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MAPPINGS BETWEEN BANACH SPACES

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Abstract. Let X be a Banach space and let $\tilde{X}$ be a class of spaces isomorphic to X with a pseudo metric $\triangle(Y,Z) = \inf\{\ln \|T\| \cdot \|T^{-1}\| : Y, Z \in \tilde{X}, T : Y \to Z$ is an isomorphism $\}$. We make use of linear functionals in dual space X^* to obtain bounds for $\triangle(X, H)$, where H denotes a Hilbert space, so that X is a uniformly nonsquare space or X is a space with uniform normal structure respectively.

1. Introduction

Let X be a normed linear space and let $S(X) = \{x \in X : ||x|| = 1\}$ and $B(X) = \{x \in X : ||x|| \leq 1\}$ be the unit sphere and unit ball of X, respectively.

In a series of papers, Schäffer made use of concept of geodesic to study the unit spheres S(X). He introduced the following two notations:

$$m(X) = \inf\{d(x, -x) : x \in S(X)\}$$

and

$$M(X) = \sup\{d(x, -x) : x \in S(X)\},$$

where $d(x, -x)$ is the shortest length of the arcs joining antipodal points $x, -x$ in $S(X)$. He called $2m(X)$ the girth, and $2M(X)$ the perimeter of X. These parameters were used to study some classic Banach spaces, reflexivity, and isomorphism of Banach spaces [6].

Gao and Lau considered a simplification of such a concept. They defined the distance of antipodal points x and $-x$ on $S(X)$ as

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$$a(x) = \inf\{\max\{\|x+y\|, \|x-y\|\} : y \in S(X)\},$$

$$g(X) = \inf\{a(x) : x \in S(X)\},$$

and

$$G(X) = \sup\{a(x) : x \in S(X)\}.$$

They also defined another distance function of x and $-x$ on $S(X)$ as

$$b(x) = \sup\{\min\{\|x+y\|, \|x-y\|\} : y \in S(X)\},$$

$$j(X) = \inf\{b(x) : x \in S(X)\},$$

and

$$J(X) = \sup\{b(x) : x \in S(X)\}.$$

These parameters were used to study some classic Banach spaces, reflexivity, uniform nonsquare, normal structure and isomorphism of Banach spaces [3]. Recently Gao introduced a new parameter $W(\epsilon)$ and showed the relationship between $W(\epsilon)$ and uniform nonsquare and normal structure [2].

In this paper, by using parameter $W(\epsilon)$ and mappings between isomorphic Banach spaces, we further study uniformly nonsquare spaces, and the spaces with uniform normal structure. Some results in [3] and [4] are improved.

2. Preliminary

Let X and Y be normed linear spaces and $T : X \to Y$ be an isomorphism. Following the notation of Schäffer [6], we define $\partial T : S(X) \to S(Y)$ by $\partial T(x) = \frac{Tx}{||Tx||}$. It is clear that ∂T is bijection.

Let X^* and Y^* be the dual space of X and Y, respectively. For any $x \in S(X)$, we use $\nabla_x \subseteq S(X^*)$ to denote the set of norm 1 supporting functionals of $S(X)$ at x and T^* the conjugate mapping of T from Y^* to X^*. For any $x \in X$ and $f \in X^*$, we use $\langle x, f\rangle$ to denote the value of linear functional f at x.

Lemma 1. *For any $x \in S(X)$, $\frac{T^*}{\|T(x)\|}$ maps $\nabla_{(\partial T)x}$ of Y^* to ∇_x of X^* and $\frac{T^*}{\|T(x)\|}$ is bijection from $\nabla_{(\partial T)x}$ of Y^* to ∇_x of X^*.*

Proof. For any $f_{(\partial T)x} \in \nabla_{(\partial T)x}$, $\langle x, \frac{T^*(f_{(\partial T)x})}{||Tx||}\rangle = \langle Tx, \frac{f_{(\partial T)x}}{||Tx||}\rangle = \langle \frac{Tx}{||Tx||}, f_{(\partial T)x}\rangle = \langle (\partial T)x, f_{(\partial T)x}\rangle = 1$. So $\frac{T^*(f_{(\partial T)x})}{||Tx||} \in \nabla_x$.

