Big\|_{L_2}^2 dF(s_1) \right]^{1/2} d\nu(t) \\
\leq\ & \sup_{t\in[0,T]} \sum_{j=1}^m \Big[\int_0^t |||a_{2j}(t,s_1)|||^2 dF(s_1) \Big]^{1/2} \\
& \times \int_0^T p(t)\nu - \text{ess}\sup_{s\in[0,t]} \Big\| g_j(t,s,X(s)) - g_j(t,s,Y(s)) \Big\|_{L_2} d\nu(t) \\
& + \sqrt{2} \sup_{t\in[T,\infty)} \sum_{j=1}^m \Big[\int_0^t |||a_{2j}(t,s_1)|||^2 dF(s_1) \Big]^{1/2} \\
& \times \int_T^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \|X(s)\|_{L_2} d\nu(t) \\
& + \sqrt{2} \sup_{t\in[T,\infty)} \sum_{j=1}^m \Big[\int_0^t |||a_{2j}(t,s_1)|||^2 dF(s_1) \Big]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_T^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \quad \|Y(s)\|_{L_2} d\nu(t) \\
&+2\sqrt{2}\int_T^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{j=1}^m \Big[\int_0^s |||b_{2j}(s,s_1)|||^2 dF(s_1)\Big]^{1/2} \ d\nu(t).
\end{aligned}
$$

For the second term of the right hand side of the above inequality we have

$$
\begin{aligned}
&\int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \Big\| \sum_{i=1}^n \int_0^s f_i(s,s_1,X(s_1)) - f_i(s,s_1,Y(s_1)) \Big\|_{L_2} ds_1 \ d\nu(t) \\
\leq \ & \int_0^T p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{i=1}^n \int_0^s |||a_{1i}(s,s_1)||| \ \Big\| f_i(s,s_1,X(s_1)) \\
&-f_i(s,s_1,Y(s_1)) \Big\|_{L_2} ds_1 \ d\nu(t) \\
&+\int_T^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{i=1}^n \int_0^s \Big\| f_i(s,s_1,X(s_1)) \Big\|_{L_2} ds_1 \ d\nu(t) \\
&+\int_T^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{i=1}^n \int_0^s \Big\| f_i(s,s_1,Y(s_1)) \Big\|_{L_2} ds_1 \ d\nu(t) \\
\leq \ & \sup_{t\in[0,T]} \sum_{i=1}^n \int_0^t |||a_{1i}(t,s_1)||| ds_1 \ \int_0^T p(t)\nu \\
&-\text{ess} \sup_{s\in[0,t]} \Big\| f_i(t,s,X(s)) - f_i(t,s,Y(s)) \Big\|_{L_2} d\nu(t) \\
&+\sup_{t\in[0,T]} \sum_{i=1}^n \int_0^t |||a_{1i}(t,s_1)||| ds_1 \ \int_T^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \|X(s)\|_{L_2} d\nu(t) \\
&+\sup_{t\in[0,T]} \sum_{i=1}^n \int_0^t |||a_{1i}(t,s_1)||| ds_1 \ \int_T^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \|Y(s)\|_{L_2} d\nu(t) \\
&+2\int_T^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{i=1}^n \ \int_0^s |||b_{1i}(s,s_1)||| \ ds_1 \ d\nu(t).
\end{aligned}
$$

Thus we have by (ii) and (iv),

$$
\begin{aligned}
&\int_0^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \Big\|(GX)(s) - (GY)(s)\Big\|_{L_2} d\nu(t) \\
&\leq \; M_1\left[\int_0^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \int_0^s \Big\|f_i(t,s,X(s)) - f_i(t,s,Y(s))\Big\|_{L_2} d\nu(t)\right] \\
&\quad + \int_0^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \Big\|g_j(t,s,X(s)) - g_j(t,s,Y(s))\Big\|_{L_2} d\nu(t) \\
&\quad + M_1\epsilon_2 \; + 2M_1\epsilon_3 \; + 2\epsilon_4, \qquad (3.8)
\end{aligned}
$$

where

$$\int_T^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \Big\|X(s)\Big\|_{L_2} d\nu(t) \; < \epsilon_2,$$

$$\int_T^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \Big\|Y(s)\Big\|_{L_2} d\nu(t) \; < \epsilon_3,$$

and

$$
\begin{aligned}
&\sqrt{2}\int_T^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \sum_{j=1}^m \Big[\int_0^s |||b_{2j}(s,s_1)|||^2 dF(s_1)\Big]^{1/2} \; d\nu(t) \\
&\quad + \int_T^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \sum_{i=1}^n \int_0^s |||b_{1i}(s,s_1)|||ds_1 \; d\nu(t) \; < \epsilon_4,
\end{aligned}
$$

whenever T is sufficiently large. By (3.8), for any given $\epsilon > 0$,

$$\|GX - GY\|_p \; < \epsilon,$$

whenever $|x-y| < \delta$, which prove the continuity of the operator.

