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INFINITE TIME HORIZON OPTIMAL CONTROL OF THE SEMILINEAR HEAT EQUATION

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Abstract. We consider here the infinite horizon control problem for the semilinear heat equation with Lipschitz nonlinearity and quadratic cost functional. We prove that the associated value function is locally Lipschitz using observability inequalities for the linear backward parabolic equations. Optimality conditions and feedback representation of the the optimal controls are also considered.

1. Introduction

In this paper we study the infinite horizon quadratic control problem for the semilinear heat equation. Since the results presented in Section 2 are valid for a larger class of state systems and cost functionals, we consider first the problem in an abstract form, and then we proceed in Section 3 to the particular case of parabolic systems and quadratic performance index.

Let H be a real Hilbert space, $A : D(A) \subset H \to H$ a linear unbounded operator such that $-A$ generates a C_0 semigroup, and $F : H \to H$ a Lipschitz nonlinear mapping. If U is a Hilbert space and $B \in L(U, H)$ is a bounded linear operator from U to H, for each control $u \in L^2(0, \infty; U)$ and each initial value $x \in H$ we denote by $y(\cdot) = y(\cdot, x, u) \in C([0, \infty); H)$ the unique mild solution of the state system:

$$\begin{cases} y'(s) + Ay(s) + Fy(s) = Bu(s) \text{ on } [0, \infty) \\ y(0) = x. \end{cases} \tag{1.1}$$

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For $g : H \to [0, \infty]$ strongly lower semicontinous, we try to minimize:

$$J(x, u) = \int_0^\infty \left(\frac{1}{2}|u(s)|^2 + g(y(s)) \right) ds \tag{1.2}$$

over all controls $u \in L^2(0, \infty; U)$, provided that $y(\cdot) = y(\cdot, x, u)$ is the solution of (1.1). Along the paper, we will denote by $|\cdot|$ ($\langle \cdot, \cdot \rangle$) the norm (respectively the inner product) in all abstract Hilbert spaces H, U, etc.

We define the value function $V : H \to [0, \infty]$ to be:

$$V(x) = \inf \left\{ J(x, u) : \ u \in L^2(0, \infty; U) \right\}. \tag{1.3}$$

In order to study this problem, we also need to consider the finite horizon problems:

$$\inf\{J(t, x, u) : \ u \in L^2(0, t; U)\},$$

where we denoted

$$J(t, x, u) = \int_0^t \left(\frac{1}{2}|u(s)|^2 + g(y(s, x, u)) \right) ds \tag{1.4}$$

We define $v : [0, \infty) \times H \to [0, \infty]$ by:

$$v(t, x) = \inf \left\{ J(t, x, u) : \ u \in L^2(0, t; U) \right\}. \tag{1.5}$$

It is well known that for the linear quadratic case, which means that $F \equiv 0$ and $g(y) = \frac{1}{2}|Cy|^2$ for $C \in L(H, Y)$ (where Y is a third Hilbert space), we have that

$$v(t, x) \to V(x) \text{ as } t \to \infty, \text{ for each } x \in H, \tag{1.6}$$

provided that $V(x) < \infty$ for all $x \in H$ (the pair (A, B) is C stabilizable). In this case the proof of (1.6) requires considerations on Riccati equations and uses essentially the uniform boundedness principle. We obtain that $V(x) = \frac{1}{2}\langle Qx, x \rangle$ where $Q \in \Sigma^+(H)$ is the minimal solution of the algebraic Riccati equation:

$$A^*Q + QA + QBB^*Q = C^*C.$$

(See [3] for details.)

All these arguments are only valid for the linear quadratic case. In the present paper, in Section 2, we prove that in case $-A$ generates a compact semigroup (1.6) is also true for the semilinear case, and even if $V(x) = \infty$ for some $x \in H$.

In Section 3, we consider further assumptions on A, F, B and g, such that (1.1) becomes in fact the semilinear heat equation with internal control and (1.2) is a quadratic generalized energy. We know the state system is stabilizable (since it is null controllable), so $V(x) < \infty$ $(\forall)$ $x \in H$. Using observability inequalities for the backward linear parabolic systems (which allowed us to conclude that the state system is null controllable), we prove that $V(\cdot)$ is locally Lipschitz on H. We deliberately avoid a viscosity approach, since it was considered in greater generality in [6]. Our purpose is to prove the regularity of V, the necessary conditions of optimality, as well as the feedback representation of the optimal controls (in this particular case), which does not overlap with [6].