Conversely, for any $f_x \in \nabla_x$, consider $||Tx||(T^*)^{-1}(f_x) \in Y^*$, we have $\frac{T^*(||Tx||(T^*)^{-1}(f_x))}{||Tx||} = f_x$, and $\langle (\partial T)x, ||Tx||(T^*)^{-1}(f_x)\rangle = \langle Tx, (T^*)^{-1}(f_x)\ \rangle = \langle x, (T^*)(T^*)^{-1}(f_x)\rangle = \langle x, f_x\rangle = 1$. So $||Tx||(T^*)^{-1}(f_x) \in \nabla_{(\partial T)x}$. $\square$

For any $x, y \in S(X), f_x \in \nabla_x$, let $f_{(\partial T)x} = (\frac{T^*}{||Tx||})^{-1}(f_x)$. From Lemma 1, $f_{(\partial T)x} \in \nabla_{(\partial T)x}$. Furthermore, we have following lemma:

Lemma 2. $\frac{1}{||T||\cdot||T^{-1}||}\langle y, f_x\rangle \leq \langle (\partial T)y, f_{(\partial T)x}\rangle \leq ||T||\cdot||T^{-1}||\langle y, f_x\rangle$.

Proof. $\langle y, f_x\rangle = \langle y, \frac{T^*(f_{(\partial T)x})}{||Tx||}\rangle = \langle Ty, \frac{f_{(\partial T)x}}{||Tx||}\rangle = \frac{||Ty||}{||Tx||}\ \langle \frac{Ty}{||Ty||}, f_{(\partial T)x}\ \rangle = \frac{||Ty||}{||Tx||}\langle (\partial T)y, f_{(\partial T)x}\rangle$. Since $1 = ||x|| = ||T^{-1}(Tx)|| \leq ||T^{-1}||\cdot||Tx||$, $||Tx|| \geq \frac{1}{||T^{-1}||}$. Similarly $||Ty|| \geq \frac{1}{||T^{-1}||}$. We have $\langle y, f_x\rangle \leq ||T||\cdot\|y\|\cdot||T^{-1}||$ $\langle (\partial T)y, f_{(\partial T)x}\rangle = ||T||\cdot||T^{-1}||\langle (\partial T)y, f_{(\partial T)x}\rangle$ and $\langle y, f_x\rangle \geq \frac{1}{||T||\cdot\|x\|\cdot||T^{-1}||}$ $\langle (\partial T)y, f_{(\partial T)x}\rangle = \frac{1}{||T||\cdot||T^{-1}||}\langle (\partial T)y, f_{(\partial T)x}\rangle$. $\square$

Lemma 3. [3] *For any* $x, y \in S(X)$, $\frac{1}{||T||\cdot||T^{-1}||}(\|x \pm y\| + 2) \leq \|(\partial T)y \pm (\partial T)x\| + 2 >\leq ||T||\cdot||T^{-1}||(\|x \pm y\| + 2)$.

3. Main Theorem

Definition 1. [5] A normed linear space X is called *uniformly nonsquare* if there exists a $\delta > 0$ such that either $\frac{1}{2}(x + y) \leq 1 - \delta$ or $\frac{1}{2}(x - y) \leq 1 - \delta$.

Definition 2. [1] A bounded, convex subset K of a Banach space X is said to have *normal structure* if every convex subset H of K that contains more than one point contains a point $x_0 \in H$, such that $\sup\{\|x_0 - y\|, y \in H\} < d(H)$, where $d(H) = \sup\{\|x - y\|, x, y \in H\}$ denotes the diameter of H. A Banach space X is said to have *normal structure* if every bounded, convex subset of X has normal structure. A Banach space X is said to have weak normal structure if for each weakly compact convex set K in X that contains more than one point has normal structure. X is said to have *uniform normal structure* if there exists $0 < c < 1$ such that for any subset K as above, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\|, y \in K\} < c\cdot(d(K))$.

