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WELL-POSEDNESS OF GENERALIZED BEST APPROXIMATION PROBLEMS

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Abstract. Given a closed subset A of a Banach space X, a point $x \in X$ and a continuous function $f : X \to R^1$, we consider the problem of finding a solution to the minimization problem $\min\{f(x-y) : \ y \in A\}$. For a fixed function f, we define an appropriate complete metric space $\mathcal{M}$ of all pairs (A, x) and construct a subset Ω of $\mathcal{M}$ which is a countable intersection of open everywhere dense sets such that for each pair in Ω our minimization problem is well posed.

1. Introduction

Let $(X, ||\cdot||)$ be a Banach space and let $f : X \to R^1$ be a continuous function. Assume that

$$\inf\{f(x) : \ x \in X\} \text{ is attained at a unique point } x_* \in X, \tag{1.1}$$

$$\lim_{||u|| \to \infty} f(u) = \infty, \tag{1.2}$$

$$\text{if } \{x_i\}_{i=1}^{\infty} \subset X \text{ and } \lim_{i \to \infty} f(x_i) = f(x_*), \text{ then } \lim_{i \to \infty} x_i = x_*, \tag{1.3}$$

and that for each integer $n \geq 1$, there exists an increasing function $\phi_n : (0,1) \to (0,1)$ such that

$$f(\alpha x + (1-\alpha)x_*) \leq \phi_n(\alpha) f(x) + (1 - \phi_n(\alpha)) f(x_*) \tag{1.4}$$

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for all $x \in X$ satisfying $||x|| \le n$ and all $\alpha \in (0,1)$. Clearly (1.4) holds if f is convex.

Given a closed subset A of X and a point $x \in X$, we consider the minimization problem

$$\min\{f(x-y) : \ y \in A\}. \tag{P}$$

This problem was studied by many mathematicians mostly in the case where $f(x) = ||x||$. In this special case it is well known that if A is convex and X is reflexive, then problem (P) always has at least one solution. This solution is unique when X is strictly convex. If $f = ||\cdot||$, A is merely closed, but X is uniformly convex, then according to classical results of Stechkin [18] and Edelstein [8], the set of all points in X having a unique nearest point in A is a dense G_δ subset of X. Since then there has been a lot of activity in this direction. In particular, it is known [10, 13] that the following properties are equivalent for any Banach space X:

(A) X is reflexive and has a Kadec-Klee norm.

(B) For each closed nonempty subset A of X, the set of points in $X \setminus A$ with nearest points in A is dense in $X \setminus A$.

(C) For each closed nonempty subset A of X, the set of points in $X \setminus A$ with nearest points in A is generic (that is, a dense G_δ subset) in $X \setminus A$.

A more recent result of De Blasi, Myjak and Papini [6] establishes well-posedness of problem (P) for $f = ||\cdot||$, a uniformly convex X, a closed A and a generic $x \in X$.

In this connection we recall that the minimization problem (P) is said to be well posed if it has a unique solution, say a_0, and every minimizing sequence of (P) converges to a_0. In other words, if $\{y_i\}_{i=1}^{\infty} \subset A$ and $\lim_{i\to\infty} f(x-y_i) = f(x-a_0)$, then $\lim_{i\to\infty} y_i = a_0$.

In the generic approach, instead of considering the existence of a solution to problem (P) for a single point $x \in X$, one investigates it for the whole space X and shows that solutions exist for most points in X. Such an approach is common in many areas of Analysis. We mention, for instance, the theory of dynamical systems [2, 7, 12], optimization [9, 15, 17], and optimal control [19]. Note that in all the above-mentioned studies of problem (P) [6, 8, 10, 13, 18], the function f is the norm of the space X. There are some additional results in the literature where either f is a Minkowski functional [5, 14] or the function $||x-y||$, $y \in A$, is perturbed by some convex function [1].

However, the fundamental restriction in all these results is that they hold only under certain assumptions on either the space X or the set A. In view of the Lau-Konjagin result mentioned above (see also [14]), these assumptions cannot be removed. On the other hand, many generic results in nonlinear functional analysis hold in any Banach space. Therefore a natural question is whether generic existence results for best approximation problems can be obtained for general Banach spaces. Positive answers to this question in the special case where $f = ||\cdot||$ can be found in [3, 4, 16]. In the present paper we answer this question in the affirmative for a general function f satisfying (1.1)-(1.4).

