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STRONG CONVERGENCE OF CQ ITERATION FOR ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS

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Abstract. Kim and Xu proved the strong convergence theorems of modified Mann iterations for asymptotically nonexpansive mappings and semigroups on bounded closed convex subset C of a Hilbert space H by the CQ iteration method. The purpose of this paper is to modify the CQ iteration scheme of Kim and Xu, and to prove strong convergence theorems for asymptotically nonexpansive semigroups on closed convex subset C of a Hilbert space H without the condition of boundedness of subset C. The convergence rate of iteration process presented in this paper is faster than the iteration process of Kim and Xu. While, in this article, we did not use the demi-closedness principle .

1. Introduction

Let X be a real Banach space, C a nonempty closed convex subset of X, and T: $C \to C$ a mapping. Recall that T is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [4] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \le k_n\|x - y\|$ for all integers $n \ge 1$ and $x, y \in C$. A point $x \in C$ is said to be a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Recall also that a one-parameter family $\Im = \{T(t) : t \ge 0\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitzian semigroup on C(see,[16]) if the following conditions are satisfied:

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(i) $T(0)x = x, x \in C$,
(ii) $T(t+s)x = T(t)T(s)x, \quad t, s \geq 0, x \in C$,
(iii) for each $x \in C$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$,
(iv) there exists a bounded measurable function $L(t) : (0, \infty) \to [0, \infty)$ such that, for each $t > 0$, $\|T(t)x - T(t)y\| \leq L(t)\|x - y\|, \quad x, y \in C$.

A Lipschitzian semigroup $\Im$ is called nonexpansive if $L(t) = 1$ for all $t > 0$, and asymptotically nonexpansive if $\limsup_{t\to\infty} L(t) \leq 1$, respectively. We use $F(\Im)$ to denote the common fixed points set of the semigroup $\Im$, that is $F(\Im) = \{x \in C : T(s)x = x, \forall s > 0\}$. Note that for an asymptotically nonexpansive semigroup $\Im$, we can always assume that the Lipschitzian constants $L(t)$ are such that $L(t) \geq 1$ for all $t > 0$, $L(t)$ is nonincreasing in t, and $\lim_{t\to\infty} L(t) = 1$; otherwise we replace $L(t)$, for each $t > 0$, with $\tilde{L}(t) := \max\{\sup_{s\geq t} L(s),\ 1\}$.

Construction of fixed points of nonexpansive mappings (and of common fixed points of nonexpansive semigroups) is an important subject in the theory of nonexpansive mappings and finds application in a number of applied areas, in particular, in image recovery and signal processing (see,[2,9,12,17,18]). However, the sequence $\{T^n x\}_{n=0}^{\infty}$ of iterates of the mapping T at a point $x \in C$ may not converge even in the weak topology. Thus averaged iterations prevail. Indeed, Mann's iterations do have weak convergence. More precisely, a Mann's iteration procedure is a sequence $\{x_n\}$ which is generated in the following recursive way:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily. For example, Reich [10] proved that if X is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T. However we note that Mann's iterations have only weak convergence even in a Hilbert space [3].

Attempts to modify the Mann iteration method (1.1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi[8] proposed the following modification of Mann iteration method (1.1) for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.2}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H.

Nakajo and Takahashi [8] also propose the following iteration process for a nonexpansive semigroup $\Im = \{T(s) : 0 \le s < \infty\}$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.3}$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one and if $\{t_n\}$ is a positive real divergent sequence, then the sequence $\{x_n\}$ generated by (1.2)(resp.(1.3)) converges strongly to $P_{F(T)}x_0$ (resp.$P_{F(\Im)}x_0$).

The adaptation of Mann's iteration (1.1) to asymptotically nonexpansive mappings T is given below

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \quad n \ge 0. \tag{1.4}$$

Weak convergence of the sequence $\{x_n\}$ generated by (1.4) is proved by Schu [11] (see also, Tan and Xu [15]).

Attempts to modify the Mann iteration method (1.4) so that strong convergence is guaranteed have recently been made. In 2006, Kim and Xu[6] proposed the following modification of the Mann iteration method (1.4) for asymptotically nonexpansive mapping in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.5}$$

where C is bounded closed convex subset and $\theta_n = (1-\alpha_n)(k_n^2-1)(diamC)^2 \to 0 \ \ as \ n \to \infty$.