We now prove that

$$\lim_{\epsilon\to 0} \sup_{x\in U} \int_0^\infty \Big\|p(t+\epsilon)(GX)(t+\epsilon) - p(t)(GX)(t)\Big\|_{L_2} d\nu(t) \; = \; 0 \qquad (3.9)$$

where $U \in \mathcal{M}_{L_1^p}$. Note that for any given $\epsilon > 0$, $x \in U \subset K(0,r)$, we have

$$
\begin{aligned}
&\int_0^\infty \Big\| p(t+\epsilon)(GX)(t+\epsilon) - p(t)(GX)(t) \Big\|_{L_2} d\nu(t) \\
&\quad = \int_0^\infty \Bigg\| p(t+\epsilon) \Bigg\{ h(t+\epsilon, X(t+\epsilon)) + \sum_{j=1}^m \int_0^{t+\epsilon} g_j(t+\epsilon, s, X(s); w) d\beta(s) \\
&\qquad + \sum_{i=1}^n \int_0^{t+\epsilon} f_i(t+\epsilon, s, X(s); w)\, ds \Bigg\} - p(t) \Bigg\{ h(t, X(t)) \\
&\qquad + \sum_{j=1}^m \int_0^t g_j(t, s, X(s); w)\, d\beta(s) + \sum_{i=1}^n \int_0^t f_i(t, s, X(s); w)\, ds \Bigg\} \Bigg\|_{L_2} d\nu(t) \\
&\leq \int_0^\infty |p(t+\epsilon) - p(t)| \Big\| h(t, X(t) \Big\|_{L_2} d\nu(t) \\
&\quad + \int_0^\infty p(t) \Big\| h(t+\epsilon, X(t+\epsilon)) - h(t, X(t)) \Big\|_{L_2} d\nu(t) \\
&\quad + \int_0^\infty |p(t+\epsilon) - p(t)| \sum_{j=1}^m \Big\| \int_0^t g_j(t, s, X(s); w) d\beta(s) \Big\|_{L_2} d\nu(t) \\
&\quad + \int_0^\infty p(t) \sum_{j=1}^m \Big\| \int_t^{t+\epsilon} \{ g_j(t+\epsilon, s, X(s); w) - g_j(t, s, X(s); w) \} d\beta(s) \Big\|_{L_2} d\nu(t) \\
&\quad + \int_0^\infty |p(t+\epsilon) - p(t)| \sum_{i=1}^n \Big\| \int_0^t f_i(t, s, X(s); w) ds \Big\|_{L_2} d\nu(t) \\
&\quad + \int_0^\infty p(t) \sum_{i=1}^n \Big\| \int_t^{t+\epsilon} \{ f_i(t+\epsilon, s, X(s); w) - f_i(t, s, X(s); w) \} ds \Big\|_{L_2} d\nu(t).
\end{aligned}
\tag{3.10}
$$

Moreover, we see that

$$
\begin{aligned}
&\int_0^\infty |p(t+\epsilon) - p(t)| \ \Big\| h(t, X(t)) \Big\|_{L_2} \ d\nu(t) \\
&\quad + \int_0^\infty |p(t+\epsilon) - p(t)| \sum_{j=1}^m \Big\| \int_0^t g_j(t, s, X(s); w)\ d\beta(s) \Big\|_{L_2} d\nu(t) \\
&\quad + \int_0^\infty |p(t+\epsilon) - p(t)| \sum_{i=1}^n \Big\| \int_0^t g_j(t, s, X(s); w) d\beta(s) \Big\|_{L_2} d\nu(t) \ \leq 2r
\end{aligned}
$$

and

$$\Big\|\int_0^t g_j(t,s,X(s);w)d\beta(s)\Big\|_{L_2}\Big\|\int_0^t f_i(t,s,X(s);w)d\beta(s)\Big\|_{L_2}\in L_1([0,+\infty)),$$
$$i=1,2,...,n,\ \ j=1,2,...,m. \tag{3.11}$$

Using now (3.10),(3.11) and (iv) and the above statment, we have (3.9).