For a reflexive Banach space X, the normal structure and weak normal structure coincide.

Definition 3. [6] Let X be a given normed linear space and $\tilde{X}$ be the class of spaces isomorphic to X. The *pseudo metric* D on $\tilde{X}$ is defined as: $D(X, Y) = \inf\{\ln(||T||\cdot||T^{-1}||) : T : X \to Y$ is an isomorphism$\}$.

Recently, Gao introduced a parameter $W(\epsilon)$ for a Banach space:

Definition 4. [2] Let X be a given normed linear space, $x, y \in S(X)$, $f_x \in \nabla_x$, and $\gamma(x,y) = \sup\{\langle \frac{x-y}{2}, f_x\rangle : f_x \in \nabla_x\}$. Then $W_X(\epsilon) = \inf\{\gamma(x,y), \|x-y\| \geq \epsilon\}, 0 \leq \epsilon \leq 2$, is called the *modulus* of W-convexity.

Remark. In Definition 4, $\|x-y\| \geq \epsilon$ may be replaced by $\|x-y\| > \epsilon$ or $\|x-y\| = \epsilon$.

Proposition 1. [2] *$W_X(\epsilon)$ is an increasing function of ϵ, $0 \leq \epsilon \leq 2$.*

Theorem 1. [2] *Let X be a Banach space.*

(i) *If $W_X(2^-) > 0$, then X is uniformly nonsquare.*
(ii) *If $W_X(1^-) > 0$ then X has a uniform normal structure.*

We will make use of the parameter $W_X(\epsilon)$ to obtain bounds of pseudo metric D for uniformly nonsquare and normal structure.

Theorem 2. *If $\Delta(X,Y)$ satisfies the condition*

$$W_X(\frac{3}{e^{\Delta(X,Y)}} - 2) > \frac{e^{\Delta(X,Y)} - 1}{2e^{\Delta(X,Y)}},$$

then both X with continuous $W(\epsilon)$ and Y has a uniform normal structure.

Proof. Since $e^{\Delta(X,Y)} \geq 1$, $W_X(\frac{3}{e^{\Delta(X,Y)}} - 2) > \frac{e^{\Delta(X,Y)}-1}{2e^{\Delta(X,Y)}}$ implies $W_X(1) > 0$, by Proposition 1, and hence X has a uniform normal structure by Theorem 1.

To prove Y has a uniform normal structure, if $x, y \in S(X)$, and T is an isomorphism from$X$ to Y such that $\|(\partial T)x - (\partial T)y\| \geq \|T\| \cdot \|T^{-1}\|(2+\epsilon) - 2$. From Lemma 3, we have $\|x-y\| \geq \epsilon$, hence $\langle \frac{x-y}{2}, f_x\rangle > W_X(\epsilon)$ for some $f_x \in \nabla_x$.

Let $f_{(\partial T)x} = (\frac{T^*}{||Tx||})^{-1}(f_x) \in \nabla_{(\partial T)x}$, then $\langle \frac{(\partial T)x - (\partial T)y}{2}, f_{(\partial T)x}\rangle = \frac{1}{2} - \langle \frac{(\partial T)y}{2}, f_{(\partial T)x}\rangle \geq \frac{1}{2} - \|T\| \cdot \|T^{-1}\|\langle \frac{y}{2}, f_x\rangle = \frac{1}{2} + \|T\| \cdot \|T^{-1}\|\langle \frac{x-y}{2}, f_x\rangle - \frac{1}{2}\|T\| \cdot \|T^{-1}\| \geq \|T\| \cdot \|T^{-1}\| W_X(\epsilon) - \frac{1}{2}(\|T\| \cdot \|T^{-1}\| - 1)$.