They also proposed the following modification of the Mann iteration method (1.4) for asymptotically nonexpansive semigroup in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \overline{\theta}_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.6}$$

where C is bounded closed convex subset and

$$\overline{\theta}_n = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2 - 1](diamC)^2 \to 0 \ \ as\ n \to \infty.$$

It is purpose of this paper to modify iteration process (1.6) for asymptotically nonexpansive semigroup with nonempty common fixed points set $F(\Im)$ on a closed convex subset C in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \ chosen\ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{z \in C_{n-1}\bigcap Q_{n-1} : \|y_n - z\|^2 \le \|x_n - z\|^2 + \overline{\theta}_n\}, n \ge 1 \\ C_0 = \{z \in C : \|y_0 - z\|^2 \le \|x_0 - z\|^2 + \tilde{\theta}_0\}, \\ Q_n = \{z \in C_{n-1}\bigcap Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, n \ge 1 \\ Q_0 = C \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.7}$$

where

$$\overline{\theta}_n = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2 - 1](\sup_{x\in A}\|x_n - x\|)^2,$$

and

$$A = \{\ y \in F(T) : \|y - p_0\| \le 1\},\ p_0 = P_{F(T)}(x_0).$$

We shall prove that iteration process (1.7) converges strongly to $P_{F(\Im)}(x_0)$ provided the sequence $\{\alpha_n\}$ is bounded from above one.

2. Main result

In 2006, Kim and Xu proved the following theorem.

Theorem 2.1.([6]) *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\Im = \{T(t) : 0 \le t < \infty\}$ be an asymptotically nonexpansive semigroup on C. Assume that $\{\alpha_n\}$ is a sequences in (0,1) such that $\alpha_n \le a$, for all n and for some $0 < a < 1$ and $\{t_n\}$ is a positive real divergent sequence. Define a sequence $\{x_n\}$ in C by the algorithm (1.6). Then $\{x_n\}$ converges strongly to $P_{F(\Im)}(x_0)$.*

In order to prove our strong convergence theorem, we will use the following lemma which is given by Kim and Xu in [6].

Lemma 2.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\Im = \{T(t) : 0 \le t < \infty\}$ be an asymptotically nonexpansive semigroup on C. Then it holds that*

$$\limsup_{s\to\infty}\limsup_{t\to\infty}\sup_{x\in C}\|\frac{1}{t}\int_0^t T(u)xdu - T(s)(\frac{1}{t}\int_0^t T(u)xdu)\| = 0.$$

The main result of this paper is following.

Theorem 2.3. *Let $\Im = \{T(t) : 0 \le t < \infty\}$ be an asymptotically nonexpansive semigroup with nonempty common fixed points set $F(\Im)$ defined on a closed convex subset C of a Hilbert space H. Assume that $\{\alpha_n\}$ is a sequences in [0,1] such that $\alpha_n \le a$, for all n and for some $a \in (0,1)$. Define a sequence $\{x_n\}$ in C by the algorithm (1.7). Then $\{x_n\}$ converges strongly to $P_{F(\Im)}(x_0)$.*

Proof. We observe that C_n is convex for all $n \ge 0$. Indeed, the defining inequality in C_n is equivalent to the following inequality

$$\langle 2(x_n - y_n), z\rangle \le \|x_n\|^2 - \|y_n\|^2 + \theta_n$$

which is affine(and hence convex) in z.

In addition, it is obvious that Q_n is also convex for all $n \ge 0$ while

$$A = \{\ y \in F(\Im) : \|y - p_0\| \le 1\},$$

is bounded closed convex subset of H, where $p_0 = P_{F(T)}(x_0)$.

Now, we show that $A \subset C_n$ for all n. Indeed, for any $p \in A$ we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\frac{1}{t_n}\int_0^{t_n} \|T(u)x_n - p\| du)^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2\|x_n - p\|^2 \\
&\le \|x_n - p\|^2 + (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2 - 1]\|x_n - p\|^2 \\
&\le \|x_n - p\|^2 + (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2 - 1](\max_{x\in A}\|x_n - x\|)^2 \\
&\le \|x_n - p\|^2 + \tilde{\theta_n},
\end{aligned}$$

so that $p \in C_n$ for all $n \ge 0$, that is, we have proved the $A \subset C_n$ for all $n \ge 0$.

Next we show that $A \subset C_n \bigcap Q_n$, for all $n \ge 0$. It suffices to show that $A \subset Q_n$, for all $n \ge 0$. We prove this by induction. For $n = 0$, we have

$A \subset F(T) \subset Q_0$. Assume that $A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1} \rangle \geq 0, \quad \forall z \in Q_n \cap C_n,$$

as $A \subset C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in A$. This together with the definition of Q_{n+1} implies that $A \subset Q_{n+1}$. Hence the $A \subset C_n \bigcap Q_n$ holds for all n.