To this end, as in [3, 4, 16], we change our point of view and consider another framework the main feature of which is that the set A in problem (P) can also vary. In our first result (Theorem 2.1), we fix x and consider the space $S(X)$ of all nonempty closed subsets of X equipped with an appropriate complete metric, say h. We then show that the collection of all sets $A \in S(X)$ for which problem (P) is well posed contains an everywhere dense G_δ set. In the second result (Theorem 2.2), we consider the space of pairs $S(X) \times X$ with the metric $h(A, B) + ||x - y||$, $A, B \in S(X)$, $x, y \in X$. Once again we show that the family of all pairs $(A, x) \in S(X) \times X$ for which problem (P) is well posed contains an everywhere dense G_δ set. In our third result (Theorem 2.3), we show that for any separable closed subset X_0 of X there exists an everywhere dense G_δ subset $\mathcal{F}$ of $(S(X), h)$ such that any $A \in \mathcal{F}$ has the following property: There exists a G_δ dense subset F of X_0 such that for any $x \in F$, problem (P) is well posed.

In our final result (Theorem 2.4) we show that a continuous coercive convex $f : X \to R^1$ which has a unique minimizer and a certain well-posedness property (on the whole space X) has a unique minimizer and the same well-posedness property on a generic closed subset of X.

2. Main results

We recall that $(X, ||\cdot||)$ is a Banach space, $f : X \to R^1$ is a continuous function satisfying (1.1)-(1.3) and that for each integer $n \geq 1$, there exists an increasing function $\phi_n : (0, 1) \to (0, 1)$ such that (1.4) is true.

For each $x \in X$ and each $A \subset X$ set

$$\rho(x, A) = \inf\{\rho(x, y) : y \in A\} \tag{2.1}$$

and

$$\rho_f(x, A) = \inf\{(f(x - y) : y \in A\}. \tag{2.2}$$

Denote by $S(X)$ the collection of all nonempty closed subsets of X. For each $A, B \in S(X)$ define

$$H(A, B) = \max\{\sup\{\rho(x, B) : x \in A\}, \sup\{\rho(y, A) : y \in B\}\} \tag{2.3}$$

and

$$\tilde{H}(A,B) = H(A,B)(1 + H(A,B))^{-1}.$$

Here we use the convention that $\infty/\infty = 1$.

It is not difficult to see that the metric space $(S(X), \tilde{H})$ is complete.

For each natural number n and each $A, B \in S(X)$ we set

$$h_n(A,B) = \sup\{|\rho(x,A) - \rho(x,B)| : x \in X \text{ and } ||x|| \le n\} \tag{2.4}$$

and

$$h(A,B) = \sum_{n=1}^{\infty} [2^{-n} h_n(A,B)(1 + h_n(A,B))^{-1}].$$

Once again it is not difficult to see that h is a metric on $S(X)$ and that the metric space $(S(X), h)$ is complete. Clearly, $\tilde{H}(A,B) \ge h(A,B)$ for all $A, B \in S(X)$.

We equip the set $S(X)$ with the pair of metrics $\tilde{H}$ and h. The topologies induced by the metrics $\tilde{H}$ and h on $S(X)$ will be called the strong topology and the weak topology, respectively.

We now state our four main results. The proofs of the first three will be given in Section 4. The proof of the last result will be given at the end of Section 2.

Theorem 2.1. *Let $\tilde{x} \in X$. Then there exists a set $\Omega \subset S(X)$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X)$ such that for each $A \in \Omega$ the following property holds:*

(C1) There exists a unique $\tilde{y} \in A$ such that $f(\tilde{x} - \tilde{y}) = \rho_f(\tilde{x}, A)$. Moreover, for each $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in A$ satisfies $f(\tilde{x} - x) \le \rho_f(\tilde{x}, A) + \delta$, then $||x - \tilde{y}|| \le \epsilon$.

