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ON THE GENERALIZED HYERS-ULAM STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION WITH A GENERAL INVOLUTION

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Abstract. In this paper we prove the generalized Hyers-Ulam stability of the quadratic functional equation

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(y), \quad x,y\in G,$$

where σ is an involution of the normed space G.

1. Introduction

In [16] Ulam proposed the following stability problem: Under what conditions does there exist an additive mapping near an approximately additive mapping?

The first partial solution to Ulam's problem was given by Hyers in [4]: If $f : E_1 \longrightarrow E_2$ is a mapping satisfying

$$\|f(x+y)-f(x)-f(y)\| \leq \delta,$$

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for all $x, y \in E_1$, where E_1 and E_2 are Banach spaces and δ is a given positive number, then there exists a unique additive mapping $T : E_1 \longrightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \delta,$$

for all $x \in E_1$. The proof of this result follows the same spirit if E_1 is an abelian semigroup.
In 1978, a generalization of Hyers' Theorem was formulated and proved by Rassias [9] in the setting when E_1 is a normed space, E_2 is a Banach space and the Cauchy difference is allowed to be unbounded.

Theorem 1.1. *Let $f : E_1 \longrightarrow E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geq 0$ and $p < 1$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$ (for all $x, y \in E_1 \setminus \{0\}$ if $p < 0$). Then there exists a unique linear mapping $T : E_1 \longrightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in E_1$ (for all $x \in E_1 \setminus \{0\}$ if $p < 0$).

Rassias during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [3] following the same approach as in Rassias [9], gave an affirmative solution to Rassias'question for $p > 1$. It was showed by Gajda [3] as well as by Rassias and Semrl [12] that a similar Theorem in the spirit of Theorem 1.1 for the case $p = 1$ cannot be proved.

Stability problems of various functional equations have been extensively investigated by a number of authors. The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. For more detailed definitions and further developments of stability concepts one is referred to [2], [6], [8], [11], [13], [14].

In this paper we prove the stability of the quadratic functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y), \; x, y \in G, \tag{1.1}$$

where σ: $G \longrightarrow G$ is an involution of G, i.e., $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$.

The functional equation

$$f(x+y) = f(x) + f(y), \; x, y \in G, \tag{1.2}$$

corresponds to $\sigma = I$, and the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \; x, y \in G, \tag{1.3}$$

corresponds to $\sigma = -I$. Reflection in a subspace of $\mathbb{R}^n$ provides a third example. Some other examples are the transpose involution and the symmetric involution in the additive group of 2×2 matrices.

The quadratic equation (1.1) has been solved by Stetkær [15].

The stability problem for the quadratic equation (1.3) was proved firstly by Skof in [14]. In [1] Cholewa extended the Skof's result in the following way, where G is an abelian group and E is a Banach space.

Theorem 1.2. *Let $\eta > 0$ be a real number and $f : G \longrightarrow E$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \eta \textit{ for all } x, y \in G. \tag{1.4}$$

Then for every $x \in G$ the limit $q(x) = \lim_{n \longrightarrow +\infty} \frac{f(2^n x)}{2^{2n}}$ exists and $q : G \longrightarrow E$ is the unique solution of (1.3) satisfying

$$\|f(x) - q(x)\| \leq \frac{\eta}{2},\ x \in G. \tag{1.5}$$

In [2] Czerwik obtained a generalization of the Skof-Cholewa's result.

Theorem 1.3. *Let $p \neq 2$, $\theta > 0$, $\delta > 0$ be real numbers. Suppose that the function $f : E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta + \theta(\|x\|^p + \|y\|^p) \textit{ for all } x, y \in E_1.$$

Then there exists exactly one quadratic function $q : E_1 \longrightarrow E_2$ such that

$$\|f(x) - q(x)\| \leq c + k\theta\|x\|^p$$

for all $x \in E_1$ if $p \geq 0$ and for all $x \in E_1 \setminus \{0\}$ if $p \leq 0$, where
$c = \frac{\|f(0)\|}{3}$, $k = \frac{2}{4-2^p}$ and $q(x) = \lim_{n \longrightarrow +\infty} \frac{f(2^n x)}{4^n}$, for $p < 2$ as well as,
$c = 0$, $k = \frac{2}{2^p-4}$ and $q(x) = \lim_{n \longrightarrow +\infty} 4^n f(2^{-n}x)$, for $p > 2$.

In dealing with a general involution σ of G one provides first of all a unified study for the stability of equations (1.2) and (1.3) and secondly a generalization of both of these equations. In particular, one wants to see how the involution σ enter into the approximative solutions formulas.

2. Hyers-Ulam stability of equation(1.1)

In this section we investigate the Hyers-Ulam stability for the equation (1.1). This generalizes the result obtained for $\sigma = I$ and $\sigma = -I$.

