Nonlinear Funct. Anal. & Appl., Vol. 7, No. 1 (2002), pp. 1–14

TWO-STEP APPROXIMATION SCHEMES FOR MULTIVALUED QUASI VARIATIONAL INCLUSIONS

Muhammad Aslam Noor

Abstract. It is well known that the multivalued quasi variational inclusions are equivalent to the implicit resolvent equations. In this paper, we use the resolvent equations technique to suggest and analyze a class of two-step iterative methods for solving the multivalued quasi variational inclusions. We also discuss some special cases, which can be obtained from our results. The results obtained in this paper represent an improvement and a significant refinement of previously known results.

1. Introduction

Multivalued quasi variational inclusion, which was introduced and studied by Noor [6-8], is a useful and important extension of the variational principles with a wide range of applications in industry, physical, regional, social, pure and applied sciences. Quasi variational inclusions provide us with a unified, natural, novel, innovative and general technique to study a wide class of problems arising in different branches of mathematical and engineering sciences, There are a substantial number of numerical methods including projection method and its variant forms, Wiener-Hopf equations, auxiliary principle and descent for solving various classes of variational inequalities and complementarity problems. It is well known that the projection methods, Wiener-Hopf equations techniques and auxiliary principle techniques cannot be extended and modified for solving variational inclusions. This fact motivated to develop another technique, which involves the use of the resolvent operator associated with maximal monotone operator. Using the resolvent operator technique, one shows that the variational inclusions are equivalent to the fixed point problem.

Received January 22, 2001.

2000 Mathematics Subject Classification: 49J40, 90C33.

Key words and phrases: Variational inclusions, resolvent equations, algorithms, iterative methods, convergence.

This alternative formulation was used to develop a number of numerical methods for solving various classes of variational inclusions and related problems, see [3-23] and the references therein.

Equally important is the area of mathematical sciences known as the resolvent equations. The resolvent equations technique is being used to develop powerful and efficient numerical techniques for solving variational inclusions and related optimization problems. The resolvent equations technique provides a simple and convenient device for formulating a wide variety of important problems in a unified manner. In recent years, the resolvent equations have been generalized and extended in many directions using novel and innovative techniques, both for their own sake and its applications. A useful and important generalization is called the implicit resolvent equation, which was introduced and studied by Noor[8] associated with multivalued quasi variational inclusions. Noor[8] has shown that the multivalued quasi variational inclusions are equivalent to the implicit resolvent equations by using the resolvent operator method. In this paper, we use this equivalence is used to suggest and analyze a number of two-step iterative methods for solving the multivalued quasi variational inclusions and related optimization problems. This paper is a continuous of our earlier works. We remark that if the nonlinear term is the indicator function of a closed convex set in H, then the resolvent equations are equivalent to the Wiener-Hopf equations, which were introduced by Shi [23] and Robinson [22] for the standard variational inequalities. For applications, formulation and numerical methods for the Wiener-Hopf equations, see Noor[12-15, 19, 22, 23] and references therein. Since the multivalued mixed variational inequalities include the classical variational inequalities, quasi variational inequalities and generalized quasi complementarity problems as special cases, our results also hold true for these problems.

In Section 2, we formulate the problems and review some basic results and concepts. In Section 3, we use the implicit resolvent equations technique to suggest and analyze a class of two-step iterative schemes for the multivalued quasi variational inclusions. We also study the convergence criteria of these methods.

2. Formulations and Basic Results

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. Let $C(H)$ be a family of all nonempty compact subsets of H. Let $T, V : H \longrightarrow C(H)$ be the multivalued operators and $g : H \longrightarrow H$ be a single-valued operator. Let $A(\cdot, \cdot) : H \times H \longrightarrow H$ be a maximal monotone operator with respect to the first argument. For a given nonlinear operator $N(\cdot,\cdot) : H \times H \longrightarrow H$, consider the problem of finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$0 \in N(w,y) + A(g(u),u), \tag{2.1}$$

which is called the multivalued quasi variational inclusions, see Noor[6-8]. A number of problems arising in structural analysis, mechanics and economics can be studied in the framework of the multivalued quasi variational inclusions; see , for example,[2,6-10,16]. We now discuss some special cases of the problem (2.1).