Fix now $U\subset K(0,r)$, where $r=N_1/(1-M_1)$.
Then We prove that

$$\lim_{A\to 0}\ \sup_{x\in U}\ \int_A^\infty p(t)\ \|(GX)(t)\|_{L_2}\ d\nu(t)\ =\ 0. \tag{3.12}$$

By (3.2), (3.3), (i) and (ii) we get the following estimate

$$\begin{aligned}
&\int_A^\infty p(t)\|(GX)(t)\|_{L_2}\ d\nu(t)\\
&\quad\le k\int_A^\infty p(t)\nu-\text{ess}\sup_{s\in[0,t]}\|X(s)\|_{L_2}d\nu(t)\\
&\quad+\int_A^\infty p(t)\nu-\text{ess}\sup_{s\in[0,t]}|h(s,0)|d\nu(t)\\
&\quad+\sqrt{2}\int_A^\infty p(t)\nu\\
&\quad-\text{ess}\sup_{s\in[0,t]}\sum_{j=1}^m\Big[\int_0^s |||a_{2j}(s,s_1)|||^2\ \|X(s_1)\|^2_{L_2}dF(s_1)\Big]^{1/2}\ d\nu(t)\\
&\quad+\sqrt{2}\int_A^\infty p(t)\nu-\text{ess}\sup_{s\in[0,t]}\sum_{j=1}^m\Big[\int_0^s |||b_{2j}(s,s_1)|||^2\ dF(s_1)\Big]^{1/2}\ d\nu(t)\\
&\quad+\int_A^\infty p(t)\nu-\text{ess}\sup_{s\in[0,t]}\sum_{i=1}^n\int_0^s |||a_{1i}(s,s_1)|||\ \|X(s_1)\|_{L_2}ds_1\ d\nu(t)\\
&\quad+\int_A^\infty p(t)\nu-\text{ess}\sup_{s\in[0,t]}\sum_{i=1}^n\int_0^s |||b_{1i}(s,s_1)|||\ ds_1\ d\nu(t).
\end{aligned}\tag{3.13}$$

Now we see that

$$k\int_A^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \|X(s)\|_{L_2}\, d\nu(t)$$
$$+\sqrt{2}\int_A^\infty p(t)\nu$$
$$-\text{ess}\sup_{s\in[0,t]}\sum_{j=1}^m\Big[\int_0^s |||a_{2j}(s,s_1)|||^2\ \|X(s_1)\|_{L_2}^2 dF(s_1)\Big]^{1/2}\, d\nu(t)$$
$$+\int_A^\infty p(t)\nu$$
$$-\text{ess}\sup_{s\in[0,t]}\sum_{i=1}^n\int_0^s |||a_{1i}(s,s_1)|||\|X(s_1)\|_{L_2} ds_1 d\nu(t) \le M_1 r. \qquad (3.14)$$

Therefore by (3.13), (3.14), (iii) and (iv) we get (3.12). Hence by (3.9),(3.12) and the assumptions (ii),(iv) we obtain

$$\mu'(GU) \le C\mu'(U),$$

where $C = \sum_{i=1}^n L_{1i} + \sum_{j=1}^m L_{2j} + L_3$ which proves that G is μ'- contraction. Therefore by Theorem 2.4, we complete the proof of Theorem 3.3. □

Remark 3.4. Let $h(t,\ X(t;w)) \in D([0,+\infty))$. Then by theorem(3.3), the solution $X(t;w)$ of equation (3.1) are such that $X(t;w) \in D([0,+\infty))$ and

$$\lim_{T\to\infty}\int_T^\infty p(t)\nu - \text{ess}\sup_{s\in[0,t]} \|X(s)\|_{L_2}\, d\nu(t) = 0.$$

Now we consider the stochastic functional integral equation of the form

$$\begin{aligned} X(t;w) &= h(t,X(t;w)) + \sum_{i=1}^l \int_0^t a_i(t,\tau;w) f_i(\tau, X(\tau;w))d\tau \\ &\quad + \sum_{j=1}^m \int_0^t b_j(t,\tau;w) g_j(\tau, X(\tau;w)) d\beta(\tau;w), \end{aligned} \qquad (3.15)$$

where we assume the following hypothesis:

(i) $w \in \Omega$, the supporting set of the complete probability measure space (Ω, A, P) with A being the σ-algebra and P probability measure,
(ii) $X(t;w)$ is the unknown random process,
(iii) $h(t,X)$ is a map from $R_+ \times R$ into R,
(iv) $a_i(t,\tau;w),\ b_j(t,\tau;w)$ are stochastic kernels which are random valued functions defined for $0 \le \tau \le t < \infty$ and $w \in \Omega, i = 1,2,...,l,\ j = 1,2,...,m$,

(v) $f_i(t,X),\ i=1,2,...,n,\ g_j(t,X),\ j=1,2,...,m,$ are maps from $R_+\times R_+\times R$ into R,

(vi) $t\in R_+$ and $\beta(t;w)$ is a stochastic process.

The first part of the integral equation (3.15) is to be understood as an ordinary Lebesgue integral with probabilistic characterization, while the second part is an Ito-Doob stochastic integral.