To state our second result we endow the Cartesian product $S(X) \times X$ with the pair of metrics d_1 and d_2 defined by

$$d_1((A,x),(B,y)) = h(A,B) + \rho(x,y),\ d_2((A,x),(B,y)) = \tilde{H}(A,B) + \rho(x,y),$$

$$x, y \in X,\ A, B \in S(X).$$

We will refer to the topologies induced on $S(X) \times X$ by d_2 and d_1 as the strong and weak topologies, respectively.

Theorem 2.2. *There exists a set* $\Omega \subset S(X)\times X$ *which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of* $S(X)\times X$ *such that for each* $(A,\tilde{x}) \in \Omega$ *the following property holds:*

(C2) There exists a unique $\tilde{y} \in A$ *such that* $f(\tilde{x}-\tilde{y}) = \rho_f(\tilde{x}, A)$*. Moreover, for each* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $z \in X$ *satisfies* $||z-\tilde{x}|| \le \delta$*,* $B \in S(X)$ *satisfies* $h(A,B) \le \delta$*, and* $y \in B$ *satisfies* $f(z-y) \le \rho_f(z,B)+\delta$*, then* $||y-\tilde{y}|| \le \epsilon$*.*

In most classical generic results the set A was fixed and x varied in a dense G_δ subset of X. In our first two results the set A is also variable. However, our third result shows that for every fixed A in a dense G_δ subset of $S(X)$, the set of all $x \in X$ for which problem (P) is well posed contains a dense G_δ subset of X.

Theorem 2.3. *Assume that* X_0 *is a closed separable subset of* X*. Then there exists a set* $\mathcal{F} \subset S(X)$ *which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of* $S(X)$ *such that for each* $A \in \mathcal{F}$ *the following property holds:*

(C3) There exists a set $F \subset X_0$ *which is a countable intersection of open everywhere dense subsets of* X_0 *with the relative topology such that for each* $\tilde{x} \in F$ *there exists a unique* $\tilde{y} \in A$ *for which* $f(\tilde{x}-\tilde{y}) = \rho_f(\tilde{x}, A)$*. Moreover, if* $\{y_i\}_{i=1}^{\infty} \subset A$ *satisfies* $\lim_{i\to\infty} f(\tilde{x}-y_i) = \rho_f(\tilde{x}, A)$*, then* $y_i \to \tilde{y}$ *as* $i \to \infty$*.*

Now we will show that Theorem 2.1 implies the following result.

Theorem 2.4. *Assume that* $g : X \to R^1$ *is a continuous convex function such that* $\inf\{g(x) :\ x \in X\}$ *is attained at a unique point* $y_* \in X$*,* $\lim_{||u||\to\infty} g(u) = \infty$*, and if* $\{y_i\}_{i=1}^{\infty} \subset X$ *and* $\lim_{i\to\infty} g(y_i) = g(y_*)$*, then* $y_i \to y_*$ *as* $i \to \infty$*. Then there exists a set* $\Omega \subset S(X)$ *which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of* $S(X)$ *such that for each* $A \in \Omega$ *the following property holds:*

(C4) There is a unique $y_A \in A$ *such that* $g(y_A) = \inf\{g(y) :\ y \in A\}$*. Moreover, for each* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $y \in A$ *satisfies* $g(y) \le g(y_A)+\delta$*, then* $||y-y_A|| \le \epsilon$*.*

Proof of Theorem 2.4. Define $f(x) = g(-x)$, $x \in X$. Clearly f is convex and satisfies (1.1)-(1.3). Therefore Theorem 2.1 is valid with $\tilde{x} = 0$ and there exists a set $\Omega \subset S(X)$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X)$ such that for each $A \in \Omega$ the following property holds:

There is a unique $\tilde{y} \in A$ such that

$$g(\tilde{y}) = f(-\tilde{y}) = \inf\{f(-y) :\ y \in A\} = \inf\{g(y) : y \in A\}.$$

