Nonlinear Funct. Anal. & Appl., Vol. 6, No. 2 (2001), pp. 281–290

A FORWARD-BACKWARD SPLITTING ALGORITHM FOR GENERAL MIXED VARIATIONAL INEQUALITIES

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Abstract. In this paper, we suggest and analyze a forward-backward splitting algorithm for solving general mixed variational inequalities by using the updating technique of the solution. The convergence of this new method requires only the pseudomonotonicity of the operator, which is a weaker condition than monotonicity. The new results are versatile and are easy to implement.

1. Introduction

Variational inequality theory has emerged as an effective and powerful tool for studying a wide class of unrelated problems arising in various branches of social, physical, engineering, pure and applied sciences in a unified and general framework, see, for example, [1-18]. Variational inequalities have been extended and generalized in different directions by using novel and innovative techniques and ideas, both for their own sake and for their applications. An important and useful generalization is called the mixed variational inequality or the variational inequality of the second kind, see [3-5,8-14,18] and references therein. In recent years, much attention has been given to develop efficient and implementable numerical methods including projection method and its variant forms, Wiener-Hopf (normal) equations, linear approximation, auxiliary principle, and descent framework for solving variational inequalities and related optimization problems. It is well known that the projection method and its variant forms; and Wiener-Hopf equations technique cannot be used to

Received January 12, 2001.
2000 Mathematics Subject Classification: 49J40, 90C33.
Key words and phrases: Variational inequalities, resolvent equations, splitting methods, convergence.

suggest and analyze iterative methods for solving mixed variational inequalities due to the presence of the nonlinear term. These facts motivated us to use the technique of resolvent operators. In this technique, the given operator is decomposed into the sum of two (or more) maximal monotone operators, whose resolvent are easier to evaluate than the resolvent of the original operator. Such a method is known as the operator splitting method. This can lead to the development of very efficient methods, since one can treat each part of the original operator independently. In the context of the mixed variational inequalities, Noor [5,8,14] has used the resolvent operator technique to suggest some two-step and three-step splitting type methods. A useful feature of the forward-backward splitting method for solving the mixed variational inequalities is that the resolvent step involves the subdifferential of the proper, convex and lower semicontinuous part only and the other part facilitates the problem decomposition. The simplest of these is the resolvent method [12], which requires the restrictive assumption that the underlying operator is strongly monotone and Lipschitz continuous. These strict assumptions rule out many applications of the resolvent method. To overcome this difficulty, we modify the resolvent method, which entails an additional forward step and a resolvent at each step according to double resolvent formula. We propose a new class of forward-backward splitting methods for solving the general mixed variational inequalities that are as versatile and capable of exploiting problem structure as the extraresolvent and have a scaling feature absent in the latter. The convergence of these modified extraresolvent method requires only the speudomonotonicity which is weaker than monotonicity. Consequently, we improve the convergence results of previously known methods which can be obtained as special cases from our results.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let K be a nonempty closed convex set in H. Let $\varphi : H \to R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Let $N(\cdot, \cdot) : H \times H \longrightarrow H$ be a single-valued operator.

For a given nonlinear operator $g : H \longrightarrow H$, consider the problem of finding $u \in H$ such that

$$\langle N(u,u), g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \qquad \text{for all } g(v) \in H. \quad (2.1)$$

The inequality of type (2.1) is called *the general mixed variational inequality* or the general variational inequality of the second kind [9,10]. It can be shown that a wide class of linear and nonlinear problems arising in pure and applied sciences can be studied via the general mixed variational inequalities (2.1).

We remark that if $g \equiv I$, the identity operator, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\langle N(u,u), v-u\rangle + \varphi(v) - \varphi(u) \geq 0, \qquad \text{for all } v \in H, \tag{2.2}$$

which are called the mixed variational inequalities. For the applications, numerical methods and formulations, see [1-5,8-16,18] and the references therein.