We have proved that for all $x, y \in S(X)$, therefore for all $(\partial T)x, (\partial T)y \in S(Y)$ with $\|(\partial T)x - (\partial T)y\| \geq \|T\| \cdot \|T^{-1}\|(2+\epsilon) - 2$, there exists a $f_{(\partial T)x} \in \nabla_{(\partial T)x}$ such that $\langle \frac{(\partial T)x - (\partial T)y}{2}, f_{(\partial T)x}\rangle \geq \|T\| \cdot \|T^{-1}\| W_X(\epsilon) - \frac{1}{2}(\|T\| \cdot \|T^{-1}\| - 1)$.

Therefore, by definition of W_Y, $W_Y(\|T\| \cdot \|T^{-1}\|(2+\epsilon) - 2) \geq \|T\| \cdot \|T^{-1}\| W_X(\epsilon) - \frac{1}{2}(\|T\| \cdot \|T^{-1}\| - 1)$.

Let $\epsilon = \frac{3}{\|T\| \cdot \|T^{-1}\|} - 2$, we have $W_Y(1) \geq \|T\| \cdot \|T^{-1}\| W_X(\frac{3}{\|T\| \cdot \|T^{-1}\|} - 2) - \frac{1}{2}(\|T\| \cdot \|T^{-1}\| - 1)$.

If $\triangle(X,Y)$ satisfies $W_X(\frac{3}{e^{\Delta(X,Y)}}-2) > \frac{e^{\Delta(X,Y)}-1}{2e^{\Delta(X,Y)}}$, we can take an isomorphism T such that $W_X(\frac{3}{\|T\|\cdot\|T^{-1}\|}-2) > \frac{(\|T\|\cdot\|T^{-1}\|-1)}{2\|T\|\cdot\|T^{-1}\|}$. Therefore $W_Y(1) > 0$. From [2], Y has a uniform normal structure. $\square$

Similarly we can prove the following theorem about uniformly nonsquare:

Theorem 3. *If $\Delta(X,Y)$ satisfies the condition*

$$W_X\left(\frac{4}{e^{\Delta(X,Y)}}-2\right) > \frac{e^{\Delta(X,Y)}-1}{2e^{\Delta(X,Y)}},$$

then both X with continuous $W(\epsilon)$ and Y are uniformly nonsquare spaces.

Proposition 2. *For a Hilbert space H, $W_H(\epsilon) = \frac{\epsilon^2}{4}$.*

Proof. Let $x, y \in S(H)$ with $\|x-y\| = \epsilon$. From $\|x-y\|^2 = \|x\|^2 - 2x\cdot y + \|y\|^2$ where $x\cdot y$ denotes the inner product of x and y. we have $x\cdot y = \frac{2-\epsilon^2}{2}$. Therefore $\langle\frac{x-y}{2}, f_x\rangle = \frac{1}{2} - \frac{1}{2}\langle y, f_x\rangle = \frac{1}{2} - \frac{1}{2}x\cdot y = \frac{\epsilon^2}{4}$. $\square$

Theorem 4. *If $\Delta(X,H) < \ln\frac{5-\sqrt{7}}{2}$, then X has a uniform normal structure.*

Proof. From Theorem 2, and Proposition 1, $\frac{1}{4}(\frac{3}{e^{\Delta(X,H)}}-2)^2 > \frac{e^{\Delta(X,H)}-1}{2e^{\Delta(X,H)}}$ implies X has a uniform normal structure. By solving the above quadratic equation for $e^{\Delta(X,H)}$ we have $e^{\Delta(X,H)} < \frac{5-\sqrt{7}}{2}$, hence $\Delta(X,H) \le \ln\frac{5-\sqrt{7}}{2}$ implies X has a uniform normal structure. $\square$

Similarly we can prove the following theorem about uniformly nonsquare:

Theorem 5. *If $\Delta(X,H) < \ln\frac{7-\sqrt{17}}{2}$, then X is uniformly nonsquare.*

Theorem 5 improved the Theorem 4.4 and Theorem 4.5 of [3] for $p = 2$, and Corollary 6.5 of [4] for $p = 2$.

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