By the definition of Q_n, we know that Q_n is convex and $x_n \in Q_n$ so that $x_n = P_{Q_n}(x_0)$ which together with the fact that $x_{n+1} \in C_n \bigcap Q_n \subset Q_n$ implies that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|. \tag{2.1}$$

This shows that the sequence $\{\|x_n - x_0\|\}$ is increasing.

Now, we show that $\{x_n\}$ is bounded sequence. Indeed, since $A \subset Q_n, n \geq 0$, then for any $z \in A$, we have $\langle x_n - z, x_0 - x_n \rangle \geq 0$ which implies that

$$\begin{aligned} \|x_0 - z\|^2 &= \|x_n - z + x_0 - x_n\|^2 \\ &= \|x_n - z\|^2 + \|x_0 - x_n\|^2 + 2\langle x_n - z, x_0 - x_n \rangle \qquad (2.2) \\ &\geq \|x_n - z\|^2 + \|x_0 - x_n\|^2. \end{aligned}$$

It follows from above inequality (2.2) that $\{x_n\}$ is bounded which together with the inequality (2.1) implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Noticing again that $x_n = P_{Q_n}(x_0)$ and for any positive integer m we have $x_{n+m} \in Q_{n+m-1} \subset Q_n$ which implies that $\langle x_{n+m} - x_n, x_n - x_0 \rangle \geq 0$, and noticing the identity

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v \rangle, \quad \forall u, v \in H,$$

we have

$$\begin{aligned} \|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \ n \to \infty. \qquad (2.3) \end{aligned}$$

From result (2.3) we known that $\{x_n\}$ is a Cauchy sequence in C, so that there exists a point $p \in C$ such that $\lim_{n\to\infty} x_n = p$

Now, we prove that $\lim_{n\to\infty} \|T(s)x_n - x_n\| \to 0$ for all $0 \leq s < \infty$, indeed,

$$\begin{aligned}
\|T(s)x_n - x_n\| &\leq \|T(s)x_n - T(s)(\frac{1}{t_n}\int_0^{t_n} T(u)x_n du)\| \\
&\quad +\|T(s)(\frac{1}{t_n}\int_0^{t_n} T(u)x_n du)) - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du)\| \\
&\quad +\|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \\
&\leq (L+1)\|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \\
&\quad +\|T(s)(\frac{1}{t_n}\int_0^{t_n} T(u)x_n du)) - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du)\| \\
&:= (L+1)A_n + B_n(s). \qquad (2.4)
\end{aligned}$$

We claim that

$$(i)\ \lim_{n\to\infty} A_n = 0 \ \ and \ \ (ii)\ \limsup_{s\to\infty}\limsup_{n\to\infty} B_n(s) = 0. \qquad (2.5)$$

As a matter of fact, that (ii) holds is guaranteed by Lemma 2.2, while (i) is verified by the following argument. By the definition of y_n, we have

$$\begin{aligned}
A_n &= \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \\
&= \frac{1}{1-\alpha_n}\|y_n - x_n\| \\
&\leq \frac{1}{1-\alpha_n}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \\
&\leq \frac{1}{1-a}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|). \qquad (2.6)
\end{aligned}$$

Note that the fact $x_{n+1} \in C_n$ implies that

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \overline{\theta}_n.$$

Because $\{x_n\}$ is convergent and $\overline{\theta}_n \to 0$, $\|y_n - x_{n+1}\| \to 0$. Therefore, it follows from (2.6) that $A_n \to 0$, as $n \to \infty$. We thus conclude from (2.4) that

$$\lim_{n\to\infty} \|T(s)x_n - x_n\| = 0, \quad \forall s \in [0,\infty). \qquad (2.7)$$

We have proved that $\{x_n\}$ converges in norm to a point $p \in C$ which together with the (2.7) implies that p is a common fixed point of $\Im$.

We claim that $p = z_0 = P_{F(T)}(x_0)$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exists a positive integer N, if $n > N$ then $\|x_0 - x_n\| > \|x_0 - z_0\|$ which leads to

$$\begin{aligned}\|z_0 - x_0\|^2 &= \|z_0 - x_n + x_n - x_0\|^2 \\ &= \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.\end{aligned}$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F(T)$, this is a contradiction. This completes the proof. □

Remark. In this article, we did not use the demi-closedness principle (see [6], Lemma 3.2) for the proof of main theorem. On the other hand, from the definitions of Q_n and C_n, it is easy to see that, Q_n and C_n are monotone decreasing sets, so that the convergence rate of iteration process (1.5) presented in this paper is faster than the iteration process of Kim and Xu [6]. The result of this paper is proved without the condition of boundedness of subset C.

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