Special Cases.
I. If $A(\cdot,u) = \partial\phi(\cdot,u) : H \times H \longrightarrow R \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function with respect to the first argument, then problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\langle N(w,y), g(v)-g(u)\rangle + \phi(g(v),g(u)) - \phi(g(u),g(u)) \geq 0, \text{ for all } v \in H, \tag{2.2}$$

which is called the set-valued mixed quasi variational inequality, see, for example [9,10].
II. If $A(g(u,v) \equiv A(g(u))$, for all $v \in H$, then problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$0 \in N(w,y) + A(g(u)), \tag{2.3}$$

a problem considered and studied by Noor [11] using the resolvent equations technique. See also [25] for the related work.
III. If $A(g(u)) \equiv \partial\phi(g(u))$, where $\phi : H \longrightarrow R \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function, then problem (2.1) reduces to: find $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\langle N(w,y), g(v) - g(u)\rangle + \phi(g(v)) - \phi(g(u)) \geq 0. \tag{2.4}$$

Problem (2.4) is known as the set-valued mixed variational inequality, which has been studied in [17,18]'
IV. If the function $\phi(\cdot,\cdot)$ is the indicator function of a closed convex-valued set $K(u)$ in H, that is,

$$\phi(u,u) = K_{(u)}(u) = \begin{cases} 0, & \text{if} \quad u \in K(u) \\ +\infty, & \text{otherwise} , \end{cases}$$

then problem (2.2) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$, $g(u) \in K(u)$ such that

$$\langle N(w,y), g(v) - g(u)\rangle \geq 0, \quad \text{for all} \quad v \in K(u), \tag{2.5}$$

a problem considered and studied by Noor [14], using the projection method and the implicit Wiener-Hopf equations technique.

V. If $K^*(u) = \{u \in H, \langle u, v\rangle \geq 0, \text{ for all } v \in K(u)\}$ is a polar cone of the convex-valued cone $K(u)$ in H, then problem (2.5) is equivalent to finding $u \in$, $w \in T(u)$, $y \in V(u)$ such that

$$g(u) \in K(u), \quad N(w,y) \in K^*(u) \quad \text{and} \quad \langle N(w,y), g(u)\rangle = 0, \tag{2.6}$$

which is called multivalued implicit complementarity problem, see Noor[6-8].

For suitable and appropriate choice of the operators T, $N(\cdot,\cdot), g$ and the convex set K, one can obtain a large number of variational inequalities and complementarity problems, see, for example, [3-25] and the references therein. We would like to mention that the problem of finding a zero of the sum of two maximal monotone operators, location problem, $\min_{u\in H}\{f(u) + g(u)\}$, where f, g are both convex functions, various classes of variational inequalities and complementarity problems are very special cases of problem (2.1). Thus it is clear that problem (2.1) is general and unifying one and has numerous applications in pure and applied sciences.

Related to the multivalued quasi variational inclusions, we now consider a new system of equations, which are called the implicit resolvent equations. For this purpose, we need the following concepts and notions.

Definition 2.1. [1] If T is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with T is defined by

$$J_T(u) = (I + \rho T)^{-1}(u), \quad \text{for all} \quad u \in H,$$

where I is the identity operator. It is known that the monotone operator T is maximal monotone if and only if the resolvent operator J_T is defined everywhere on the space. Furthermore, the resolvent operator J_T is single-valued and nonexpansive.

Remark 2.1. *Since the operator $A(\cdot,\cdot)$ is a maximal monotone operator with respect to the first argument, for a constant $\rho > 0$, we denote by*

$$J_{A(u)} \equiv (I + \rho A(u))^{-1}(u), \quad \textit{for all} \quad u \in H,$$

the resolvent operator associated with $A(\cdot,u) \equiv A(u)$. *For example, if* $A(\cdot,u) = \partial\phi(\cdot,u)$, *for all* $u \in H$, *and* $\phi(\cdot,\cdot) : H \times H \longrightarrow R \cup \{+\infty\}$ *is a proper, convex and lower semicontinuous with respect to the first argument, then it is well-known that* $\partial\phi(\cdot,u)$ *is a maximal monotone operator with respect to the first argument. In this case, the resolvent operator* $J_{A(u)} = J_{\phi(u)}$ *is*

$$J_{\phi(u)} = (I + \rho\partial\phi(\cdot,u))^{-1}(u) = (I + \rho\partial\phi(u))^{-1}(u), \quad \text{for all} \quad u \in H,$$

which is defined everywhere on the space H, *where* $\partial\phi(u) \equiv \partial\phi(\cdot,u)$. *For a recent state-of-the-art of the nonlinear convex analysis, see Gao[4].*