Theorem 2.1. *Let G be an abelian group, E a Banach space and $f\colon G \longrightarrow E$ a mapping which satisfies the inequality*

$$\| f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y) \| \leq \delta \textit{ for all } x, y \in G \tag{2.1}$$

for some $\delta > 0$. *Then there exists a unique mapping* q: $G \longrightarrow E$ *such that*

$$q(x) = \lim_{n\longrightarrow+\infty} \frac{1}{2^{2n}}\{f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\}$$

is a solution of the quadratic functional equation (1.1) satisfying

$$\|f(x) - q(x)\| \leq \frac{\delta}{2} \text{ for all } x \in G. \tag{2.2}$$

Proof. By letting $x = y = u$, respectively $x = y = u + \sigma(u)$ in (2.1) we obtain

$$\|f(2u) + f(u + \sigma(u)) - 4f(u)\| \leq \delta \tag{2.3}$$

and

$$\|2f(2u + 2\sigma(u)) - 4f(u + \sigma(u))\| \leq \delta. \tag{2.4}$$

Setting $x = y$ in (2.1) yields

$$\|f(x) - \frac{1}{4}\{f(2x) + f(x + \sigma(x))\}\| \leq \frac{\delta}{4} \text{ for all } x \in G. \tag{2.5}$$

Applying the inductive assumption we obtain

$$\|f(x) - \frac{1}{2^{2n}}\{f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\}\| \leq \frac{\delta}{2}(1 - \frac{1}{2^n}\} \tag{2.6}$$

for some positive integer n.
From (2.5) it follows that (2.6) is true for $n = 1$. The inductive step must now be demonstrated to hold true for the integer $n + 1$, that is

$$\begin{aligned}
&\|f(x) - \frac{1}{2^{2(n+1)}}\{f(2^{n+1}x) + (2^{n+1} - 1)f(2^n x + 2^n\sigma(x))\}\| \\
\leq\ & \frac{1}{2^{2(n+1)}}\|f(2^{n+1}x) + f(2^n x + 2^n\sigma(x)) - 4f(2^n x)\| \\
+\ & \frac{1}{2^{2(n+1)}}\|2(2^n - 1)f(2^n x + 2^n\sigma(x)) - 4(2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\| \\
+\ & \frac{1}{2^{2(n+1)}}\|4f(2^n x) + 4(2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x)) - 2^{2(n+1)}f(x)\| \\
\leq\ & \frac{\delta}{2^{2(n+1)}} + \frac{(2^n - 1)\delta}{2^{2(n+1)}} + \frac{\delta}{2}(1 - \frac{1}{2^n}) = \frac{\delta}{2}(1 - \frac{1}{2^{n+1}}).
\end{aligned}$$

This proves the validity of the inequality (2.6).

Let us define

$$q_n(x) = \frac{1}{2^{2n}}\{f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\} \tag{2.7}$$

for any positive integer n and $x \in G$. Then $\{q_n(x)\}$ is a Cauchy sequence for every $x \in G$. In fact by using (2.6), (2.7), (2.4) and (2.3), we get

$$\begin{aligned}
&\|q_{n+1}(x) - q_n(x)\| \\
&\leq \frac{1}{2^{2(n+1)}}\|f(2^{n+1}x) + f(2^n x + 2^n\sigma(x)) - 4f(2^n x)\| \\
&\quad + \frac{1}{2^{2(n+1)}}\|2(2^n - 1)f(2^n x + 2^n\sigma(x)) - 4(2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\| \\
&\leq \frac{\delta}{2^{2(n+1)}} + \frac{(2^n - 1)\delta}{2^{2(n+1)}} = \frac{\delta}{4}(\frac{1}{2})^n.
\end{aligned}$$

It easily follows that $\{q_n(x)\}$ is a Cauchy sequence for all $x \in G$. Since E is complete, we can define $q(x) = \lim_{n\longrightarrow+\infty} q_n(x)$ for any $x \in G$ and one can verify that q is a solution of (1.1). For all $x, y \in G$ we have

$$\begin{aligned}
&\|q_n(x + y) + q_n(x + \sigma(y)) - 2q_n(x) - 2q_n(y)\| \\
&= \frac{1}{2^{2n}}\|f(2^n x + 2^n y) + (2^n - 1)f(2^{n-1}x + 2^{n-1}y + 2^{n-1}\sigma(x) + 2^{n-1}\sigma(y)) \\
&\quad + f(2^n x + 2^n\sigma(y)) + (2^n - 1)f(2^{n-1}x + 2^{n-1}y + 2^{n-1}\sigma(x) + 2^{n-1}\sigma(y)) \\
&\quad - 2[f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))] \\
&\quad - 2[f(2^n y) + (2^n - 1)f(2^{n-1}y + 2^{n-1}\sigma(y))]\| \\
&\leq \frac{1}{2^{2n}}\|f(2^n x + 2^n y) + f(2^n x + 2^n\sigma(y)) - 2f(2^n x) - 2f(2^n y)\| \\
&\quad + \frac{(2^n - 1)}{2^{2n}}\|f(2^{n-1}x + 2^{n-1}y + 2^{n-1}\sigma(x) + 2^{n-1}\sigma(y)) \\
&\quad + f(2^{n-1}x + 2^{n-1}y + 2^{n-1}\sigma(x) + 2^{n-1}\sigma(y)) - 2f(2^{n-1}x + 2^{n-1}\sigma(x)) \\
&\quad - 2f(2^{n-1}y + 2^{n-1}\sigma(y))\| \\
&\leq \frac{\delta}{2^{2n}} + \frac{(2^n - 1)\delta}{2^{2n}} \\
&= \frac{\delta}{2^n}.
\end{aligned}$$