Moreover, for each $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in A$ satisfies

$$g(x) = f(-x) \le \rho_f(0, A)+\delta = \inf\{f(-y) : y \in A\}+\delta = \inf\{g(y) : \ y \in A\}+\delta,$$

then $||x - \tilde{y}|| \le \epsilon$. Theorem 2.4 is proved. □

It is easy to see that in the proofs of Theorems 2.1-2.3 we may assume without loss of generality that $\inf\{f(x) : x \in X\} = 0$. It is also not difficult to see that we may assume without loss of generality that $x_* = 0$. Indeed, instead of the function $f(\cdot)$ we can consider $f(\cdot + x_*)$. This new function also satisfies (1.1)-(1.4). Once Theorems 2.1-2.3 are proved for this new function they will also hold for the original function f because the mapping $(A, x) \to (A, x+x_*)$, $(A, x) \in S(X) \times A$, is an isometry with respect to both metrics d_1 and d_2.

3. Basic lemma

Lemma 3.1. *Let $A \in S(X)$, $\tilde{x} \in X$, and let $r, \epsilon \in (0, 1)$. Then there exist $\tilde{A} \in S(X)$, $\bar{x} \in \tilde{A}$, and $\delta > 0$ such that*

$$\tilde{H}(A, \tilde{A}) \le r, \ f(\tilde{x} - \bar{x}) = \rho_f(\tilde{x}, \tilde{A}), \tag{3.1}$$

and such that the following property holds:

For each $\tilde{y} \in X$ satisfying $||\tilde{y}-\tilde{x}|| \le \delta$, each $B \in S(X)$ satisfying $h(B, \tilde{A}) \le \delta$, and each $z \in B$ satisfying

$$f(\tilde{y} - z) \le \rho_f(\tilde{y}, B) + \delta, \tag{3.2}$$

the inequality $||z - \bar{x}|| \le \epsilon$ holds.

Proof. There are two cases: either $\rho(\tilde{x}, A) \le r$ or $\rho(\tilde{x}, A) > r$. Consider the first case where

$$\rho(\tilde{x}, A) \le r. \tag{3.3}$$

Set

$$\bar{x} = \tilde{x} \text{ and } \tilde{A} = A \cup \{\tilde{x}\}. \tag{3.4}$$

Clearly (3.1) is true. Fix an integer $n > ||\tilde{x}||$. By (1.3) there is $\xi \in (0, 1)$ such that

$$\text{if } z \in X \text{ and } f(z) \le 4\xi, \text{ then } ||z|| \le \epsilon/2. \tag{3.5}$$

Using (1.1), we choose a number $\delta \in (0, 1)$ such that

$$\delta < 2^{-n-4} \min\{\epsilon, \xi\} \tag{3.6}$$

and

$$\text{if } z \in X \text{ and } ||z|| \le 2^{n+4}\delta, \text{ then } f(z) \le \xi. \tag{3.7}$$

Let

$$\tilde{y} \in X,\ ||\tilde{y} - \tilde{x}|| \le \delta,\ B \in S(X),\ h(B, \tilde{A}) \le \delta \tag{3.8}$$

and let $z \in B$ satisfy (3.2). By (3.8) and (2.4), $h_n(\tilde{A}, B)(1 + h_n(\tilde{A}, B))^{-1} \le 2^n\delta$. This implies that $h_n(\tilde{A}, B)(1 - 2^n\delta) \le 2^n\delta$. Combined with (3.6) this inequality shows that $h_n(\tilde{A}, B) \le 2^{n+1}\delta$. Since $n > ||\tilde{x}||$, the last inequality, when combined with (3.4) and (2.4), implies that $\rho(\tilde{x}, B) \le 2^{n+1}\delta$. Hence there is $x_0 \in B$ such that $||\tilde{x} - x_0|| \le 2^{n+2}\delta$. This inequality and (3.8) imply in turn that $||\tilde{y} - x_0|| \le 2^{n+3}\delta$. The definition of δ (see (3.7)) now shows that $f(\tilde{y} - x_0) \le \xi$. Combining this inequality with (3.2), (3.6) and the inclusion $x_0 \in B$, we see that

$$f(\tilde{y} - z) \le \delta + f(\tilde{y} - x_0) \le \xi + \delta \le 2\xi. \tag{3.9}$$

It now follows from (3.5) that $||z - \tilde{y}|| \le \epsilon/2$. Hence (3.6), (3.8) and (3.4) imply that $||\bar{x} - z|| \le \epsilon$. This concludes the proof of the lemma in the first case.