Let $R_{A(u)} \equiv I - J_{A(u)}$, where I is the identity operator and $J_{A(u)} = (I + \rho A(u))^{-1}$ is the resolvent operator. For given operators $T, V : H \longrightarrow C(H)$ and $N(\cdot,\cdot) : H \times H \longrightarrow H$, consider the problem of finding $z, u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$N(w,y) + \rho^{-1}R_{A(u)}z = 0, \tag{2.7}$$

where $\rho > 0$ is a constant. Equations (2.7) are called the implicit resolvent equations, introduced and studied by Noor[8]. In particular, if $A(g(u),u) \equiv A(u))$, then $J_{A(u)} = (I + \rho A)^{-1} = J_A$ and implicit resolvent equations (2.7) are equivalent to finding $u, z \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$N(w,y) + \rho^{-1}R_A z = 0, \tag{2.8}$$

which are called the resolvent equations, introduced and studied by Noor [11]. It has been shown in [11] that the problems (2.8) and (2.3) are equivalent using the technique of general principle of duality. This equivalence was used to suggest and analyze some iterative methods for solving the generalized set-valued variational inclusions. For formulation and applications of the resolvent equations, see [6-13].

If $A(\cdot,\cdot) = \phi(\cdot,\cdot)$ is the indicator function of a closed convex-valued set $K(u)$ in H, then the resolvent operator $J_{A(u)} \equiv P_{K(u)}$, the projection of H onto $K(u)$. Consequently problem (2.7) is equivalent to finding $z, u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$N(w,y) + \rho^{-1}Q_{K(u)}z = 0, \tag{2.9}$$

where $Q_{K(u)} = I - P_{K(u)}$ and I is the identity operator. The equations of the type (2.9) are called the implicit Wiener-Hopf equations, which were introduced and studied by Noor [14]. For applications, formulation and numerical methods of the Wiener-Hopf equations, see [12-15, 22,23].

Definition 2.2. For all $u_1, u_2 \in H$, the operator $N(\cdot,\cdot)$ is said to be strongly monotone and Lipschitz continuous with respect to the first argument, if there exist constants $\alpha > 0$, $\beta > 0$ such that

$$\langle N(\cdot, w_1) - N(\cdot, w_2), u_1 - u_2 \rangle \geq \alpha \|u_1 - u_2\|^2, \text{ for all } w_1 \in T(u_1), w_2 \in T(u_2)$$
$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \beta \|u_1 - u_2\|.$$

In a similar way, we can define strong monotonicity and Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to second argument.

Definition 2.3. The set-valued operator $V : H \longrightarrow C(H)$ is said to be M-Lipschitz continuous, if there exists a constant $\xi > 0$ such that

$$M(V(u), V(v) \leq \xi \|u - v\|, \quad \text{for all} \quad u, v \in H,$$

where $M(\cdot,\cdot)$ is the Hausdorff metric on $C(H)$.

We also need the following condition.

Assumption 2.1. For all $u, v, w \in H$, the resolvent operator $J_{A(u)}$ satisfies the condition

$$\|J_{A(u)}w - J_{A(v)}w\| \leq \nu \|u - v\|, \tag{2.10}$$

where $\nu > 0$ is a constant.

It has been shown in[8] that Assumption 2.1 holds in some special cases.

3. Main Results

In this section, we use the resolvent operator technique to establish the equivalence between the multivalued quasi variational inclusions and the implicit resolvent fixed points. This equivalence is used to suggest an iterative method for solving the quasi variational inclusions. For this purpose, we need the following result, which is due to Noor[6-8].

Lemma 3.1. *(u, w, y) is a solution of (2.1) if and only if (u, w, y) satisfies the relation*

$$g(u) = J_{A(u)}[g(u) - \rho N(w, y)], \tag{3.1}$$

where $\rho > 0$ is a constant and $J_{A(u)} = (I + \rho A(u))^{-1}$ is the resolvent operator.

From Lemma 3.1, we conclude that the multivalued quasi variational inclusions (2.1) are equivalent to the implicit fixed point problem (3.1). This alternative formulation is very useful from both theoretical and numerical analysis points of view. This equivalence has been used to propose some iterative algorithm for solving multivalued quasi variational inclusions (2.1) and related problems, see Noor[6-11] and the references therein.