We note that if φ is the indicator function of a closed convex set K in H, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

then the general mixed variational inequality (2.1) is equivalent to finding $u \in H,\ g(u) \in K$ such that

$$\langle N(u,u), g(v) - g(u)\rangle \geq 0, \qquad \text{for all } g(v) \in K. \tag{2.3}$$

The inequality of the type (2.3) is known as the *general variational inequality*, which was introduced and studied by Noor [6] in 1988. It turned out that a class of unrelated odd-order and nonsymmetric free, unilateral, obstacle, non-convex programming, quasi variational inequalities and equilibrium problems can be studied by the general variational inequality (2.3), see [8,9,11,12,15].

If $K^* = \{u \in H : \langle u, v\rangle \geq 0, \text{for all } v \in K\ \}$ is a polar (dual) cone of the convex cone K in H, then problem (2.3) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad N(u,u) \in K^*, \quad \langle g(u), N(u,u)\rangle = 0, \tag{2.4}$$

which is known as the general complementarity problem. Note that if $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem is known as the quasi(implicit) complementarity problem. If $g \equiv I$, the identity operator, then problem(2.4) is the generalized complementarity problem, which has been studied extensively, see [3,11-15,18] and references therein.

For $g \equiv I$, the identity operator, the general variational inequality (2.3) becomes : find $u \in K$ such that

$$\langle N(u,u), v-u\rangle \geq 0, \qquad \text{for all } v \in K, \tag{2.5}$$

which is called the classical variational inequality. For the recent state-of-the-art, see [1-18].

We now recall the following well known concepts and results.

Definition 2.1. [2] If A is a *maximal monotone* operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \text{for all } u \in H,$$

where I is the identity operator. It is well known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is single-valued and nonexpansive, that is, for all $u, v \in H$,

$$\|J_A(u) - J_A(v)\| \leq \|u - v\|.$$

Remark 2.1. It is well known that the subdifferential $\partial\varphi$ of a proper, convex and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$ is a maximal monotone operator. We denote by

$$J_\varphi(u) = (I + \rho\partial\varphi)^{-1}(u), \quad \text{for all } u \in H,$$

the resolvent operator associated with $\partial\varphi$, which is defined everywhere on H.

Lemma 2.1. [2] *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0, \quad \textit{for all } v \in H, \tag{2.6}$$

if and only if

$$u = J_\varphi z,$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant. This property of the resolvent operator J_φ plays an important role in obtaining our results.

Definition 2.2. For all $u, v \in H$, the operator $N(\cdot, \cdot) : H \times H \to H$ is said to be:

(i) *g-monotone*, if

$$\langle N(u, u) - N(v, v), g(u) - g(v)\rangle \geq 0$$

(ii) *g-pseudomonotone*, if

$$\langle N(u, u), g(v) - g(u)\rangle \geq 0 \quad \text{implies} \quad \langle N(v, v), g(v) - g(u)\rangle \geq 0.$$

Note that for $g \equiv I$, the identity operator, Definition 2.2 reduces to the standard definition of monotonicity and pseudomonotonicity of the operator $N(\cdot, \cdot)$. It is well known [3] that monotonicity implies pseudomonotonicity, but not conversely.

3. Main Results

In this section, we suggest and analyze some new iterative methods for solving general mixed variational inequality (2.1). One can prove that the general mixed variational inequality (2.1) is equivalent to a fixed-point problem by invoking Lemma 2.1.

Lemma 3.1. *The function $u \in H$ is a solution of the mixed variational inequality* (2.1) *if and only if it satisfies the relation*

$$g(u) = J_\varphi[g(u) - \rho N(u,u)], \tag{3.1}$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the general mixed variational inequality (2.1) is equivalent to the fixed-point problem (3.1). This alternate equivalent formulation is very useful from the numerical point of view. This fixed-point formulation enables us to suggest and analyze the following iterative algorithm.

Algorithm 3.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = J_\varphi[g(u_n) - \rho N(u_n, u_n)], \quad n = 0, 1, 2, \cdots .$$

It is known that Algorithm 3.1 *converges if both the operators $N(\cdot,\cdot)$ and g are strongly monotone and Lipschitz continuous. These strict conditions rule out many applications of Algorithm* 3.1. *These facts motivated us to modify the resolvent method.*

If g is invertible, then, using the technique of updating the solution, one can rewrite (3.1) in the form:

$$g(u) = J_\varphi[g(y) - \rho N(y,y)] \tag{3.2}$$

$$g(y) = J_\varphi[g(u) - \rho N(u,u)]. \tag{3.3}$$

For the sake of simplicity, we denote by

$$w = J_\varphi[g(y) - \rho N(y,y)], \tag{3.4}$$

unless otherwise specified.