2. Approximation of the Value Function by Finite Horizon Problems

This section is devoted to the proof of (1.6) under the following assumptions:

$-A$ generates a compact C_0-semigroup, $F : H \to H$ is globally Lipschitz ($|Fx - Fy| \leq L|x - y|$ $(\forall)x, y \in H$), $g : H \to [0, \infty]$ is strongly lower semicontinous, and $B : U \to H$ is a linear bounded operator.

Compactness of the semigroup generated by $-A$ is particulary important because it implies the stability of the solutions of (1.1), namely:

$$\begin{cases} \text{if } u_n \rightharpoonup u \text{ in } L^2(0,t;H) \text{ for some } t > 0 \text{ and } x_n \to x \text{ in } H \text{ then} \\ y(\cdot, x_n, u_n) \to y(\cdot, x, u) \text{ in } C([0,t];H). \end{cases} \tag{2.1}$$

This can be proved using the linear version of Baras theorem (see [8]), as well as the continuity of the nonlinear mapping $F : H \to H$. Stability condition (2.1) allows us to conclude that for each $x \in H$ the infimum is attained in (1.5), and for any $t > 0$, $v(t, \cdot)$ is strongly lower semicontinuous on H.

In order to prove (1.6) we need the following lemma:

Lemma 2.1. *Let $T > 0$ and $\psi_1, \ldots, \psi_n, \cdots : H \to [0, \infty]$ such that ψ_n is strongly l.s.c for each n, and $\psi_1 \leq \psi_2 \leq \cdots \leq \psi_n \leq \ldots$ We define $\psi = \sup_n \psi_n$ and for each $x \in H$*

$$\varphi_n(x) = \inf \left\{ \int_0^T (\frac{1}{2}|u(s)|^2 + g(y(s,x,u)))ds + \psi_n(y(T,x,u)) \right\}$$

$$\varphi(x) = \inf \left\{ \int_0^T (\frac{1}{2}|u(s)|^2 + g(y(s,x,u)))ds + \psi(y(T,x,u)) \right\}.$$

Then $\varphi_n(x) \to \varphi(x)$ *as* $n \to \infty$ *for each* $x \in H$.

Proof. Since $\psi_n \leq \psi$ on H, it is obvious that

$$\varphi_n(x) \leq \varphi(x) \ (\forall) x \in H. \tag{2.2}$$

Let us denote $\overline{\varphi}(x) = \sup_n \varphi_n(x)$, for each $x \in H$. Using (2.2) we obtain $\overline{\varphi}(x) \leq \varphi(x)$ for each $x \in H$. Let x be a fixed element of H.

In case $\overline{\varphi}(x) = \infty$, there is nothing to prove. If $\overline{\varphi}(x) < \infty$, then, for each n we have that $\varphi_n(x) \leq \overline{\varphi}(x) < \infty$. Using stability condition (2.1) , since ψ_n is strongly l.s.c. we can conclude that for each n there exists $u_n \in L^2(0,T;H)$ such that

$$\varphi_n(x) = \int_0^T \left(\frac{1}{2}|u_n(s)|^2 + g(y_n(s))\right) ds + \psi_n(y_n(T))$$

where $y_n(\cdot) = y(\cdot, x, u_n)$. Since $\frac{1}{2}\int_0^T |u_n(s)|^2 \leq \varphi_n(x) \leq \overline{\varphi}(x) < \infty$, using again (2.1) we can conclude that (choosing eventually a subsequence), $u_n \rightharpoonup u$ in $L^2(0,T;U)$ and $y_n(\cdot) \to y(\cdot, x, u)$ in $C([0,T];H)$. Taking into account that

$$\psi(y(T,x,u)) \leq \liminf_{n\to\infty} \psi_n(y(T,x,u_n)) \tag{2.3}$$

(see [6] Lemma (5.1)), we obtain that

$$\begin{aligned}
\varphi(x) &\leq \int_0^T \left(\frac{1}{2}|u(s)|^2 + g(y(s,x,u))\right) ds + \psi(y(T,x,u)) \\
&\leq \liminf_{n\to\infty} \left\{ \int_0^T \left(\frac{1}{2}|u_n(s)|^2 + g(y_n(s))\right) ds + \psi_n(y_n(T)) \right\} \\
&= \liminf_{n\to\infty} \varphi_n(x) \leq \overline{\varphi}(x) \leq \varphi(x).
\end{aligned}$$

So $\varphi(x) = \overline{\varphi}(x)$, which means that $\varphi_n(x) \to \varphi(x)$ as $n \to \infty$. □

We are now ready to state the main result of this section:

Theorem 2.1. *If the semigroup generated by* $-A$ *is compact, then*

$$v(t,x) \to V(x) \text{ as } t \to \infty, \text{ for each } x \in H.$$

Proof. It is obvious that for each $x \in H$ we have that $v(t,x) \leq V(x)$ and $v(t,x)$ increases as t increases. Let us denote, for each $x \in H$ by

$$\overline{v}(x) = \sup_{t\geq 0} v(t,x) = \lim_{t\nearrow\infty} v(t,x) \leq V(x). \tag{2.4}$$

For fixed $T > 0$, $v(\cdot, \cdot)$ verifies the dynamic programming principle

$$v(t+T,x) = \inf\left\{\int_0^T \left(\frac{1}{2}|u(s)|^2 + g(y(s,x,u))\right) ds + v(t, y(T,x,u))\right\} \quad (2.5)$$

for any $t \geq 0$ and $x \in H$.

The proof of (2.5) is standard so we skip it. Since $v(t,\cdot) \to \overline{v}(\cdot)$ and $v(t+T,\cdot) \to \overline{v}(\cdot)$ as $t \to \infty$, using Lemma 2.1, we can conclude that

$$\overline{v}(x) = \inf\left\{\int_0^T \left(\frac{1}{2}|u(s)|^2 + g(y(s,x,u))\right) ds + \overline{v}(y(T,x,u))\right\} \quad (2.6)$$

for any $x \in H$. By the way we defined $\overline{v}(\cdot)$, since $v(t,\cdot)$ is strongly l.s.c and positive for all $t \geq 0$, we obtain that $\overline{v} : H \to [0,\infty]$ is strongly l.s.c. Let $0 = t_0 < t_1 < \ldots t_n < \ldots$ such that $t_n \to \infty$ as $n \to \infty$. Since A, F, B and g are independent on the time variable, we can use (2.6) to obtain

$$\overline{v}(x) = \inf\left\{\int_{t_n}^{t_{n+1}} \left(\frac{1}{2}|u(s)|^2 + g(y(s))\right) ds + \overline{v}(y(t_{n+1}))\right\} \quad (2.7)$$

provided that the infimum is considered over the set of solutions of the system

$$\begin{cases} y' + Ay + Fy = Bu \ \text{ on } \ [t_n, t_{n+1}] \\ y(t_n) = x. \end{cases}$$

Let now $x \in H$ be fixed. If $\overline{v}(x) = \infty$, by (2.4) it is obvious that $\overline{v}(x) = V(x)$. In case $\overline{v}(x) < \infty$, since $\overline{v}(\cdot)$ is strongly l.s.c and positive, we can use the stability condition (2.1) to conclude that the infimum is attained in(2.7), for $n = 0$. This means there exist $(u_0, y_0) \in L^2(0, t_1; U) \times C([0, t_1]; H)$ such that

$$\begin{cases} y_0' + Ay_0 + Fy_0 = Bu_0 \ \text{ on } \ [0, t_1] \\ y_0(0) = x \end{cases}$$

and

$$\overline{v}(x) = \int_{t_0}^{t_1} \left(\frac{1}{2}|u_0(s)|^2 + g(y_0(s))\right) ds + \overline{v}(y_0(t_1)).$$

Inductively, if we have the pair $(u_{n-1}, y_{n-1}) \in L^2(t_{n-1}, t_n; U) \times C([t_{n-1}, t_n]; H)$ we can find in the same way $(u_n, y_n) \in L^2(t_n, t_{n+1}; U) \times C([t_n, t_{n+1}]; H)$ such that

$$\begin{cases} y_n' + Ay_n + Fy_n = Bu_n \ \text{ on } \ [t_n, t_{n+1}] \\ y_n(t_n) = y_{n-1}(t_n) \end{cases}$$

and

$$\overline{v}(y_{n-1}(t_n)) = \int_{t_n}^{t_{n+1}} \left(\frac{1}{2}|u_n(s)|^2 + g(y_n(s))\right) ds + \overline{v}(y_n(t_{n+1})).$$

We can now define $(\overline{u}, \overline{y}) \in L^2_{loc}([0,\infty); U) \times C([0,\infty); H)$ by $\overline{u}(s) = u_n(s)$ a.e $s \in [t_n, t_{n+1}]$ and $\overline{y}(s) = y_n(s)$ for all $s \in [t_n, t_{n+1}]$. We denoted by $L^2_{loc}([0,\infty); U)$ the set

$$\{u : [0,\infty) \to U : u \text{ is strongly measurable and } u \in L^2(0,T;U)\ (\forall)T > 0\}.$$