The relation (3.1) can be written as

$$u = u - g(u) + J_{A(u)}[g(u) - \rho N(w, u)], \tag{3.2}$$

where $\rho > 0$ is a constant.

This fixed point formulation has been used to suggest and analyze the following two-step iterative scheme for solving multivalued quasi variational inclusions (2.1).

Algorithm 3.1. [7]. Assume that $T, V : H \longrightarrow C(H)$, $g : H \longrightarrow H$ and $N(\cdot,\cdot)$, $A(\cdot,\cdot) : H \times H \longrightarrow H$ are operators. For a given $u_0 \in H$, compute the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$ and $\{\overline{y_n}\}$ by the iterative schemes

$$\begin{aligned}
&w_n \in T(u_n) : ||w_{n+1} - w_n|| \leq M(T(u_{n+1}), T(u_n))\\
&y_n \in V(u_n) : ||y_{n+1} - y_n|| \leq M(V(u_{n+1}), V(u_n))\\
&\overline{w_n} \in T(v_n) : ||\overline{w_{n+1}} - \overline{w_n}|| \leq M(T(v_{n+1}), T(v_n))\\
&\overline{y_n} \in V(v_n) : ||\overline{y_{n=1}} - \overline{y_n}|| \leq M(V(v_{n+1}), V(v_n))\\
&v_n = (1 - \beta_n)u_n + \beta_n\{u_n - g(u_n) + J_{A(u_n)}[g(u_n) - \rho N(w_n, y_n)]\}\\
&u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{v_n - g(v_n) + J_{A(v_n)}[g(v_n) - \rho N(\overline{w_n}, \overline{y_n})]\},
\end{aligned}$$

for $n = 0, 1, 2 \ldots$, where $0 \leq \alpha_n, \beta_n \leq 1$; for all $n \geq 0$, and $\sum_{n=0}^{\infty} \alpha_n$ diverges and $\rho > 0$ is a constant.

In this paper, we suggest another class of two-step iterative schemes for solving multivalued quasi variational inequalities (2.1) by using the technique of the resolvent equations. For this purpose, we need the following result, which is due to Noor[8]. We include its proof for the sake of completeness.

Theorem 3.1. *The multivalued quasi variational inclusion (2.1) has a solution $u \in H$, $w \in T(u)$, $y \in V(u)$ if and only if $z, u \in H$, $w \in T(u)$, $y \in V(u)$ is a solution of the implicit resolvent equation (2.7), where*

$$g(u) = J_{A(u)}z \tag{3.3}$$

$$z = g(u) - \rho N(w, y), \tag{3.4}$$

and $\rho > 0$ is a constant.

Proof. Let $u \in H, w \in T(u), y \in V(u)$ be a solution of (2.1). Then, invoking lemma 3.1, we have

$$g(u) = J_{A(u)}[g(u) - \rho N(w, y)]. \tag{3.5}$$

Let

$$z = g(u) - \rho N(w, y). \tag{3.6}$$

From (3.5) and (3.6), we have

$$g(u) = J_{A(u)}z$$

and

$$z = J_{A(u)}z - \rho N(w, y),$$

that is,

$$N(w, y) + \rho^{-1}R_{A(u)}z = 0,$$

the required resolvent equations (2.7). □

From Theorem 3.1, we see that both problems (2.1) and (2.7) are equivalent. This interplay between these problems plays an important and fundamental role in suggesting a number of iterative algorithms for solving the mixed variational inclusions (2.1). By a suitable and appropriate rearrangement of the resolvent equations (2.7), we now suggest and analyze a new class of two-step iterative schemes for solving multivalued quasi variational inclusions (2.1).

I. The implicit resolvent equations (2.7) can be written as

$$R_{A(u)}z = -\rho N(w, y),$$

which implies that

$$z = J_{A(u)}z - \rho N(w, y) = g(u) - \rho N(w, y), \quad \text{using (3.3)}.$$

This fixed point formulation allows us to suggest the following two-step iterative method.