By letting $n \longrightarrow +\infty$, we get the desired result.
To prove that (2.2) holds true, we take the limit as $n \longrightarrow +\infty$ in (2.6) and, similarly as above, we derive the result.
Assume now that there exist two functions $q_i : G \longrightarrow E$ $(i = 1, 2)$ that are solutions of (1.1) with $\|f(x) - q_i(x)\| \leq \frac{\delta}{2}$ for all $x \in G$.
First, we will prove by mathematical induction that

$$q_i(2^n x) + (2^n - 1)q_i(2^{n-1}x + 2^{n-1}\sigma(x)) = 2^{2n}q_i(x). \tag{2.8}$$

Setting $y = x$ in relation (1.1), we obtain (2.8) for $n = 1$. Suppose (2.8) is true for n and we will prove it for $n + 1$. Hence, we have

$$\begin{aligned}
&q_i(2^{n+1}x) + (2^{n+1} - 1)q_i(2^n x + 2^n\sigma(x)) \\
&= q_i(2^{n+1}x) + q_i(2^n x + 2^n\sigma(x)) - 4q_i(2^n x) \\
&\quad + 2(2^n - 1)q_i(2^n x + 2^n\sigma(x)) - 4(2^n - 1)q_i(2^{n-1}x + 2^{n-1}\sigma(x)) \\
&\quad + 4q_i(2^n x) + 4(2^n - 1)q_i(2^{n-1}x + 2^{n-1}\sigma(x)) \\
&= q_i(2^n x + 2^n x) + q_i(2^n x + 2^n\sigma(x)) - 4q_i(2^n x) \\
&\quad + (2^n - 1)[q_i(2^{n-1}x + 2^{n-1}\sigma(x) + 2^{n-1}x + 2^{n-1}\sigma(x)) \\
&\quad + q_i(2^{n-1}x + 2^{n-1}\sigma(x) + 2^{n-1}x + 2^{n-1}\sigma(x)) \\
&\quad - 4q_i(2^{n-1}x + 2^{n-1}\sigma(x))] \\
&\quad + 4[q_i(2^n x) + (2^n - 1)q_i(2^{n-1}x + 2^{n-1}\sigma(x))] \\
&= 0 + 0 + 2^{2(n+1)}q_i(x) \\
&= 2^{2(n+1)}q_i(x).
\end{aligned}$$

Therefore, relation (2.8) is true for any natural number n. We will prove the uniqueness of the mapping q. For all $x \in G$ and all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|q_1(x) - q_2(x)\| \\
&= \frac{1}{2^{2n}}\|q_1(2^n x) + (2^n - 1)q_1(2^{n-1}x + 2^{n-1}\sigma(x)) - q_2(2^n x) \\
&\qquad - (2^n - 1)q_2(2^{n-1}x + 2^{n-1}\sigma(x))\| \\
&\leq \frac{1}{2^{2n}}[\|q_1(2^n x) - f(2^n x)\| + (2^n - 1)\|q_1(2^{n-1}x + 2^{n-1}\sigma(x)) \\
&\qquad - f(2^{n-1}x + 2^{n-1}\sigma(x))\|] \\
&\quad + \frac{1}{2^{2n}}[\|q_2(2^n x) - f(2^n x)\| + (2^n - 1)\|q_2(2^{n-1}x + 2^{n-1}\sigma(x)) \\
&\qquad - f(2^{n-1}x + 2^{n-1}\sigma(x))\|] \\
&\leq \frac{1}{2^{2n}}[\frac{\delta}{2} + \frac{\delta}{2} + \frac{\delta(2^n - 1)}{2} + \frac{\delta(2^n - 1)}{2}] \\
&= \frac{\delta}{2^n}.
\end{aligned}$$

If we let $n \longrightarrow +\infty$, we get $q_1(x) = q_2(x)$ for all $x \in G$. This completes the proof of the theorem. $\square$

3. Stability of equation (1.1) with ($p < 1$)

In the present section, we give a generalization of Skof's, Czerwik's and Rassias's results for the functional equation (1.1).

Theorem 3.1. *Let G be a normed space and E a Banach space. If a function $f : G \longrightarrow E$ satisfies the inequality*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3.1}$$

for some $\theta \geq 0$, $p < 1$ and for all $x, y \in G$, then there exists a unique mapping $q : G \longrightarrow E$, defined by

$$q(x) = \lim_{n \longrightarrow +\infty} \frac{f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))}{2^{2n}}$$

that is a solution of the quadratic functional equation (1.1) and

$$\|f(x) - q(x)\| \leq \frac{\theta}{2 - 2^p}\{\|x\|^p + \frac{1}{2}\|x + \sigma(x)\|^p\},\ x \in G. \tag{3.2}$$