Now we turn our attention to the second case where

$$\rho(\tilde{x}, A) > r. \tag{3.10}$$

For each $t \in [0, r]$, set

$$A_t = \{v \in X :\ \rho(v, A) \le t\} \in S(X) \tag{3.11}$$

and

$$\mu(t) = \rho_f(\tilde{x}, A_t). \tag{3.12}$$

By (3.10) and (1.3),

$$\mu(t) > 0,\ t \in [0, r]. \tag{3.13}$$

Clearly $\mu(t)$, $t \in [0, r]$, is a decreasing function. Choose a number

$$t_0 \in (0, r/4) \tag{3.14}$$

such that μ is continuous at t_0. By (1.2), there exists a natural number n which satisfies the following conditions:

$$n > 4||\tilde{x}|| + 8 \tag{3.15}$$

and

$$\text{if } z \in X,\ f(x) \leq \mu(0) + 1,\ \text{then } ||z|| \leq n/4. \tag{3.16}$$

Let $\phi_n :\ (0,1) \to (0,1)$ be an increasing function for which (1.4) is true. Choose a positive number $\gamma \in (0,1)$ such that

$$\gamma < \mu(t_0)(1 - \phi(1 - 2r/n))/8. \tag{3.17}$$

Next, choose a positive number $\delta_0 < 1/4$ such that

$$2^{n+3}\delta_0 < \min\{\epsilon, \gamma\}, \tag{3.18}$$

$$[t_0 - 4\delta_0, t_0 + 4\delta_0] \subset (0, r/4), \tag{3.19}$$

and

$$|\mu(t) - \mu(t_0| \leq \gamma,\ t \in [t_0 - 4\delta_0, t_0 + 4\delta_0]. \tag{3.20}$$

Finally, choose a vector x_0 such that

$$x_0 \in A_{t_0} \text{ and } f(\tilde{x} - x_0) \leq \mu(t_0) + \gamma. \tag{3.21}$$

It follows from (3.21), (3.11) and (3.14) that

$$||x_0 - \tilde{x}|| \geq \rho(\tilde{x}, A) - \rho(x_0, A) \geq \rho(\tilde{x}, A) - t_0 \geq \rho(\tilde{x}, A) - r/2, \tag{3.22}$$

and hence by (3.10),

$$||x_0 - \tilde{x}|| > r/2. \tag{3.23}$$

It follows from (3.21) and (3.16) that

$$||x_0 - \tilde{x}|| \leq n/4. \tag{3.24}$$

There exist $\bar{x} \in \{\alpha x_0 + (1-\alpha)\tilde{x} :\ \alpha \in (0,1)\}$ and $\alpha_0 \in (0,1)$ such that

$$||\bar{x} - x_0|| = r/2 \tag{3.25}$$

and

$$\bar{x} = \alpha_0 x_0 + (1 - \alpha_0)\tilde{x}. \tag{3.26}$$

By (3.26) and (3.25), $r/2 = ||\bar{x} - x_0|| = ||\alpha_0 x_0 + (1-\alpha_0)\tilde{x} - x_0|| = (1 - \alpha_0)||\tilde{x} - x_0||$ and

$$\alpha_0 = 1 - r(2||\tilde{x} - x_0||)^{-1}. \tag{3.27}$$

The relations (3.27) and (3.24) imply that

$$\alpha_0 \le 1 - r/(2n/4) = 1 - 2r/n. \tag{3.28}$$

Set

$$\tilde{A} = A_{t_0} \cup \{\bar{x}\}. \tag{3.29}$$

Now we will estimate $f(\tilde{x} - \bar{x})$. By (3.26), (3.24), (1.4), (3.21) and (3.28),

$$f(\tilde{x} - \bar{x}) = f(\tilde{x} - (\alpha_0 x_0 + (1 - \alpha_0)\tilde{x})) = f(\alpha_0(\tilde{x} - x_0)) \le$$

$$\phi_n(\alpha_0) f(\tilde{x} - x_0) \le \phi_n(\alpha_0)(\mu(t_0) + \gamma) \le \phi_n(1 - 2r/n)(\mu(t_0) + \gamma).$$