Algorithm 3.2. For given $z_0, u_0 \in H$, $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, compute the sequences $\{z_n\}$,$\{u_n\}$, $\{w_n\}$, $\{y_n\}$. $\{\overline{w_n}\}$ and $\{\overline{y_n}\}$ by the iterative schemes

$$g(u_n) = J_{A(u_n)}z_n \tag{3.7}$$

$$g(v_n) = J_{A(v_n)}v_n \tag{3.8}$$

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n)) \tag{3.9}$$

$$w_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)) \tag{3.10}$$

$$\overline{w_n} \in T(v_n) : \|\overline{w_{n+1}} - \overline{w_n}\| \leq M(T(v_{n+1}), T(v_n)) \tag{3.11}$$

$$\overline{y_n} \in V(v_n) : \|\overline{y_{n+1}} - \overline{y_n}\| \leq M(V(v_{n+1}), V(v_n)) \tag{3.12}$$

$$v_n = (1 - \beta_n)z_n + \beta_n\{g(u_n) - \rho N(w_n, y_n)\} \tag{3.13}$$

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n\{g(v_n) - \rho N(\overline{w_n}, \overline{y_n})\}, \ \ n = 0, 1, 2, \ldots . \tag{3.14}$$

where $0 \leq \alpha_n, \beta_n \leq 1$, for all $n \geq 0$, and $\sum_{n=0}^{\infty} \alpha_n$ diverges and $\rho > 0$ is a constant.

II. The resolvent equations (2.7) may be written as

$$0 = -\rho^{-1} R_{A(u)} z - N(w, y),$$

from which it follows that

$$R_{A(u)} z = (1 - \rho^{-1}) R_{A(u)} z - N(w, y).$$

Thus

$$\begin{aligned} z &= J_{A(u)} z - N(w, y) + (1 - \rho^{-1}) R_{A(u)} z \\ &= g(u) - N(w, y) + (1 - \rho^{-1}) R_{A(u)} z, \quad \text{using (3.3)}. \end{aligned}$$

We use this fixed-point formulation to suggest the following two-step iterative schemes for solving multivalued quasi variational inclusions (2.1).

Algorithms 3.3. For given $z_0, u_0 \in H, w_0 \in T(u_0), y_0 \in V(u_0)$, compute the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, and $\{\overline{y_n}\}$ by the iterative schemes

$$\begin{aligned} & g(u_n) = J_{A(u_n)} z_n \\ & g(v_n) = J_{A(v_n)} v_n \\ & w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n)) \\ & y_n \in V(u_n) P \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)) \\ & \overline{w_n} \in T(v_n) : \|\overline{w_{n+1}} - \overline{w_n}\| \leq M(T(v_{n+1}), T(v_n)) \\ & \overline{y_n} \in V(v_n) : \|\overline{y_{n+1}} - \overline{y_n}\| \leq M(V(v_{n+1}), V(v_n)) \\ & v_n = (1 - \beta_n) z_n + \beta_n \{g(u_n) - N(w_n, y_n) + (1 - \rho^{-1}) R_{A(u_n)} z_n\} \\ & z_{n+1} = (1 - \alpha_n) z_n + \alpha_n \{g(v_n) - N(\overline{w_n}, \overline{y_n}) + (1 - \rho^{-1}) R_{A(v_n)} z_n\}, \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $0 \leq \alpha_n, \beta_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges.

Note that Algorithms 3.1-3.3 are similar to the Ishikawa iterations for solving variational inequalities. For $\beta_n = 0$, Algorithms 3.1-3.3 are called the Mann iterations for solving the multivalued quasi variational inclusions (2.1) and appear to be new ones.

We now study the convergence analysis of Algorithm 3.2. In a similar way, one can study the convergence of Algorithm 3.3.

Theorem 4.2. *Let the operator $N(\cdot, \cdot)$ be strongly monotone with constant $\alpha > 0$ and be Lipschitz continuous with constant $\beta > 0$ with respect to the first* *argument. Let $g : H \longrightarrow H$ be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$. Let $N(\cdot,\cdot)$ be a Lipschitz continuous with constant $\eta > 0$ with respect to the second argument and $V : H \longrightarrow C(H)$ be M-Lipschitz continuous $\xi > 0$. Let $T : H \longrightarrow C(H)$ be a M-Lipschitz continuous with constant $\mu > 0$. If Assumption 2.1 and*

$$|\rho - \frac{\alpha - (1-k)\eta\xi}{\beta^2\mu^2 - \eta^2\xi^2}| < \frac{\sqrt{[\alpha - (1-k)\eta\xi]^2 - k(\beta^2\mu^2 - \eta^2\xi^2)(2-k)}}{\beta^2\xi^2 - \eta^2\xi^2} \tag{3.15}$$

$$\alpha > (1-k)\eta\xi + \sqrt{k(\beta^2\xi^2 - \eta^2\xi^2)(2-k)} \tag{3.16}$$

$$\rho\eta\xi < 1 - k \tag{3.17}$$

$$k = 2(\sqrt{1 - 2\sigma - \delta^2}) + \nu, \tag{3.18}$$

hold, then there exist $z, u \in H$, $w \in T(u)$, $y \in V(u)$ satisfying the implicit resolvent equations (2.7) and the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$ and $\{\overline{y_n}\}$ generated by Algorithm 4.1 converge to $z, u, w, y, \overline{w}$ and $\overline{y}$ strongly in H respectively.