Now $(\overline{u}, \overline{y})$ is a solution of the state system

$$\begin{cases} \overline{y}' + A\overline{y} + F\overline{y} = B\overline{u} \text{ on } [0,\infty) \\ \overline{y}(0) = x \end{cases}$$

and, in the same time

$$\overline{v}(x) = \int_0^{t_n} \left(\frac{1}{2}|\overline{u}(s)|^2 + g(\overline{y}(s)\right) ds + \overline{v}(\overline{y}(t_n)) \text{ for each } n \in N.$$

Since $\overline{v}(\overline{y}(t_n)) \geq 0\ (\forall)n \in N$ we are able to conclude that

$$\int_0^{\infty} \left(\frac{1}{2}|\overline{u}(s)|^2 + g(\overline{y}(s))\right) ds \leq \overline{v}(x) < \infty.$$

Taking into account the way we defined $V(\cdot)$, we have $V(x) \leq \overline{v}(x) \leq V(x)$, so $V(x) = \overline{v}(x)$. The proof is now complete. □

Remark 2.1. *Using weak lower semicontinuity arguments we can prove in the same way that, in the linear quadratic case,* (1.6) *holds without the compactness assumption on the semigroup, and even if the pair* (A, B) *is not* C *stabilizable.*

3. Regularity of the Value Function. Optimality Conditions and Feedback Laws.

The main purpose of this section is to prove that $V(\cdot)$ is locally Lipschitz in the case when the state system (1.1) is nothing else than the abstract expression of the semilinear heat equation on the L^2 space, and g is a quadratic

term. Consequently, we will be able to prove the necessary conditions of optimality together with the feedback representation of the optimal controls.

So, along this section, we assume that $H = L^2(\Omega)$ for a regular domain Ω in R^n.We define $A : D(A) \subset H \to H$ by $Ay = -\triangle y$ for $y \in D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ and $F : H \to H$ by $(Fy)(\xi) = f(\xi, y(\xi))$ a.e. $\xi \in \Omega$, where the function $f : \Omega \times R \to R$ is measurable in ξ and C^1 in y and satisfies:

$$f(\xi, 0) = 0 \ (\forall) \ \xi \in \Omega \text{ and } \left|\frac{\partial f}{\partial y}(\xi, y)\right| \leq L \ (\forall) \ \xi \in \Omega, (\forall) \ y \in R. \tag{3.1}$$

Using (3.1) we can conclude that $F : H \to H$ defined in this way is globally Lipschitz (with Lip. constant L), and it is Gateaux differentiable:

$$(F'(y)(h))(\xi) = \frac{\partial f}{\partial y}(\xi, y(\xi))h(\xi) \text{ a.e } \xi \in \Omega \text{ if } y, h \in H = L^2(\Omega).$$

We also consider that $U = L^2(\Omega)$ and $B \in L(L^2(\Omega))$ is the internal controler $(Bu)(\xi) = m(\xi)u(\xi)$ a.e $\xi \in \Omega$ for $u \in L^2(\Omega)$, provided that m is the characteristic function of an open subset of Ω, $\omega \Subset \Omega$.

Under all these assumptions, state system (1.1) is in fact the semigroup representation of

$$\begin{cases} y_t(t, \xi) - \triangle y(t, \xi) + f(\xi, y(t, \xi)) = m(\xi)u(t, \xi), \ (\forall) \ (t, \xi) \in (0, \infty) \times \Omega \\ y(t, \xi) = 0 \ (\forall) \ (t, \xi) \in (0, \infty) \times \partial\Omega \\ y(0, \xi) = x(\xi) \ (\forall) \ \xi \in \Omega \end{cases} \tag{3.2}$$

for $u \in L^2((0, \infty) \times \Omega)$ and $x \in L^2(\Omega)$.

We further assume that $g(y) = \frac{1}{2}|Cy|^2$ where C is a bounded linear operator from H to another Hilbert space Y, so (1.2) is a quadratic cost functional.

For sure the semigroup generated by $-A$ is compact and g is lower semicontinuous, so all hypotheses assumed in Section 2 are fullfiled. Consequently, it is true that $v(t, x) \to V(x)$ as $t \to \infty$, for each $x \in H$.

It is well known (see [5]), that for global Lipschitz nonlinearities (which is the case), system (3.2) (or equivalently (1.1)) is null controllable in finite time and consequently it is stabilizable.This means that $V(x) < \infty \ (\forall) \ x \in H$.

