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INVARIANCE FOR AN ABSTRACT NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION

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Abstract. The aim of the present paper is to study the invariance of arbitrary closed set for nonlinear Volterra integrodifferential equation in Banach spaces with general semigroup.

1. Introduction

Let X be a Banach space with norm $\| \ \|$ and K a closed subset of X. The present paper is concerned with invariant sets for nonlinear Volterra integrodifferential equation in Banach spaces of the form

$$x'(t) = Ax(t) + \int_0^t \{a(t,s)f(s,\, x(s)) + g(t,s,\, x(s))\}ds + f_0(t),\ \ t \geq 0; \tag{1.1}$$

$$x(0) = x_0 \in X; \tag{1.2}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$, $t \geq 0$ in X, nonlinear functions $f\colon R^+ \times X \to X$, $g\colon R^+ \times R^+ \times X \to X$, $f_0\colon R^+ \to X$ and the kernel $a\colon R^+ \times R^+ \to R$ are continuous and x_0 is a given element of X.

Recently, in [4], the authors have proved that the equations (1.1) – (1.2) have a unique mild solution $x(t) = T(t)x_0,\ t \geq 0$ satisfying the following

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integral equation

$$\begin{aligned} T(t)x_0 &= S(t)x_0 \\ &+ \int_0^t S(t-s)\int_0^s \{a(s,\tau)f(\tau,T(\tau)x_0) + g(s,\tau,T(\tau)x_0)\}d\tau ds \\ &+ \int_0^t S(t-s)f_0(s)ds. \end{aligned} \tag{1.3}$$

The transformations $T(t),\ t \geq 0$ form a semigroup of nonlinear operators. In the present paper, we establish the condition under which the set K is invariant with respect to the invariance conditions of different types have been studied by many authors. The classical Nagumo condition

$$\liminf_{t\downarrow 0} \frac{1}{t} d(x + tf(x), K) = 0,\ for\ all\ x \in K \tag{1.4}$$

is sufficient and necessary for invariance of a closed set K for the classical differential equation

$$x'(t) = f(x(t)),\ x(0) = x_0 \tag{1.5}$$

where $d(\cdot, K)$ denotes the distance function (see [1]). In [2], Brezis proved that the condition

$$\liminf_{t\downarrow 0} \frac{1}{t} d(T(t)x, K) = 0,\ for\ all\ x \in K \tag{1.6}$$

is sufficient and necessary for the semilinear differential equations

$$x'(t) = Ax(t) + f(x(t)),\ x(0) = x_0 \tag{1.7}$$

when X is a Hilbert space, K is a closed, convex set and $T(\cdot)$ is a general nonlinear semigroup of contractions. This result is extended by Martin [6] when X is a Banach space and K is closed set. In [8] Pavel proved that if X is Banach space, K is closed subset of X, then $T(t)K \subset K,\ t \geq 0$ if and only if

$$\liminf_{t\downarrow 0} \frac{1}{t} d(S(t)x + tf(x), K) = 0\ for\ all\ x \in K \tag{1.8}$$

provided that the operators $S(t),\ t \geq 0$ are compact. The condition (1.8) is called semigroup Nagumo's condition. We observe that if K is contained in the domain $D(A)$ of the operator A, then (1.8) becomes

$$\liminf_{t\downarrow 0} \frac{1}{t} d(x + t(Ax + f(x)), K) = 0,\ for\ all\ x \in K \tag{1.9}$$

The paper is organized as follows. In section 2, we present the preliminaries and statements of our main results. Section 3 deals with the proofs of main theorem.

2. Preliminaries and Statement of Results

Before proceeding to the statements of our main results, we shall setforth some preliminaries and hypotheses on the functions involved in (1.1) – (1.2) that will be used throughout the paper.

Let (X, r) be a complete metric space and $T(t),\ t \geq 0$ a family of nonlinear operators on X satisfying:

$$T(t+s) = T(t)T(s) \ for \ t, s \geq 0 \tag{2.1}$$

$$\exists\ \omega \in R,\ M > 0,\ r(T(t)u, T(t)v) \leq Me^{\omega t} r(u, v),\ t \geq 0, u, v \in X \tag{2.2}$$

$$\forall u \in X,\ t \to T(t)u \ is \ contonuous \ on \ [0, \infty). \tag{2.3}$$

We now state lemma and theorem before stating our main results.