Proof. If the Assumption (2.1) and the conditions (3.15)-(3.17) hold, then it has been shown in [6, Theorem 3.1, pp.106] that there exists a solution $u \in H, w \in T(u), y \in V(u)$ satisfying the multivalued quasi variational inclusions (2.1). From Theorem 3.1, it follows that $z, u \in H$ is a solution of the resolvent equations(2.7). Then

$$g(u) = J_{A(u)}z \tag{3.19}$$

$$z = (1-\alpha_n)z + \alpha_n\{g(u) - \rho N(w,y)\} \tag{3.20}$$

$$= (1-\beta_n)z + \beta_n\{g(u) - \rho N(w,y)\} \tag{3.21}$$

From (3.14) and (3.20), we have

$$\begin{aligned} \|z_{n+1} - z\| &\le (1-\alpha_n)\|z_n - z\| + \alpha_n\{\|g(v_n) - g(u) - \rho(N(\overline{w_n}, \overline{y_n}))\|\} \\ &\le (1-\alpha_n)\|z_n - z\| + \alpha_n\|v_n - u(g(v_n) - g(u))\| \\ &\quad + \alpha_n\|v_n - u - \rho(N(\overline{w_n}, \overline{y_n}) - N(w, \overline{y_n}))\| \\ &\quad + \rho\alpha_n\|N(w, \overline{y_n}) - N(w,y)\| \end{aligned} \tag{3.22}$$

Since $N(\cdot,\cdot)$ is a strongly monotone Lipschitz continuous operator with respect to the first argument, it follows that

$$\begin{aligned}
&\|v_n - u - \rho\{N(\overline{w_n}, \overline{y_n}) - N(w, \overline{y_n})\|^2 \\
&= \|v_n - u\|^2 - 2\rho\langle v_n - u, N(\overline{w_n}, \overline{y_n}) - N(w, \overline{y_n})\rangle \\
&\quad + \rho^2\|N(\overline{w_n}, \overline{y_n}) - N(w, \overline{y_n})\|^2 \\
&\leq (1 - 2\rho\alpha + \rho^2\beta^2\mu^2)\|v_n - u\|^2.
\end{aligned} \tag{3.23}$$

Using the Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to the second argument and the M-Lipschitz continuity of V, we have

$$\|N(w, \overline{y_n}) - N(w, y)\| \leq \eta\|\overline{y_n} - y\| \leq \eta M(V(v_n), V(u)) \leq \eta\xi\|v_n - u\|. \tag{3.24}$$

From the strongly monotonicity and Lipschitz continuity of the operator g, we have

$$\|v_n - u - (g(v_n) - g(u))\|^2 \leq (1 - 2\sigma + \delta^2)\|v_n - u\|^2, \tag{3.25}$$

where $\sigma > 0$ and $\delta > 0$ are the strongly monotonicity and Lipschitz continuity constants of the operator g respectively. Combining (3.22)-(3.25), we obtain

$$\begin{aligned}
\|z_{n+1} - z\| &\leq (1 - \alpha_n)\|z_n - z\| + \alpha_n\{(\sqrt{1 - 2\sigma + \delta^2}) + \rho\eta\xi \\
&\quad + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}\}\|v_n - u\| \\
&= (1 - \alpha_n)\|z_n - z\| + \alpha_n\left\{\frac{k - \nu}{2} + \rho\eta\xi + t(\rho)\right\}\|v_n - u\|, \quad \text{using (3.18)},
\end{aligned} \tag{3.26}$$

where

$$t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}. \tag{3.27}$$

Also from (3.3)-(3.8) and Assumption 2.1, we have

$$\begin{aligned}
\|v_n - u\| &\leq \|v_n - u - (g(v_n) - g(u))\| + \|J_{A(v_n)}v_n - J_{A(u)}z\| \\
&\leq \|v_n - u - (g(v_n) - g(u))\| + \|J_{A(v_n)}v_n - J_{A(v_n)}z\| \\
&\quad + \|J_{A(v_n)}z - J_{A(u)}z\| \\
&\leq \left(\frac{k - \nu}{2}\right)\|v_n - u\| + \nu\|v_n - u\| + \|v_n - z\|,
\end{aligned}$$

which implies that

$$\|v_n - u\| \leq \left\{\frac{1}{1 - \frac{k+\nu}{2}}\right\}\|v_n - z\|. \tag{3.28}$$