A sufficient condition for null controllability (in finite time t) is the observability inequality:

$$\begin{aligned} &\|p(0)\|^2_{L^2(\Omega)} \\ &\leq C(t, \|a\|_{L^\infty((0,t)\times\Omega)}) \left(\int_0^t \int_\omega |p(\tau, \xi)|^2 d\xi d\tau + \int_0^t \int_\Omega |g(\tau, \xi)|^2 d\xi d\tau \right) \end{aligned} \tag{3.3}$$

provided that p is a solution of the the backward parabolic equation:

$$\begin{cases} p_\tau(\tau,\xi) + \triangle p(\tau,\xi) + a(\tau,\xi)p(\tau,\xi) = g(\tau,\xi) \ (\forall) \ (\tau,\xi) \in (0,t)\times\Omega \\ p(\tau,\xi) = 0 \ (\forall) \ (\tau,\xi) \in (0,t)\times\partial\Omega. \end{cases} \tag{3.4}$$

We know that (3.3) is true from [5]. In this case, eventually changing the observability constant to take into account the norm of the bounded linear operator C, we obtain that

$$|p(0)|^2 \leq C(t, \|a\|_{L^\infty((0,t)\times\Omega)}) \int_0^t \left(|B^*p(\tau)|^2 + |Cy(\tau)|^2\right) d\tau \tag{3.5}$$

if p is a solution of

$$\begin{cases} p_\tau(\tau,\xi) + \triangle p(\tau,\xi) + a(\tau,\xi)p(\tau,\xi) = (C^*Cy(\tau))(\xi) \ (\forall) \ (\tau,\xi) \in (0,t)\times\Omega \\ p(\tau,\xi) = 0 \ (\forall) \ (\tau,\xi) \in (0,t)\times\partial\Omega. \end{cases} \tag{3.6}$$

Using (3.5) and Schauder's Fixed Point Theorem, we obtain that for each $x \in H$ and $t > 0$, there exists $u \in L^2(0,t;H)$ such that

$$\begin{cases} y' + Ay + Fy = Bu \text{ on } [0,t] \\ y(0) = x, \ y(t) = 0 \end{cases}$$

and $\int_0^t \left(\frac{1}{2}|u(s)|^2 + \frac{1}{2}|Cy(s)|^2\right) ds \leq \frac{1}{2}C(t,L)|x|^2$, where L is the Lipschitz constant in (3.1). Since $F0 = 0$, we can conclude that $V(x) \leq \frac{1}{2}C(t,L)|x|^2$. This holds for any $t > 0$. In case we choose $C(t,L)$ to be the best constant in (3.5) it is obvious that $C(t_1,L) \geq C(t_2,L)$ for $t_1 \leq t_2$ so we can denote by

$$C(L) = \inf_{t\geq 0} C(t,L) = \lim_{t\to\infty} C(t,L). \tag{3.7}$$

Using this notation we have that $V(x) \leq \frac{1}{2}C(L)|x|^2$.

We now intend to prove that $v(t,\cdot)$ is locally Lipschitz, uniformly for $t \geq 0$. The idea is the same as in [7] , so we need the following lemma proved in [7].

Lemma 3.1. *Under the hypotheses assumed in the beginning of the section, if (u^*, y^*) is an optimal pair for the problem*

$$\inf\left\{\int_0^t (\frac{1}{2}|u|^2 + \frac{1}{2}|Cy|^2)d\tau + l_0(y(0)) : \ y' + Ay + Fy = Bu \ \textit{on} \ [0,t]\right\}$$

where $l_0 : H \to R$ is continuous, then there exists $p \in C([0,t];H)$ satisfying

$$\begin{cases} p' - A^*p - (F'(y^*))^*p = C^*Cy^* \ \text{on } [0,t] \\ u^* = B^*p \ \text{a.e. on } [0,t] \\ p(t) = 0 \end{cases}$$

and also

$$\langle p(0), h\rangle \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(l_0(y^*(0)+\varepsilon h) - l_0(y^*(0))\right) \ (\forall)\ h \in H.$$

We can now state the main result concerning the regularity of the value function V:

Theorem 3.1. *For any $R > 0$, if $|x|, |y| \leq R$, then*

$$|V(x) - V(y)| \leq C(L)R|x-y|.$$

For any $x \in H$ the infimum is attained in (1.3).

Proof. Let $R > 0$ and $x, y \in H$ such that $|x|, |y| \leq R$.

We know that $v(t,\cdot)$ is locally Lipschitz (see [1]) since $y \to \frac{1}{2}|Cy|^2$ is locally Lipschitz. So $\overline{f}(\lambda) = v(t, x+\lambda(y-x))$ is Lipschitz on $[0,1]$. This means that $\overline{f}(\cdot)$ is differentiable almost everywhere on $[0,1]$ and

$$v(t,y) - v(t,x) = \overline{f}(1) - \overline{f}(0) = \int_0^1 \overline{f}'(\lambda)d\lambda. \tag{3.8}$$

Let $\lambda_0 \in (0,1)$ such that $\overline{f}$ is differentiable at λ_0. This means that

$$\begin{aligned} &\lim_{\lambda\to 0} \frac{\overline{f}(\lambda+\lambda_0) - \overline{f}(\lambda_0)}{\lambda} \\ &= \lim_{\lambda\to 0} \frac{v(t, x+(\lambda+\lambda_0)(y-x)) - v(t, x+\lambda_0(y-x))}{\lambda} \end{aligned}$$

exists and it is equal to $\overline{f}'(\lambda_0)$.