The fixed-point formulation (3.2) allows us to suggest the following iterative method, which is known as the extraresolvent method.

Algorithm 3.2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} g(u_{n+1}) &= J_\varphi[g(y_n) - \rho N(y_n, y_n)] = w_n, \\ g(y_n) &= J_\varphi[g(u_n) - \rho N(u_n, u_n)], \quad n = 0, 1, 2, \cdots . \end{aligned}$$

For $N(u, u) \equiv Tu$, where $T : H \longrightarrow H$ is a single-valued operator, then Algorithm 3.2 becomes:

Algorithm 3.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} g(u_{n+1}) &= J_\varphi[g(y_n) - \rho T y_n], \\ g(y_n) &= J_\varphi[u_n - \rho T u_n], \quad n = 0, 1, 2, \cdots \end{aligned}$$

which can be written as

$$g(y_{n+1}) = J_\varphi[J - \varphi[g(u_n) - \rho T u_n] - \rho T g^{-1} J_\varphi[g(u_n) - \rho T u_n]], \quad n = 0, 1, 2, \cdots .$$

This is known as the well known forward-backward splitting algorithm, see [8].

If $g \equiv I$, where I is the identity operator, then Algorithm 3.2 becomes

Algorithm 3.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} u_{n+1} &= J_\varphi[y_n - \rho N(y_n, y_n)], \\ y_n &= J_\varphi[u_n - \rho N(u_n, u_n)], \quad n = 0, 1, 2, \cdots . \end{aligned}$$

If φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto K. Consequently Algorithm 3.3 is known as the modified extragradient method for solving variational inequalities, the convergence of which requires the monotonicity. In some cases, the operator may not be a monotone operator. To overcome this drawback of these extraresolvent methods, we suggest another method, which converges for the pseudomonotone operators and this is main motivation of this paper.

We define the residue vector $R(u)$ by the relation

$$R(u) = g(u) - J_\varphi[g(y) - \rho N(y, y)] = g(u) - w. \tag{3.5}$$

¿From Lemma 3.1, it follows that $u \in H$ is a solution of the general mixed variational inequality (2.1) if and only if $u \in H$ is a zero of the equation

$$R(u) = 0. \tag{3.6}$$

For a constant $\gamma \in (0, 2)$, equation (3.6) can be written as

$$g(u) = g(u) - \gamma R(u).$$

This formulation is used to suggest a new extraresolovent method for solving the general mixed variational inequality (2.1).

Algorithm 3.5. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) - \gamma R(u_n), \qquad n = 0, 1, 2, \cdots . \tag{3.7}$$

We remark that if φ is the indicator function of a closed convex set K in H, then the resolvent operator $J_\varphi \equiv P_K$ the projection of H onto K. Consequently, the relation (3.4) becomes

$$R_K(u) = g(u) - P_K[g(y) - \rho N(y, y)],$$

and Algorithm 3.5 becomes Algorithm 3.6, which is called the modified extragradient method for solving the general variational inequalities (2.3).

Algorithm 3.6. *For a given $u_0 \in H$, $g(u_0) \in K$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) - \gamma R_K(u_n), \quad n = 0, 1, 2, \cdots .$$

If $g \equiv I$, the identity operator, then Algorithm 3.5 reduces to:

Algorithm 3.7. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n - \gamma R(u_n), \quad n = 0, 1, 2, \cdots$$

where

$$R(u_n) = u_n - J_\varphi[y_n - \rho N(y_n, y_n)], \quad n = 0, 1, 2, \cdots .$$

If φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto K. Consequently Algorithm 3.7 becomes:

Algorithm 3.8. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n - \gamma\{u_n - P_K[y_n - \rho N(y_n, y_n)]\}, \quad n = 0, 1, 2, \cdots$$

which is known as the modified extragradient method for solving variational inequalities.