Proof. Letting $x = y$ in (3.1) yields

$$\|f(2x) + f(x + \sigma(x)) - 4f(x)\| \leq 2\theta\|x\|^p. \tag{3.3}$$

Replacing now x and y by $2x$, respectively by $x + \sigma(x)$ in (3.1), we get

$$\|f(4x) + f(2x + 2\sigma(x)) - 4f(2x)\| \leq 2^{p+1}\theta\|x\|^p, \tag{3.4}$$

respectively

$$\|2f(2x + 2\sigma(x)) - 4f(x + \sigma(x))\| \leq 2\theta\|x + \sigma(x)\|^p. \tag{3.5}$$

Now, by applying the inductive argument, we obtain

$$\begin{aligned} &\|f(x) - \frac{1}{2^{2n}}\{f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\}\| \\ &\leq \frac{\theta}{2}\|x\|^p[1 + 2^{p-2} + 2^{2(p-2)} + ... + 2^{(n-1)(p-2)}] \\ &\quad + \frac{\theta}{2^{p+1}}\|x + \sigma(x)\|^p[2^{p-2} + (3)2^{2(p-2)} + (7)2^{3(p-2)} + ... \\ &\quad + (2^{n-1} - 1)2^{(n-1)(p-2)}]. \end{aligned} \tag{3.6}$$

The property for $n = 1$ follows from that inequality (3.3). For $n = 2$, we get from (3.3), (3.4) and (3.5) that

$$
\begin{aligned}
&\|f(x)-\frac{1}{16}\{f(4x)+3f(2x+2\sigma(x))\}\| \\
&\leq \frac{1}{16}\|f(4x)+f(2x+2\sigma(x))-4f(2x)\|+\frac{1}{16}\|2f(2x+2\sigma(x))-4f(x+\sigma(x))\| \\
&\quad +\frac{1}{16}\|4f(2x)+4f(x+\sigma(x))-16f(x)\| \\
&\leq \frac{2^{p+1}\theta}{16}\parallel x\parallel^p+\frac{2\theta}{16}\parallel x+\sigma(x)\parallel^p+\frac{4(2\theta)}{16}\parallel x\parallel^p \\
&= \frac{\theta}{2}\|x\|^p(1+2^{p-2})+\frac{\theta}{2^{p+1}}\|x+\sigma(x)\|^p 2^{p-2}.
\end{aligned}
$$

Assume now that (3.6) holds for n and we shall prove it for the case $n+1$. We have

$$
\begin{aligned}
&\|f(x)-\frac{1}{2^{2(n+1)}}\{f(2^{n+1}x)+(2^{n+1}-1)f(2^n x+2^n\sigma(x))\}\| \\
&\leq \frac{1}{2^{2(n+1)}}\|f(2^{n+1}x)+f(2^n x+2^n\sigma(x))-4f(2^n x)\| \\
&\quad +\frac{1}{2^{2(n+1)}}\|4f(2^n x)+4(2^n-1)f(2^{n-1}x+2^{n-1}\sigma(x))-2^{2(n+1)}f(x)\| \\
&\quad +\frac{1}{2^{2(n+1)}}\|2(2^n-1)f(2^n x+2^n\sigma(x))-4(2^n-1)f(2^{n-1}x+2^{n-1}\sigma(x))\| \\
&\leq \frac{1}{2^{2(n+1)}}2\theta\|2^n x\|^p \\
&\quad +\frac{\theta}{2}\|x\|^p(1+2^{p-2}+2^{2(p-2)}+....+2^{(n-1)(p-2)}) \\
&\quad +\frac{\theta}{2^{p+1}}\|x+\sigma(x)\|^p(2^{p-2}+(3)2^{2(p-2)}+(7)2^{3(p-2)}+.... \\
&\quad +(2^{n-1}-1)2^{(n-1)(p-2)})+\frac{2^n-1}{2^{2(n+1)}}2\theta\|2^{n-1}x+2^{n-1}\sigma(x)\|^p \\
&= \frac{\theta}{2}\|x\|^p(1+2^{p-2}+2^{2(p-2)}+....+2^{(n)(p-2)}) \\
&\quad +\frac{\theta}{2^{p+1}}\|x+\sigma(x)\|^p(2^{p-2}+(3)2^{2(p-2)}+(7)2^{3(p-2)}+....+(2^n-1)2^{(n)(p-2)}),
\end{aligned}
$$

which proves the validity of inequality (3.6).

Let us define

$$
q_n(x)=\frac{f(2^n x)+(2^n-1)f(2^{n-1}x+2^{n-1}\sigma(x))}{2^{2n}}
$$

for any positive integer n and $x \in G$. Then $\{q_n(x)\}$ is a Cauchy sequence for every $x \in G$. In fact, by using (3.3), (3.4), (3.5) and (3.6) one has