Let us denote $h_0 = y - x$ and $x_0 = x + \lambda_0(y-x)$.In this notation ,

$$\overline{f}'(\lambda_0) = \lim_{\lambda\to 0} \frac{v(t, x_0+\lambda h_0) - v(t,x_0)}{\lambda}.$$

We remind that we defined

$$v(t,x)=\inf\left\{\int_0^t(\frac{1}{2}|u(s)|^2+\frac{1}{2}|Cy(s,x,u)|^2)ds\colon u\in L^2(0,t;H)\right\}.$$

Since stability condition (2.1) holds, we conclude that there exists an optimal pair $(u^*,y^*)\in L^2(0,t;H)\times C([0,t];H)$ such that

$$v(t,x_0)=\int_0^t(\frac{1}{2}|u^*(s)|^2+\frac{1}{2}|Cy^*(s)|^2)ds$$

where $y^*(\cdot)=y(\cdot,x_0,u^*)$. By the definition of the value function $v(t,\cdot)$, it turns out that the pair (u^*,y^*) also attains the infimum for the problem

$$\inf\left\{\int_0^t(\frac{1}{2}|u|^2+\frac{1}{2}|Cy|^2)d\tau-v(t,y(0))\colon\ y'+Ay+Fy=Bu \text{ on } [0,t]\right\}$$

According to Lemma 3.1, there exists $p\in C([0,t];H)$ such that

$$\begin{cases} p'-A^*p-(F'(y^*))^*p=C^*Cy^* \text{ on } [0,t]\\ u^*=B^*p \text{ a.e. on } [0,t]\\ p(t)=0 \end{cases} \tag{3.9}$$

and also

$$\langle p(0),h\rangle\le \liminf_{\lambda\to 0}\frac{-(v(t,x_0+\lambda h)-v(t,x_0))}{\lambda}\ (\forall)\ h\in H. \tag{3.10}$$

Taking $h=h_0$ in (3.10) we get

$$\langle p(0),h_0\rangle\le \lim_{\lambda\to 0}\frac{-(v(t,x_0+\lambda h_0)-v(t,x_0))}{\lambda}=-\overline{f}'(\lambda_0).$$

For $h=-h_0$ we obtain

$$\begin{aligned} -\langle p(0),h_0\rangle &\le \liminf_{\lambda\to 0}\frac{-(v(t,x_0-\lambda h_0)-v(t,x_0))}{\lambda}\\ &=\liminf_{\lambda\to 0}\frac{v(t,x_0-\lambda h_0)-v(t,x_0)}{-\lambda}\\ &=\lim_{\lambda\to 0}\frac{v(t,x_0+\lambda h_0)-v(t,x_0)}{\lambda}=\overline{f}'(\lambda_0). \end{aligned}$$

So $-\overline{f}'(\lambda_0) \leq \langle p(0), h_0\rangle \leq -\overline{f}'(\lambda_0)$, therefore $\overline{f}'(\lambda_0) = -\langle p(0), h_0\rangle$

We can conclude now that $|\overline{f}'(\lambda_0)| \leq |p(0)||h_0| = |p(0)||x-y|$. In the same time, since p is a solution of the adjoint system, taking (3.5) into account,we obtain

$$|p(0)|^2 \leq C(t,L)\int_0^t \left(|u^*|^2 + |Cy^*|^2\right) ds = C(t,L)2v(t,x_0) \leq C(t,L)C(L)|x_0|^2 \leq C(t,L)C(L)R^2$$

since $x_0 \leq R$. This means that

$$|\overline{f}'(\lambda_0)| \leq (C(t,L)C(L))^{\frac{1}{2}}\, R|x-y|,$$

anytime $\overline{f}$ is differentiable at λ_0. Going back to (3.8) we conclude that

$$|v(t,x) - v(t,y)| \leq (C(t,L)C(L))^{\frac{1}{2}}\, R|x-y| \text{ for } |x|,|y| \leq R. \tag{3.11}$$