Combining (3.26) and (3.28), we obtain

$$\begin{aligned}\|z_{n+1}-z\| &\leq (1-\alpha_n)\|z_n-z\| + \alpha_n\left\{\frac{\frac{k-\nu}{2}+\rho\eta\xi+t(\rho)}{1-\frac{k+\nu}{2}}\right\}\|v_n-z\| \\ &= (1-\alpha_n)\|z_n-z\| + \alpha_n\theta\|v_n-z\| \\ &= \{1-\alpha_n(1-\theta)\}\|v_n-z\|,\end{aligned} \tag{3.29}$$

where

$$\theta = \frac{\frac{k-\nu}{2}+\rho\eta\xi+t(\rho)}{1-\frac{k+\nu}{2}}. \tag{3.30}$$

In a similar way, from (3.7) and (3.19), we have

$$\|v_n-z\| \leq (1-\beta_n)\|z_n-z\| + \beta_n\left\{\frac{k-\nu}{2}+\rho\eta\xi+t(\rho)\right\}\|u_n-u\|, \tag{3.31}$$

Also from (3.7) and (3.19), we have

$$\begin{aligned}\|u_n-u\| &\leq \|u_n-u-(g(u_n)-g(u))\| + \|J_{A(u_n)}z_n - J_{A(u_n)}z\| \\ &\quad + \|J_{A(u_n)}z - J_{A(u)}z\| \\ &\leq \left(\frac{k+\nu}{2}\right)\|u_n-u\| + \|z_n-z\|,\end{aligned}$$

from which it follows that

$$\|u_n-u\| \leq \left(\frac{1}{1-\frac{k+\nu}{2}}\right)\|z_n-z\|. \tag{3.32}$$

Combining (3.31) and (3.32), we have

$$\begin{aligned}\|v_n-z\| &\leq (1-\beta_n)\|z_n-z\| + \beta_n\theta\|z_n-z\| \\ &\leq (1-\beta_n(1-\theta))\|z_n-z\| \leq \|z_n-z\|.\end{aligned} \tag{3.33}$$

From (3.29) and (3.33), we have

$$\begin{aligned}\|z_{n+1}-z\| &\leq \{1-\alpha_n(1-\theta)\}\|z_n-z\| \\ &= \prod_{i=0}^{\infty}\{1-(1-\theta)\alpha_i\}\|z_0-z\|.\end{aligned} \tag{3.34}$$

From (3.15)-(3.17), it follows that $\theta < 1$. Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1-\theta > 0$, it implies that $\sum_{i=0}^{\infty}[1-(1-\theta)\alpha_i] = 0$. Hence the sequence $\{z_n\}$ converges strongly to z. From (3.32) and (3.33), we see that the sequence $\{v_n\}$ and $\{u_n\}$ converges strongly to z and u respectively. Using the technique of Ding[4] and Noor[8], one can easily show that the sequences $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, and $\{\overline{y_n}\}$ converge strongly to w, y, $\overline{w}$ and $\overline{y}$ respectively. Now by using the continuity of the operators T,V,g, $J_{A(u)}$ and Theorem 3.1, we have

$$z = g(u) - \rho N(w,y) = J_{A(u)} - \rho N(w,y) \in H.$$

It remains to show that $w \in T(u)$, $y \in V(u)$, $\overline{w} \in T(v)$ and $\overline{y} \in V(v)$. In fact,

$$\begin{aligned} d(w,T(u)) &\leq \|w - w_n\| + d(w_n, T(u)) \\ &\leq \|w - w_n\| + M(T(u_n), T(u)) \\ &\leq \|w - w_n\| + \mu\|u_n - u\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned}$$

where $d(w,T(u)) = inf\{\|w-z\| : z \in T(u)\}$. Since the sequences $\{w_n\}$ and $\{u_n\}$ are the Cauchy sequences, it follows that $d(w,T(u)) =$. This implies that $w \in T(u)$. In a similar way, one can show that $y \in V(u)$, $\overline{w} \in T(v)$, and $\overline{y} \in V(v)$. By invoking Theorem 3.1, we have $z, u \in H, w \in T(u), y \in V(u)$ which satisfies the implicit resolvent equations (2.7) and the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$ and $\{\overline{y_n}\}$ converge strongly to $z, u, w, y, \overline{w}$ and $\overline{y}$ in H respectively, the required result. □