$$\begin{aligned}
&\|q_{n+1}(x) - q_n(x)\| \\
&\leq \frac{1}{2^{2(n+1)}}\|f(2^{n+1}x) + f(2^n x + 2^n\sigma(x)) - 4f(2^n x)\| \\
&\quad + \frac{1}{2^{2(n+1)}}\|2(2^n-1)f(2^n x + 2^n\sigma(x)) - 4(2^n-1)f(2^{n-1}x + 2^{n-1}\sigma(x))\| \\
&\leq \frac{1}{2^{2(n+1)}}2\theta\|2^n x\|^p + \frac{2^n-1}{2^{2(n+1)}}2\theta\|2^{n-1}x + 2^{n-1}\sigma(x)\|^p \\
&= \frac{\theta}{2}2^{n(p-2)}(\|x\|^p + \frac{1}{2^p}(2^n-1)\|x+\sigma(x)\|^p) \\
&= \frac{\theta}{2}2^{n(p-1)}(\frac{1}{2^n}\|x\|^p + \frac{1}{2^p}\frac{(2^n-1)}{2^n}\|x+\sigma(x)\|^p) \\
&\leq \frac{\theta}{2}2^{n(p-1)}(\|x\|^p + \frac{1}{2^p}\|x+\sigma(x)\|^p).
\end{aligned}$$

Since $2^{p-1} < 1$, it follows that $\{q_n(x)\}$ is a Cauchy sequence for every $x \in G$. However, E is a complete normed space, thus there exists the limit function

$$q(x) = \lim_{n \longrightarrow +\infty} q_n(x)$$

for any $x \in G$.

Let x, y be any two points of G. From (3.1) it follows that

$$\begin{aligned}
&\|q_n(x+y) + q_n(x+\sigma(y)) - 2q_n(x) - 2q_n(y)\| \\
&= \frac{1}{2^{2n}}\|f(2^n x + 2^n y) + (2^n-1)f(2^{n-1}x + 2^{n-1}y + 2^{n-1}\sigma(x) + 2^{n-1}\sigma(y)) \\
&\quad + f(2^n x + 2^n\sigma(y)) + (2^n-1)f(2^{n-1}x + 2^{n-1}y + 2^{n-1}\sigma(x) + 2^{n-1}\sigma(y)) \\
&\quad - 2[f(2^n x) + (2^n-1)f(2^{n-1}x + 2^{n-1}\sigma(x))] \\
&\quad - 2[f(2^n y) + (2^n-1)f(2^{n-1}y + 2^{n-1}\sigma(y))]\| \\
&\leq \frac{1}{2^{2n}}\|f(2^n x + 2^n y) + f(2^n x + 2^n\sigma(y)) - 2f(2^n x) - 2f(2^n y)\| \\
&\quad + \frac{(2^n-1)}{2^{2n}}\|f(2^{n-1}x + 2^{n-1}y + 2^{n-1}\sigma(x) + 2^{n-1}\sigma(y)) \\
&\quad + f(2^{n-1}x + 2^{n-1}y + 2^{n-1}\sigma(x) + 2^{n-1}\sigma(y)) - 2f(2^{n-1}x + 2^{n-1}\sigma(x)) \\
&\quad - 2f(2^{n-1}y + 2^{n-1}\sigma(y))\|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2^{2n}}\theta(\|2^n x\|^p + \|2^n y\|^p) \\
&\quad + \frac{(2^n-1)}{2^{2n}}\theta(\|2^{n-1}x + 2^{n-1}\sigma(x)\|^p + \|2^{n-1}y + 2^{n-1}\sigma(y)\|^p) \\
&= 2^{n(p-1)}\theta\{\frac{1}{2^n}(\|x\|^p + \|y\|^p) + \frac{2^n-1}{2^p 2^n}(\|x+\sigma(x)\|^p + \|y+\sigma(y)\|^p)\} \\
&\leq 2^{n(p-1)}\theta\{\|x\|^p + \|y\|^p + \frac{1}{2^p}(\|x+\sigma(x)\|^p + \|y+\sigma(y)\|^p)\}.
\end{aligned}$$

By letting $n \longrightarrow +\infty$ we get the equality

$$q(x+y) + q(x+\sigma(y)) = 2q(x) + 2q(y) \text{ for all } \ x, y \in G.$$

It remains to show that q and f satisfy the inequality (3.2). By using (3.6), we obtain

$$\begin{aligned}
&\|f(x) - \frac{1}{4^n}\{f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\}\| \\
&\leq \frac{\theta}{2}\|x\|^p(1 + 2^{p-2} + 2^{2(p-2)} + ... + 2^{(n-1)(p-2)}) \\
&\quad + \frac{\theta}{2^{p+1}}\|x+\sigma(x)\|^p(2^{p-2} + (3)2^{2(p-2)} + (7)2^{3(p-2)} + ... \\
&\quad + (2^{n-1}-1)2^{(n-1)(p-2)}) \\
&= \frac{\theta}{2}\|x\|^p(1 + \frac{1}{2}2^{p-1} + \frac{1}{2^2}2^{2(p-1)} + ... + \frac{1}{2^{n-1}}2^{(n-1)(p-1)}) \\
&\quad + \frac{\theta}{2^{p+1}}\|x+\sigma(x)\|^p((\frac{1}{2})2^{p-1} + (\frac{3}{4})2^{2(p-1)} + (\frac{7}{8})2^{3(p-1)} + ... \\
&\quad + (\frac{2^{n-1}-1}{2^{n-1}})2^{(n-1)(p-1)}) \\
&\leq \frac{\theta}{2}\|x\|^p(1 + 2^{p-1} + 2^{2(p-1)} + ... + 2^{(n-1)(p-1)}) \\
&\quad + \frac{\theta}{2^{p+1}}\|x+\sigma(x)\|^p(2^{p-1} + 2^{2(p-1)} + 2^{3(p-1)} + ... + 2^{(n-1)(p-1)}) \\
&\leq \frac{\theta}{2-2^p}\{\|x\|^p + \frac{1}{2}\|x+\sigma(x)\|^p\}.
\end{aligned}$$