Thus

$$f(\tilde{x} - \bar{x}) \le \phi_n(1 - 2r/n)(\mu(t_0) + \gamma) \le \mu(t_0)\phi_n(1 - 2r/n) + \gamma. \tag{3.30}$$

By (3.29), (3.12), (3.17) and (3.30), for each $x \in \tilde{A} \setminus \{\bar{x}\} \subset A_{t_0}$,

$$f(\tilde{x} - x) \ge \mu(t_0) > f(\tilde{x} - \bar{x}) \tag{3.31}$$

and therefore

$$f(\tilde{x} - \bar{x}) = \rho_f(\tilde{x}, \tilde{A}). \tag{3.32}$$

There exists $\delta \in (0, \delta_0)$ such that

$$2^{n+4}\delta < \delta_0 \tag{3.33}$$

and

$$|f(z) - f(\tilde{x} - \bar{x})| \le \gamma/4 \text{ for all } z \in X \text{ satisfying } ||z - (\tilde{x} - \bar{x})|| \le 2^{n+3}\delta. \tag{3.34}$$

By (3.29), (2.3), (3.25), (3.21), (3.14) and (3.11),

$$\tilde{H}(\tilde{A}, A) \le H(\tilde{A}, A) \le r. \tag{3.35}$$

The relations (3.35) and (3.32) imply (3.1). Assume now that

$$\tilde{y} \in X, \ ||\tilde{y} - \tilde{x}|| \le \delta \tag{3.36}$$

and

$$B \in S(X) \text{ and } h(\tilde{A}, B) \le \delta. \tag{3.37}$$

First we will show that

$$\rho_f(\tilde{y}, B) \le \mu(t_0)\phi_n(1 - 2r/n) + 2\gamma. \tag{3.38}$$

By (3.37) and the definition of h (see (2.4)), $h_n(\tilde{A}, B)(1 + h_n(\tilde{A}, B))^{-1} \le 2^n\delta$. When combined with (3.33), this inequality implies that

$$h_n(\tilde{A}, B) \le 2^n\delta(1 - 2^n\delta)^{-1} \le 2^{n+1}\delta. \tag{3.39}$$

It follows from (3.30) and the definition of n (see (3.16), (3.15)) that $||\tilde{x} - \bar{x}|| \le n/2$ and $||\bar{x}|| \le n$. Combined with (3.29) and (3.39) this implies that $\rho(\bar{x}, B) \le 2^{n+1}\delta$. Therefore there exists $\bar{y} \in B$ such that $||\bar{x} - \bar{y}|| \le 2^{n+2}\delta$. Combining this inequality with (3.36), we see that $||(\bar{y} - \tilde{y}) - (\bar{x} - \tilde{x})|| \le ||\bar{x} - \bar{y}|| + ||\tilde{y} - \tilde{x}|| \le 2^{n+3}\delta$. It follows from this inequality and (3.34) that $f(\tilde{y} - \bar{y}) \le f(\tilde{x} - \bar{x}) + \gamma/4$. By the last inequality and (3.30), $f(\tilde{y} - \bar{y}) \le \mu(t_0)\phi_n(1 - 2r/n) + 2\gamma$. This implies (3.38).

Assume now that $z \in B$ satisfies (3.2). To complete the proof of the lemma it is sufficient to show that $||\bar{x} - z|| \le \epsilon$. Assume the contrary. Then

$$||\bar{x} - z|| > \epsilon. \tag{3.40}$$

We will show that there exists $\bar{z} \in \tilde{A}$ such that

$$||z - \bar{z}|| \le 2^{n+2}\delta. \tag{3.41}$$

We have already shown that (3.39) holds. By (3.2), (3.38), (3.17) and (3.33),

$$f(\tilde{y} - z) \le \rho_f(\tilde{y}, B) + \delta \le \phi_n(1 - 2r/n)\mu(t_0) + 2\gamma + \delta \le \mu(0) + 1/2.$$