Since $v(t,x) \to V(x)$ as $t \to \infty$ for any $x \in H$ and $C(t,L) \to C(L)$ for $t \to \infty$, we can pass to limit in (3.11) to conclude

$$|V(x) - V(y)| \leq C(L)R|x-y| \text{ if } |x|,|y| \leq R.$$

The existence of optimal pairs is easily obtained by choosing an appropriate subsequence of a minimizing sequence for (1.3) and then using stability condition (2.1) and Fatou's lemma. □

Once we proved Theorem 3.1 we can use dynamic programming arguments to obtain the optimality conditions and the expected feedback law, namely:

Theorem 3.2. *If $(y^*, u^*) \in C([0,\infty); H) \times L^2(0,\infty; H)$ is an optimal pair for the control problem* (1.3) *then there exists a unique $p \in C([0,\infty; H))$ which satisfies the conditions*

$$\begin{cases} p' - A^*p - (F'(y^*))^*p = C^*Cy^* \text{ on } [0,\infty) \\ u^* = B^*p \text{ a.e. on } [0,\infty) \\ p(\infty) = 0 \end{cases} \tag{3.12}$$

and also

$$p(t) \in -\partial V(y^*(t)) \text{ for every } t \geq 0. \tag{3.13}$$

Remark 3.1. *We denoted by $\partial V(\cdot)$ the generalized gradient in the sense of F.Clarke of the function $V(\cdot)$. For details regarding the generalized gradient, we refer the reader to* [2].

Proof of Theorem 3.2. Let $(u^*, y^*) \in L^2(0,\infty;H) \times C([0,\infty;H))$ such that

$$V(x) = \int_0^\infty \left(\frac{1}{2}|u^*|^2 + \frac{1}{2}|Cy^*|^2\right) ds,\ y^*(\cdot) = y(\cdot, x, u^*).$$

We know that for any $t > 0$

$$V(x) = \inf\left\{\int_0^t (\frac{1}{2}|u(s)|^2 + \frac{1}{2}|Cy(s,x,u)|^2)ds + V(y(t,x,u))\right\} \tag{3.14}$$

and, furthermore, (u^*, y^*) is an optimal pair on $[0,t]$, i.e.

$$V(x) = \int_0^t (\frac{1}{2}|u^*(s)|^2 + \frac{1}{2}|Cy^*(s)|^2)ds + V(y^*(t)). \tag{3.15}$$

We have to say that we can either prove (3.14) directly or see that $V(\cdot) = \overline{v}(\cdot)$ and (2.6) holds. Also, (3.15) can be proved by usual dynamic programming arguments, once we know (3.14). By [2], there exists $p^t \in C([0,t];H)$ such that

$$\begin{cases} (p^t)' - A^*p^t - (F'(y^*))^*p^t = C^*Cy^* \text{ on } [0,t] \\ u^* = B^*p^t \text{ a.e. on } [0,t] \\ p^t(t) \in -\partial V(y^*(t)). \end{cases} \tag{3.16}$$

Let us consider $0 < t_1 < t_2$. We conclude that there exist $p^{t_1} \in C([0,t_1];H)$, $p^{t_2} \in C([0,t_2];H)$, satisfying the corresponding conditions (3.16) for t_1, t_2. Since $B^*p^{t_1} = u^* = B^*p^{t_2}$ a.e on $[0,t_1]$, if we denote $p = p^{t_1} - p^{t_2} \in C([0,t_1];H)$ we have $B^*p = 0$ a.e. on $[0,t_1]$. In the same time p satisfies the adjoint system

$$p' - A^*p - (F'(y^*))^*p = 0,$$

so if we use observability inequality (3.3) on subintervals $[s,t_1] \subset [0,t_1]$ (rather then on$[0,t_1]$) for $g(t,\xi) = 0,\ a(t,\xi) = -\frac{\partial f}{\partial y}(\xi, y^*(t,\xi))$, namely

$$|p(s)|^2 \le C(t_1 - s, L)\int_s^{t_1} |B^*p(\tau)|^2 d\tau$$

we conclude that $p^{t_1}(s) = p^{t_2}(s)$ for each $s \in [0, t_1]$. Considering any $t > 0$ and going over the same argument, we obtain that there exist a unique $p \in C([0,\infty); H)$ such that

$$\begin{cases} p' - A^*p - (F'(y^*))^*p = C^*Cy^* \text{ on } [0,\infty) \\ u^* = B^*p \text{ a.e. on } [0,\infty) \end{cases} \tag{3.17}$$

and $p(t) \in -\partial V(y^*(t))$ for every $t > 0$. Taking into account that the multivalued mapping $y \to \partial V(y)$ is strongly closed and p is continuous at 0, we obtain that $p(0) \in -\partial V(x)$ as well, so (3.13) is proved. □