Consequently, we obtain inequality (3.2). The uniqueness of the mapping q can be proved by using a similar argument as in the precedent paragraph. This completes the proof of the theorem. $\square$

If we replace in Theorem 3.1 the mapping σ by I, (resp. by $-I$), we obtain immediately the following corollaries.

Corollary 3.2. *Let G be a normed space and E a Banach space. If a function $f : G \longrightarrow E$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3.7}$$

for some $\theta \geq 0$, $p < 1$ and for all $x, y \in G$, then there exists a unique mapping $q : G \longrightarrow E$, given by

$$q(x) = \lim_{n \longrightarrow +\infty} \frac{f(2^n x)}{2^n},$$

that is a solution of the additive functional equation (1.2) satisfying the inequality

$$\|f(x) - q(x)\| \leq \frac{\theta\|x\|^p(2+2^p)}{2-2^p},\ x \in G. \tag{3.8}$$

Corollary 3.3. *Let G be a normed space and E a Banach space. If a function $f : G \longrightarrow E$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3.9}$$

for some $\theta \geq 0$, $0 < p < 1$ and for all $x, y \in G$, then there exists a unique mapping $q : G \longrightarrow E$, given by

$$q(x) = \lim_{n \longrightarrow +\infty} \frac{f(2^n x)}{2^{2n}},$$

that is a solution of the quadratic functional equation (1.3) satisfying the inequality

$$\|f(x) - q(x)\| \leq \frac{\theta\|x\|^p}{2-2^p}, \tag{3.10}$$

for all $x \in G$.

4. Stability of equation (1.1) with ($p > 2$)

In this section, we prove the generalized Hyers-Ulam stability of the functional equation (1.1) with $p > 2$.

Theorem 4.1. *Let G be a normed space and E a Banach space. Assume that $f : G \longrightarrow E$ satisfies the inequality*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{4.1}$$

for some $\theta \geq 0$, $p > 2$ and for all $x, y \in G$. Then there exists a unique mapping $q : G \longrightarrow E$, given by

$$q(x) = \lim_{n \longrightarrow +\infty} 2^{2n}\{f(\frac{x}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})\},$$

that is a solution of the quadratic functional equation (1.1) such that

$$\|f(x)-q(x)\|\leq\frac{2\theta}{2^p-4}\{\|x\|^p+\frac{1}{2^p}\|x+\sigma(x)\|^p\},\ x\in G. \tag{4.2}$$

Proof. Suppose that f satisfies inequality (4.1). Replacing x,y by $\frac{x}{2^{n+1}}$, (resp. by $\frac{x}{2^{n+2}}+\frac{\sigma(x)}{2^{n+2}}$), we easily obtain

$$\| f(\frac{x}{2^n})+f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})-4f(\frac{x}{2^{n+1}}) \|\leq\frac{2\theta}{2^{(n+1)p}}\| x \|^p, \tag{4.3}$$

$$\| 2f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})-4f(\frac{x}{2^{n+2}}+\frac{\sigma(x)}{2^{n+2}}) \|\leq\frac{2\theta}{2^{(n+2)p}}\| x+\sigma(x) \|^p, \tag{4.4}$$

for all $n\in\mathbb{N}_0$.

Now, we will show by induction that

$$\begin{aligned}&\| f(x)-2^{2n}\{f(\frac{x}{2^n})+(\frac{1}{2^n}-1)f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})\} \|\\&\leq\frac{2\theta}{2^p}\| x \|^p\ [1+2^{2-p}+2^{2(2-p)}+...+2^{(n-1)(2-p)}]\\&\quad+\frac{2\theta}{2^{2p}}\| x+\sigma(x) \|^p\ [(1-\frac{1}{2})+(1-\frac{1}{2^2})2^{2-p}+(1-\frac{1}{2^3})2^{2(2-p)}\\&\quad+...+(1-\frac{1}{2^n})2^{(n-1)(2-p)}].\end{aligned} \tag{4.5}$$