Lemma 2.1. [11] *Let $K \subset X$ be a closed set and $C \geq 0$. Assume for $\forall x \in K$*

$$\liminf_{t \downarrow 0} \frac{1}{t} d(T(t)x, K) \leq C. \tag{2.4}$$

Then $\forall x \in X,\ t \geq 0$

$$d(T(t)x, K) \leq Me^{\omega t} d(x, K) + \frac{CM}{\omega}(e^{\omega t} - 1). \tag{2.5}$$

Theorem 2.2. [7] *Let u, p, q and r be nonnegative continuous functions defined on R_+, p be positive and sufficiently smooth on R_+ and $u_0 \geq 0$ be a constant. If*

$$u(t) \leq u_0 + \int_0^t p(s)q(s)ds + \int_0^t p(s) \int_0^s r(\tau)u(\tau)d\tau ds \ for \ t \in R_+,$$

then

$$u(t) \leq (u_0 + \int_0^t p(s)q(s)ds) exp\{\int_0^t p(s) \int_0^s r(\tau)d\tau ds\} \ for \ t \in R_+.$$

For convenience, we list the following hypotheses used in our further discussion.

(H_1) The function $f_0 \colon R^+ \to X$ is continuously differentiable.

(H_2) The kernel function $a: R^+ \times R^+ \to R$ is continuous, continuously differentiable and

$$|a(t,s)| \leq N, \tag{2.6}$$

for $t, s \in R^+$.

(H_3) The function $f: R^+ \times X \to X$ is Lipschitz continuous i.e. there exists Lipschitz constant L_1 such that

$$\|f(t,x_1) - f(t,x_2)\| \leq L_1 \|x_1 - x_2\|, \tag{2.7}$$

for $x_1,\ x_2 \in X$.

(H_4) The function $g: R^+ \times R^+ \times X \to X$ is continuously differentiable and Lipschitz continuous i.e. there exists Lipschitz constant L_2 such that

$$\|g(t,s,x_1) - g(t,s,x_2)\| \leq L_2 \|x_1 - x_2\|, \tag{2.8}$$

for $x_1, x_2 \in X$.

We have established the following result.

Theorem 2.3. *Let X be a Banach space and K be a closed subset of X. Suppose that the hypotheses $(H_1) - (H_4)$ hold and for some $M > 0,\ \omega \in R$ and for all $t \geq 0$ the estimation $\|S(t)\| \leq Me^{\omega t}$ is valid. If there exists a constant $C \geq 0$ such that*

$$\liminf_{t \downarrow 0} \frac{1}{t} d(S(t)x + t\{\int_0^t (a(t,s)f(s,x(s)) + g(t,s,x(s)))ds + f_0(t)\},\ K) \leq C, \tag{2.9}$$

$\forall x \in K$, then

$$d(T(t)x, K) \leq MC\frac{e^{\omega t} - 1}{\omega},\ \forall t \geq 0,\ \forall x \in K. \tag{2.10}$$

Conversely, if there exists a constant $C \geq 0$ such that

$$d(T(t)x, K) \leq C\frac{e^{\omega t} - 1}{\omega},\ \forall t \geq 0,\ \forall x \in K, \tag{2.11}$$

then

$$\liminf_{t \downarrow 0} \frac{1}{t} d(S(t)x + t\{\int_0^t (a(t,s)f(s,x(s)) + g(t,s,x(s)))ds + f_0(t)\},\ K) \leq C, \tag{2.12}$$

$\forall x \in K$.

The next corollary is the special case for the $C = 0$ in Theorem 2.3.

Corollary 2.4. *Suppose the hypotheses of Theorem 2.3 hold. A closed set K is invariant for equations* (1.1) − (1.2) *if and only if*

$$\liminf_{t\downarrow 0}\frac{1}{t}d(S(t)x+t\{\int_0^t(a(t,s)f(s,x(s))+g(t,s,x(s)))ds+f_0(t)\},\ K)=0, \tag{2.13}$$

for $\forall x\in K$ holds.

3. Proof for the Theorem 2.3

Proof for the Theorem 2.3. First, we prove that (2.11) $\Rightarrow$ (2.12). Assume that (2.11) holds. For $t>0,\ x\in K$, we have

$$\begin{aligned}
&\frac{1}{t}d(S(t)x+t\{\int_0^t(a(t,s)f(s,x(s))+g(t,s,x(s)))ds+f_0(t)\},\ K)\\
\le&\frac{1}{t}\|S(t)x+t\{\int_0^t(a(t,s)f(s,x(s))+g(t,s,x(s)))ds+f_0(t)\}-T(t)x\|\\
&+\frac{1}{t}d(T(t)x,\ K)\\
\le&\|\int_0^t\{a(t,s)f(s,x(s))+g(t,s,x(s))\}ds\\
&-\frac{1}{t}\int_0^t S(t-s)\int_0^s\{a(s,\tau)f(\tau,T(\tau)x)+g(s,\tau,T(\tau)x)\}d\tau ds\|\\
&+\|f_0(t)-\frac{1}{t}\int_0^t S(t-s)f_0(s)ds\|+\frac{1}{t}d(T(t)x,\ K).
\end{aligned}$$