For $\gamma = 1$, Algorithm 3.8 is known as extragradient method. For appropriate and suitable choice of the operators $N(\cdot,\cdot), g$ and the space H, one can obtain a number of new and known iterative methods for solving various classes of variational inequalities and related optimization problems.

One can study the convergence analysis of Algorithm 3.4 by using the technique of Noor [5,8]. However, to convey an idea and for the sake of completeness, we include its proof.

Lemma 3.2. *Let $\bar{u} \in H$ be a solution of* (2.1)*. If $N(\cdot,\cdot) : H \times H \to H$ is a g-pseudomonotone operator, then*

$$\langle g(u) - g(\bar{u}), R(u)\rangle \geq \|R(u)\|^2, \text{ for all } u \in H. \tag{3.8}$$

Proof. Let $\bar{u} \in H$ be a solution of (2.1). Then

$$\langle N(\bar{u}, \bar{u}), g(v) - g(\bar{u})\rangle + \varphi(g(v)) - \varphi(g(\bar{u})) \geq 0, \quad \text{for all } g(v) \in H,$$

which implies that

$$\langle N(v, v), g(v) - g(u)\rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \quad \text{for all } g(v) \in H, \tag{3.9}$$

since $N(\cdot,\cdot)$ is g-pseudomonotone.

Taking $g(v) = w$ in (3.9), we have

$$\rho\langle N(g^{-1}(w), g^{-1}(w)), w - g(\bar{u})\rangle + \rho\varphi(w) - \rho\varphi(g(\bar{u})) \geq 0. \tag{3.10}$$

Setting $z = g(u) - \rho N(g^{-1}(w), g^{-1}(w)), u = w, v = g(\bar{u})$ in (2.6), we obtain

$$\langle g(u) - \rho N(g^{-1}(w), g^{-1}(w)) - w, w - g(\bar{u})\rangle + \rho\varphi(g(\bar{u})) - \rho\varphi(w) \geq 0. \tag{3.11}$$

Adding (3.11), (3.10) and using (3.4), we have

$$\langle -R(u), g(\bar{u}) - g(u) + R(u)\rangle \geq 0,$$

which implies

$$\langle g(u) - g(\bar{u}), R(u)\rangle \geq \|R(u)\|^2,$$

the required result. □

Lemma 3.3. *Let $\bar{u} \in H$ be the solution of* (2.1) *and u_{n+1} be the approximate solution obtained from Algorithm* 3.5*, then*

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2. \qquad (3.12)$$

Proof. Since $\bar{u}$ is a solution of (2.1) and u_{n+1} satisfies the relation (3.7), so

$$\begin{aligned}\|g(u_{n+1}) - g(\bar{u})\|^2 =& \|g(u_n) - g(\bar{u}) - \gamma R(u_n)\|^2 \\ \leq& \|g(u_n) - g(\bar{u})\|^2 - 2\gamma\langle g(u_n) - g(\bar{u}), R(u)\rangle + \gamma^2\|R(u_n)\|^2 \\ =& \|g(u_n) - g(\bar{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2.\end{aligned}$$

□

Theorem 3.1. *Let $g : H \to H$ be invertible and H be a finite dimensional space, then the approximate solution u_{n+1} obtained from Algorithm* 3.5 *converges to a solution $\bar{u}$ of the general mixed variational inequality* (2.1)*.*

Proof. Let $\bar{u} \in H$ be a solution of (2.1). From (3.12), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \gamma(2-\gamma)\|R(u_n)\|^2 \leq \|g(u_0) - g(\bar{u})\|^2$$

and consequently

$$\lim_{n\to\infty} R(u_n) = 0.$$

Let $\bar{u}$ be a cluster point of $\{u_n\}$ and suppose that the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converges to $\bar{u}$. Since $R(u)$ is continuous, it follows that

$$R(\bar{u}) = \lim_{j\to\infty} R(u_{n_j}) = 0$$

and $\bar{u}$ is the solution of the general mixed variational inequality (2.1) by invoking Lemma 3.1 and

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point and

$$\lim_{n\to\infty} g(u_n) = g(\bar{u}).$$

Since g is invertible, so

$$\lim_{n\to\infty} (u_n) = \bar{u},$$

which is the solution of the general mixed variational inequality (2.1). □

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