For $n=1$, we have

$$\begin{aligned}\| f(x)-4[f(\frac{x}{2})+(\frac{1}{2}-1)f(\frac{x}{4}+\frac{\sigma(x)}{4})] \|&\\\leq\| f(x)+f(\frac{x}{2}+\frac{\sigma(x)}{2})-4f(\frac{x}{2}) \|&\\+\| 2f(\frac{x}{4}+\frac{\sigma(x)}{4})-f(\frac{x}{2}+\frac{\sigma(x)}{2}) \|&\\\leq\frac{2\theta}{2^p}\| x \|^p+\frac{\theta}{2^{2p}}\| x+\sigma(x) \|^p&\\=\frac{2\theta}{2^p}\| x \|^p+(1-\frac{1}{2})\frac{2\theta}{2^{2p}}\| x+\sigma(x) \|^p&\end{aligned}$$

which proves (4.5) for $n = 1$. Assume that (4.5) holds for n and $x \in G$, and we will prove it for $n+1$. We obtain

$$\begin{aligned}
&\| f(x) - 2^{2(n+1)}\{f(\frac{x}{2^{n+1}}) + (\frac{1}{2^{n+1}} - 1)f(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}})\} \| \\
&\leq \| f(x) - 2^{2n}\{f(\frac{x}{2^{n}}) + (\frac{1}{2^{n}} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})\} \| \\
&\quad + 2^{2n} \| f(\frac{x}{2^{n}}) + (\frac{1}{2^{n}} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}) \\
&\quad - 4[f(\frac{x}{2^{n+1}}) + (\frac{1}{2^{n+1}} - 1)f(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}})] \| \\
&\leq \| f(x) - 2^{2n}\{f(\frac{x}{2^{n}}) + (\frac{1}{2^{n}} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})\} \| \\
&\quad + 2^{2n} \| f(\frac{x}{2^{n}}) + f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}) - 4f(\frac{x}{2^{n+1}}) \| \\
&\quad + 2^{2n} \| (\frac{1}{2^{n+1}} - 1)2f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}) - 4(\frac{1}{2^{n+1}} - 1)f(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}) \| \\
&\leq \frac{2\theta}{2^p} \| x \|^p [1 + 2^{2-p} + 2^{2(2-p)} + ... + 2^{(n-1)(2-p)}] \\
&\quad + \frac{2\theta}{2^{2p}} \| x + \sigma(x) \|^p [(1 - \frac{1}{2}) + (1 - \frac{1}{2^2})2^{2-p} + (1 - \frac{1}{2^3})2^{2(2-p)} + ... \\
&\quad + (1 - \frac{1}{2^n})2^{(n-1)(2-p)}] \\
&\quad + 2^{2n}\frac{2\theta}{2^{(n+1)p}} \| x \|^p + 2^{2n}(1 - \frac{1}{2^{n+1}})\frac{2\theta}{2^{(n+2)p}}[\| x + \sigma(x) \|^p] \\
&= \frac{2\theta}{2^p} \| x \|^p [1 + 2^{2-p} + 2^{2(2-p)} + ... + 2^{n(2-p)}] \\
&\quad + \frac{2\theta}{2^{2p}} \| x + \sigma(x) \|^p [(1 - \frac{1}{2}) + (1 - \frac{1}{2^2})2^{2-p} + (1 - \frac{1}{2^3})2^{2(2-p)} + ... \\
&\quad + (1 - \frac{1}{2^{n+1}})2^{n(2-p)}],
\end{aligned}$$

which proves the validity of the inequality (4.5).

Let us denote by $q_n(x)$ the sequence of functions defined by

$$q_n(x) = 2^{2n}\{f(\frac{x}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})\}, \tag{4.6}$$

for $x \in G$ and $n \in \mathbb{N}$. We will show that $\{q_n(x)\}$ is a Cauchy sequence for every $x \in G$.

For $n \in \mathbb{N}$, we obtain by (4.3) and (4.4) that

$$\begin{aligned}
&\| q_{n+1}(x) - q_n(x) \| \\
&= 2^{2n} \| 4[f(\frac{x}{2^{n+1}}) + (\frac{1}{2^{n+1}} - 1)f(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}})] \\
&\quad - [f(\frac{x}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})] \| \\
&\leq 2^{2n} \| f(\frac{x}{2^n}) + f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}) - 4f(\frac{x}{2^{n+1}}) \| \\
&\quad + 2^{2n} \| (\frac{1}{2^{n+1}} - 1)2f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}) - 4(\frac{1}{2^{n+1}} - 1)f(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}) \| \\
&\leq 2^{n(2-p)}\frac{2\theta}{2^p} \| x \|^p + (1 - \frac{1}{2^{n+1}})2^{n(2-p)}\frac{2\theta}{2^{2p}} \| x + \sigma(x) \|^p \\
&\leq 2^{n(2-p)}\frac{2\theta}{2^p}[\| x \|^p + \frac{1}{2^p} \| x + \sigma(x) \|^p].
\end{aligned}$$