References

1. H. Brezis, *Opérateur Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam (1973).
2. V. F. Demyanov, G. E. Stavroulakis, L. N. Polyakova and P. D. Panagiotopoulos, *Quasidifferentiability and Nonsmooth Modeling in Mechanics, Engineering and Economics*, Kluwer Academic Publishers, Holland (1996).
3. X. P. Ding, *Perturbed proximal point algorithms for generalized quasi variational inclusions*, J. Math. Anal. Appl. **210** (1997), 88–101.
4. D. Y. Gao, *Duality Principles in Nonconvex Systems: Theory, Methods and Applications*, Kluwer Academic Publishers, Dordrecht, Holland (1999).
5. F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York (1995).
6. M. Aslam Noor, *Set-valued quasi variational inequalities*, K. J. Comput. Appl. Math. **7** (2000), 101-113.
7. M. Aslam Noor, *Iterative schemes for multivalued quasi variational inequalities*, J. Global Optim. **19** (2001), 141-150.
8. M. Aslam Noor, *Multivalued quasi variational inclusions and implicit resolvent equations*, Nonlinear Anal. (2001), (in press).

9. M. Aslam Noor, *Generalized multivalued variational inequalities (III)*, Nonlinear Studies **6(2)** (1999), 191-206.
10. M. Aslam Noor, *Set-valued mixed quasi variational inequalities and implicit resolvent equations*, Math. Computer Modelling **29** (1999), 1-11.
11. M. Aslam Noor, *Generalized set-valued variational inclusions and resolvent equations*, J. Math. Anal. Appl. **228** (1998), 206-220.
12. M. Aslam Noor, *Some recent advances in variational inequalities*, Part I, basic concepts, New Zealand J. Math. **26** (1997), 53-80.
13. M. Aslam Noor, *Some recent advances in variational inequalities*, Part II, other concepts, New Zealand, J. Math. **26** (1997), 229–255.
14. M. Aslam Noor, *Generalized set-valued variational inequalities*, Le Matematiche **52** (1997), 3-24.
15. M. Aslam Noor, *Wiener-Hopf equations technique for variational inequalities*, Korean J. Comput. Appl. Math. **7** (2000), 581-598.
16. M. Aslam Noor and K. Inayat Noor, *Sensitivity analysis of quasi variational inclusions*, J. Math. Anal. Appl. **236** (1999), 290-299.
17. M. Aslam Noor and K. Inayat Noor, *Multivalued variational inequalities and resolvent equations*, Math. Computer Modelling **26(7)** (1997), 109–121.
18. M. Aslam Noor, K. Inayat Noor and Th. M. Rassias, *Set-valued resolvent equations and mixed variational inequalities*, J. Math. Anal. Appl. **220** (1998), 741–759.
19. M. Aslam Noor, K. Inayat Noor and Th. M. Rassias, *Some aspects of variational inequalities*, J. Comut. Appl. Math. **47** (1993), 285–312.
20. M. Aslam Noor and W. Oettli, *On general nonlinear complementarity problems and quasiequilibria*, Le Matematiche **49** (1994), 313–331.
21. S. M. Robinson, *Generalized equations and their solutions*, Math. Program. Studies **10** (1979), 128–141.
22. S. M. Robinson, *Normal maps induced by linear transformations*, Math. Oper. Res. **17** (1992), 691–714.
23. P. Shi, *Equivalence of variational inequalities with Wiener-Hopf equations*, Proc. Amer. Math. Soc. **111** (1991), 339–346.
24. G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes*, C.R. Acad. Sci. Paris **258** (1964), 4413–4416.
25. L. U. Uko, *Strongly nonlinear generalized equations*, J. Math. Anal. Appl. **220** (1998), 65–76.

M. A. Noor
Mathematics, Etisalat College of Engineering
P. O. Box 980, Sharjah
United Arab Emirates
E-mail address: noor@ece.ac.ae