Now, from hypotheses $(H_1)-(H_4)$, we have

$$\begin{aligned}
&\lim_{t\to 0}\|\int_0^t(a(t,s)f(s,x(s))+g(t,s,x(s)))ds\\
&-\frac{1}{t}\int_0^t S(t-s)\int_0^s\{a(s,\tau)f(\tau,T(\tau)x)+g(s,\tau,T(\tau)x)\}d\tau ds\|=0,\\
&\lim_{t\to 0}\|f_0(t)-\frac{1}{t}\int_0^t S(t-s)f_0(s)ds\|=0.
\end{aligned}$$

Therefore, by using (2.11), we obtain

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}d(S(t)x+t\{\int_0^t(a(t,s)f(s,x(s))+g(t,s,x(s)))ds+f_0(t)\},\ K)\\
&\qquad\le\lim_{t\to 0}C\frac{e^{\omega t}-1}{\omega t}=C.
\end{aligned}$$

Now, we prove that (2.9) $\Rightarrow$ (2.10). Assume that (2.9) holds. It is clear that $T(t),\ t \geq 0$ satifies (2.1) and (2.3). We show that (2.2) also holds. For $t \geq 0,\ x, y \in X$, we obtain from (H_3) and (H_4)

$$\begin{aligned}
&d(T(t)x, T(t)y) \\
&\quad = \|T(t)x - T(t)y\| \\
&\quad \leq \|S(t)(x-y)\| + \| \int_0^t S(t-s) \int_0^s \{a(s,\tau)\,[f(\tau, T(\tau)x) - f(\tau, T(\tau)y)] \\
&\quad + [g(s,\tau,T(\tau)x) - g(s,\tau,T(\tau)y)]\} d\tau ds\| \\
&\quad \leq Me^{\omega t}\|(x-y)\| + Me^{\omega t}[NL_1 + L_2] \int_0^t e^{-\omega s} \int_0^s \|T(\tau)x - T(\tau)y\| d\tau ds.
\end{aligned}$$

$$\begin{aligned}
\|T(t)x - T(t)y\| \leq{}& Me^{\omega t}\|(x-y)\| \\
&+ Me^{\omega t}[NL_1 + L_2] \int_0^t e^{-\omega s} \int_0^s \|T(\tau)x - T(\tau)y\| d\tau ds.
\end{aligned}$$

Putting $u(t) = \|T(t)x - T(t)y\| e^{-\omega t}$, we have

$$u(t) \leq M\|x-y\| + M[NL_1 + L_2] \int_0^t e^{-\omega s} \int_0^s e^{\omega\tau} u(\tau) d\tau ds. \tag{3.1}$$

Applying Theorem 2.2 to equation (3.1), we get

$$\begin{aligned}
u(t) &\leq M\|x-y\| exp\{\int_0^t M[NL_1 + L_2] e^{-\omega s}\ \frac{[e^{\omega s} - 1]}{\omega} ds\} \\
&\leq M\|x-y\| exp\{\frac{M[NL_1 + L_2]t}{\omega}\},
\end{aligned}$$

and therefore,

$$\|T(t)x - T(t)y\| \leq Me^{\lambda t}\|x-y\|$$

where

$$\lambda = \{\omega + \frac{M[NL_1 + L_2]}{\omega}\}.$$

This yields

$$d(T(t)x, T(t)y) \leq Me^{\lambda t} d(x,y).$$

Hence (2.2) holds for $T(t)$ with ωt is replaced by λt. We observe that semigroup $T(.)$ satisfies (2.4). Now,

$$
\begin{aligned}
&\frac{1}{t} d(T(t)x, K) \\
&\quad \leq \frac{1}{t} \|T(t)x - S(t)x - t\{\int_0^t (a(t,s)f(s,x(s)) + g(t,s,x(s)))ds + f_0(t)\}\| \\
&\quad + \frac{1}{t} d(S(t)x + t\{\int_0^t (a(t,s)f(s,x(s)) + g(t,s,x(s)))ds + f_0(t)\},\ K) \\
&\quad \leq \|\frac{1}{t}\int_0^t S(t-s)\int_0^s \{a(s,\tau)f(\tau,T(\tau)x) + g(s,\tau,T(\tau)x)\}d\tau ds \\
&\quad - \{\int_0^t (a(t,s)f(s,x(s)) + g(t,s,x(s)))ds\}\| \\
&\quad + \|\frac{1}{t}\int_0^t S(t-s)f_0(s)ds - f_0(t)\| \\
&\quad + \frac{1}{t} d(S(t)x + t\{\int_0^t (a(t,s)f(s,x(s)) + g(t,s,x(s)))ds + f_0(t)\},\ K).
\end{aligned}
$$

Taking limit as $t \to 0$ and using hypotheses $(H_1) - (H_4)$ and (2.9), we get

$$\lim_{t \to 0} \frac{1}{t} d(T(t)x, K) \leq C.$$

Hence, by Lemma, we have for $\forall x \in K,\ t \geq 0$

$$d(T(t)x, K) \leq MC\frac{e^{\omega t} - 1}{\omega}.$$

This proves the Theorem 2.3. □

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