Since $2^{(2-p)} < 1$, the desired conclusion follows. However, E is a Banach space, thus we can define

$$q(x) = \lim_{n \longrightarrow +\infty} q_n(x) \tag{4.7}$$

for any $x \in G$. We will show that q is a solution of equation (1.1). Let us consider $x, y \in G$. Then

$$\begin{aligned}
&\| q_n(x+y) + q_n(x + \sigma(y)) - 2q_n(x) - 2q_n(y) \| \\
&= 2^{2n} \| f(\frac{x}{2^n} + \frac{y}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x}{2^{n+1}} + \frac{y}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}} + \frac{\sigma(y)}{2^{n+1}}) \\
&\quad + f(\frac{x}{2^n} + \frac{\sigma(y)}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x}{2^{n+1}} + \frac{y}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}} + \frac{\sigma(y)}{2^{n+1}}) \\
&\quad - 2[f(\frac{x}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})] \\
&\quad - 2[f(\frac{y}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{y}{2^{n+1}} + \frac{\sigma(y)}{2^{n+1}})] \| \\
&\leq 2^{2n} \| f(\frac{x}{2^n} + \frac{y}{2^n}) + f(\frac{x}{2^n} + \frac{\sigma(y)}{2^n}) - 2f(\frac{x}{2^n}) - 2f(\frac{y}{2^n}) \| \\
&\quad + 2^{2n}(1 - \frac{1}{2^n}) \| f(\frac{x}{2^{n+1}} + \frac{y}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}} + \frac{\sigma(y)}{2^{n+1}}) \\
&\quad + f(\frac{x}{2^{n+1}} + \frac{y}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}} + \frac{\sigma(y)}{2^{n+1}}) - 2f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}) - 2f(\frac{y}{2^{n+1}} + \frac{\sigma(y)}{2^{n+1}}) \| \\
&\leq \theta 2^{n(2-p)}[\| x \|^p + \| y \|^p + \frac{1}{2^p} \| x + \sigma(x) \|^p + \frac{1}{2^p} \| y + \sigma(y) \|^p].
\end{aligned}$$

This implies that q is a solution of equation (1.1). The uniqueness of q can be derived by using some computations similar to the ones of the proof of Theorem 2.1. Some computations used in page 9 and inequality (4.5) imply (4.2). This ends the proof of Theorem 4.1. $\square$

Corollary 4.2. *Let G be a normed space and E a Banach space. Assume that $f : G \longrightarrow E$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{4.8}$$

for some $\theta \geq 0$, $p > 2$ and for all $x, y \in G$. Then there exists a unique mapping $q : G \longrightarrow E$, given by

$$q(x) = \lim_{n \longrightarrow +\infty} 2^n\{f(\frac{x}{2^n})\}$$

that is a solution of the additive functional equation (1.2), such that

$$\|f(x) - q(x)\| \leq \frac{8\theta}{2^p - 4}\|x\|^p, \ x \in G. \tag{4.9}$$

Corollary 4.3. *Let G be a normed space and E a Banach space. Assume that $f : G \longrightarrow E$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{4.10}$$

for some $\theta \geq 0$, $p > 2$ and for all $x, y \in G$. Then there exists a unique mapping $q : G \longrightarrow E$, given by

$$q(x) = \lim_{n \longrightarrow +\infty} 2^{2n} f(\frac{x}{2^n})$$

that is a solution of the quadratic functional equation (1.3) with

$$\|f(x) - q(x)\| \leq \frac{2\theta}{2^p - 4}\|x\|^p, \ x \in G. \tag{4.11}$$

It is a natural and interesting problem to study the stability of equation (1.1), when $p \in]1, 2[$.

References

[1] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math., **27**(1984), 76-86.

[2] S. Czerwik (ed.), *Stability of Functional Equations in Mathematical Analysis,* Hadronic Press, Inc., Palm Harbor, Florida, 2003.

[3] Z. Gajda, *On stability of additive mappings,* Internat. J. Math. Sci., **14**(1991), 431-434.

[4] D. H. Hyers, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci. U.S.A., **27**(1941), 222-224.

[5] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms,* Aequationes Math. **44**(1992), 125-153.

[6] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables,* Birkhauser, Basel, 1998.

[7] S.-M. Jung, *Stability of the quadratic equation of Pexider type,* Abh. Math. Sem. Univ. Hamburg. **70**(2000), 175-190.

[8] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis,* Hadronic Press, Inc., Palm Harbor, Florida, 2003.

[9] Th. M. Rassias, *On the stability of linear mappings in Banach spaces,* Proc. Amer. Math. Soc., **72**(1978), 297-300.

[10] Th. M. Rassias, *The problem of S. M. Ulam for approximately multiplicative mappings,* J. Math. Anal. Appl., **246**(2000), 352-378.

[11] Th. M. Rassias, *On the stability of the functional equations and a problem of Ulam,* Acta Applicandae Mathematicae. **62**(2000), 23-130.

[12] Th. M. Rassias and P. Semrl, *On the behavior of mappings which does not satisfy Hyers-Ulam stability,* Proc. Amer. Math. Soc., **114**(1992), 989-993.

[13] Th. M. Rassias and J. Tabor, *Stability of Mappings of Hyers-Ulam Type,* Hardronic Press, Inc., Palm Harbor, Florida 1994.

[14] F. Skof, *Local properties and approximations of operators,* Rend. Sem. Math. Fis. Milano., **53**(1983), 113-129.

[15] H. Stetkær, *Functional equations on abelian groups with involution*, Aequationes Math. **54**(1997) 144-172.

[16] S. M. Ulam, *A Collection of Mathematical Problems,* Interscience Publ. New York, 1961, Problems in Modern Mathematics, Wiley, New York 1964.