Since p satisfies (3.17), we know that a similar inequality to (3.5) holds on the interval $[t, t+1]$ for each $t > 0$. In other words we have

$$|p(t)|^2 \le C(1,L) \int_t^{t+1} \left(|B^*p(\tau)|^2 + |Cy^*(\tau)|^2 \right) d\tau.$$

Taking into account that $B^*p = u^*$ a.e. on $[0,\infty)$ we conclude that

$$|p(t)|^2 \le C(1,L) \int_t^{t+1} \left(|u^*(\tau)|^2 + |Cy^*(\tau)|^2 \right) d\tau. \tag{3.18}$$

Since $\int_0^\infty \left(|u^*(\tau)|^2 + |Cy^*(\tau)|^2 \right) d\tau = 2V(x) < \infty$, using Lebesgue's dominated convergence theorem we get that

$$\int_t^{t+1} \left(|u^*(\tau)|^2 + |Cy^*(\tau)|^2 \right) d\tau \to 0 \text{ for } t \to \infty,$$

which implies, together with (3.18) that

$$p(\infty) = \lim_{t\to\infty} p(t) = 0. \tag{3.19}$$

We see that (3.17) and (3.19) together are nothing else but the necessary conditions of optimality, namely (3.12)

Since $u^*(\cdot) = B^*p(\cdot)$ a.e. on $[0,\infty)$, condition (3.13) is a feedback representation formula for the optimal control u^*. In fact we have

$$u^*(\cdot) \in -B^*\partial V(y^*(\cdot)) \text{ a.e. on } [0,\infty).$$

Remark 3.2. *The idea to use dynamic programming arguments to obtain optimality conditions for the infinite horizon control problems was previously used in* [2](*pages* $210-213$). *Here the author considers a monotonic nonlinearity* (*possibly discontinous and multivalued*) *which allows to conclude that the trajectories of the state systems have exponential decay if we consider the feedback control* $u=-B^*y$. *This means that the state system is stabilizable, and it is possible to prove directly the regularity of* V. *This approach still works in our case if the Lipschitz constant* L *is strictly smaller then the first eigenvalue of the Laplacean with Dirichlet boundary conditions.*

However, for general C^1 *and globally Lipschitz nonlinearities this approach does not work and this is the case were the results in this paper apply. In fact, null controllability* (*a consequence of observability inequalities*) *is the only way to prove that the state system is stabilizable. The same observability inequalities allowed us to conclude that the dual state is unique* (*compared to* [2] *where* $\overline{R(B)}=H$ *was needed for this*) *and that the extra condition* $p(\infty)=0$ *holds.*

Remark 3.3. *For the particular case of the semilinear heat equation, the convergence result* (1.6) *can be improved, namely we can prove that*

$$v(t,x)\nearrow V(x)\ \textit{uniformly for}\ x\in B_R\ (\forall)\ R>0 \tag{3.20}$$

where $B_R=\{x\in H:\ |x|\leq R\}$. *In order to prove* (3.20) *we just have to use Dini's criterion on the weak (sequentially) compact set* B_R *as well as the monotone convergence* (1.6). *Weak sequential continuity of* $v(t,\cdot)$ *and* $V(\cdot)$ *follows from the fact that* $v(t,\cdot)$ *is defined by* (1.5) *and* $V(\cdot)$ *satisfies* (3.14) *and* V *(regarded as the penalization for time* t *in* (3.14)*) is locally Lipschitz. Using the results in* [4] *Section VII, we can conclude that* $v(t,\cdot)$ *and* $V(\cdot)$ *are* D*-continuous, where* $D=(I+A)^{-1}$ *is self adjoint and compact. Compactness of* D *implies that* D*-continuity is nothing else than weak sequential continuity.* (*See* [4] *for details regarding* D*-continuity.*) *Taking in account all these we are now able to use Dini's criterion to conclude* (3.20).

Remark 3.4. *The results presented in Section* 3 *are still valid for different kinds of homogenous boundary conditions. Namely, we can replace the Dirichlet condition by*

$$y_\nu(t,\xi)+\beta y(t,\xi)=0\ (\forall)\ (t,\xi)\in(0,\infty)\times\Omega$$

where y_ν *is the outward normal derivative and* $\beta\geq 0$.

Remark 3.5. *We want to emphasize that condition* $f(\cdot,0)\equiv 0$ *means in fact that* $y_e\equiv 0$ *is a steady-state(equilibrium) solution of* (3.2).

For the general case ($f(\cdot,0)\not\equiv 0$)*, the stabilizability of an equilibrium solution* y_e *can be reduced to the stabilizability of the trivial solution substituting* y *by* $y+y_e$ *in the state system.*

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