Let $t\in R_+$ be fixed. We suppose that the stochastic kernels $a_i,\ i=1,2,...,n, b_j, j=1,2,...,m,\ 0\le\tau\le t$ and p-essentially bounded, continuous for $t\in R_+$ and integrable in Lebesgue sense for $\tau\in[0,t]$ map the set

$$\Delta=\ \{(t,\tau):\ 0\le\tau\le t<\infty\}\text{ into }L_\infty(\Omega,A,P).$$

For $0\le\tau\le t<\infty$, define

$$|||a_i(t,\tau)|||=P-\operatorname*{ess\,sup}_{w\in\Omega}|a_i(t,\tau;w)|, i=1,2,...,n$$

and

$$|||b_j(t,\tau)|||=P-\operatorname*{ess\,sup}_{w\in\Omega}|b_j(t,\tau;w)|, j=1,2,...,m.$$

Following the proof of theorem 3.3 one can get the following result.

Theorem 3.5. Let functions $f_i,\ i=1,2,...,n,\ g_j,\ i=1,2,...,m$ in the stochastic functional integral equation (3.15) be sublinear, that is,

(i) $|f_i(t,x(t;w))|\le u_{1i}(t)|x(t;w)|+v_{1i}(t)P-$ a.s., $|g_j(t,x(t;w))|\le u_{2j}(t)|x(t;w)|+v_{2j}(t)P-$ a.s., where non negative functions $u_{1i},\ v_{1i}, \imath=1,2,...,n$ and $u_{2j},\ v_{2j},\ j=1,2,...,m$ are defined for $t\in R_+$ and

(ii) $|h(t,X(t;w))-h(t,Y(t;w))|\le k|X(t;w)-Y(t;w)|$ P-a.s., for $k\in[0,1)$. Let the quantities M_1 and N_1 be defined as follows

$$\begin{aligned}M_1 &= k+\sup_{t\in[0,\infty)}\sum_{i=1}^{n}\int_0^t|||a_i(t,s)|||u_{1i}(s)ds\\ &\quad+\sqrt{2}\sup_{t\in[0,\infty)}\sum_{j=1}^{m}\left[\int_0^t|||b_j(t,s)|||^2(u_{2j}(s))^2dF(s)\right]^{1/2},\end{aligned}$$

$$N_1 = \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} |h(s,0)| \; d\nu(t)$$

$$+ \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{i=1}^{n} \int_0^s |||a_i(s,s_1)|| \; v_{1i}(s_1) ds_1 \; d\nu(t)$$

$$+\sqrt{2} \int_0^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \sum_{j=1}^{m} \left[\int_0^s |||b_j(s,s_1)|||^2 (v_{2j}(s_1))^2 dF(s_1) \right]^{1/2} d\nu(t),$$

(iii) $0 \le M_1 < 1, \; N_1 < \infty$,

(iv) If the mappings $X(\cdot;w) \to f_i(\cdot, x(\cdot;w)), \; i = 1,2,...,n$ and $X(\cdot;w) \to g_j(\cdot, x(\cdot;w))$,
$j = 1,2,...,m$, $L_1^p(R_+, \; L_2(\Omega, A, P), p)$ into $L_1^p(R_+, \; L_2(\Omega, A, P), p)$ are continuous in the topology generated by $\| \cdot \|_p$,

(v) $\|h(t, X(t)\|_{L_2} \; \in L_1([0,\infty))$, and

(vi) there exist $L_{1i}, \; i = 1,2,...,n, \; L_{2j}, \; j = 1,2,...,m,$ and L_3 satisfying such that

$$0 \le \sum_{i=1}^{n} L_{1i} + \sum_{j=1}^{m} L_{2j} + L_3 < 1,$$

$$m\Big(\int_0^t a_i(t,\tau;w) f_i(\tau, U(\tau)) d\tau \Big) \le L_{1i} m(U(t)), \; i = 1,2,...,n$$

$$m\Big(\int_0^t b_j(t,\tau;w) g_j(\tau, U(\tau)) d\beta(\tau;w) \Big) \le L_{2j} m(U(t)), j = 1,2,...,m$$

$$m\Big(h(t, U(t) \Big) \le L_3 m(U(t)),$$

$$U(t) = \{X(s) \in L_2(\Omega, A, P), s \ge 0, \; x \in U \subset K(0,r) :$$
$$\nu - \text{ess}\sup_{s\in[0,t]} \; \|X(s)\|_{L_2} \; \le \|U\| \; p(t)\}, t \ge 0, \text{and } r = N_1/(1 - M_1),$$

then there exists at least one solution $x \in L_1^p$ of equation (3.15) such that

$$\lim_{T\to\infty} \int_T^\infty p(t)\nu - \text{ess} \sup_{s\in[0,t]} \; \|X(s)\|_{L_2} \; d\nu(t) = \; 0.$$

Remark 3.6. If $p(t) = \; 0$ for $t \in R_+$ random solution $X(t;w)$ equations (3.1) and (3.15) are asymptotically stable in the sense of definition.

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