Hence $||z - \tilde{y}|| \le n/4$ by (3.16), and by (3.36) and (3.15),

$$||z|| \le n/4 + ||\tilde{y}|| \le n/4 + ||\tilde{x}|| + ||\tilde{y} - \tilde{x}|| \le n.$$

Thus $||z|| \le n$. The inclusion $z \in B$ and (3.39) now imply that $\rho(z, \tilde{A}) \le h_n(B, \tilde{A}) \le 2^{n+1}\delta$. Therefore there exists $\bar{z} \in \tilde{A}$ such that (3.41) holds. It follows from (3.41), (3.40), (3.29), (3.33) and (3.18) that

$$\bar{z} \in A_{t_0}. \tag{3.42}$$

By (3.41) and (3.36), $||z + \tilde{x} - \tilde{y} - \bar{z}|| \le ||\tilde{x} - \tilde{y}|| + ||z - \bar{z}|| \le 2^{n+2}\delta + \delta \le 2^{n+3}\delta$. It follows from this inequality, (3.42), (3.11) and (3.33) that

$$\rho(z + \tilde{x} - \tilde{y}, A) \le ||z + \tilde{x} - \tilde{y} - \bar{z}|| + \rho(\bar{z}, A) \le 2^{n+3}\delta + t_0 \le t_0 + \delta_0.$$

Thus $z + \tilde{x} - \tilde{y} \in A_{t_0+\delta_0}$. By this inclusion, (3.11), (3.12) and (3.20),

$$f(\tilde{y} - z) = f(\tilde{x} - (z + \tilde{x} - \tilde{y})) \geq \rho_f(\tilde{x}, A_{t_0+\delta_0}) = \mu(t_0 + \delta_0) \geq \mu(t_0) - \gamma.$$

Hence, by (3.2), (3.38), (3.18) and (3.33),

$$\mu(t_0) - \gamma \leq f(\tilde{y} - z) \leq \rho_f(\tilde{y}, B) + \delta \leq \phi_n(1 - 2r/n)\mu(t_0) + 2\gamma + \delta \leq$$

$$\phi_n(1 - 2r/n)\mu(t_0) + 3\gamma.$$

Thus $\mu(t_0) - \gamma \leq \phi_n(1 - 2r/n)\mu(t_0) + 3\gamma$, which contradicts (3.17). This completes the proof of Lemma 3.1. □

4. Proofs of Theorems 2.1-2.3

The cornerstone of our proofs is the property established in Lemma 3.1. Since this property is close, but not identical to the hypotheses of the variational principle in [9], we will present direct proofs of our results.

By Lemma 3.1, for each $(A, x) \in S(X) \times X$ and each integer $k \geq 1$, there exist $A(x, k) \in S(X)$, $\bar{x}(A, k) \in A(x, k)$, and $\delta(x, A, k) > 0$ such that

$$\tilde{H}(A, A(x, k)) \leq 2^{-k}, \ f(x - \bar{x}(A, k)) = \rho_f(x, A(x, k)), \tag{4.1}$$

and the following property holds:

(P1) For each $y \in X$ satisfying $||y - x|| \leq 2\delta(x, A, k)$, each $B \in S(X)$ satisfying $h(B, A(x, k)) \leq 2\delta(x, A, k)$, and each $z \in B$ satisfying $f(y - z) \leq \rho_f(y, B) + 2\delta(x, A, k)$, the inequality $||z - \bar{x}(A, k)|| \leq 2^{-k}$ holds.

For each $(A, x) \in S(X) \times X$ and each integer $k \geq 1$, define

$$V(A, x, k) = \{(B, y) \in S(X) \times X : \tag{4.2}$$

$$h(B, A(x, k)) < \delta(x, A, k) \text{ and } ||y - x|| < \delta(x, A, k)\}$$

and

$$U(A, x, k) = \{B \in S(X) : \ h(B, A(x, k)) < \delta(x, A, k)\}. \tag{4.3}$$

Now set

$$\Omega = \cap_{n=1}^{\infty} \cup \{V(A, x, k) : \ (A, x) \in S(X) \times X, \ k \geq n\}, \tag{4.4}$$

and for each $x \in X$ let

$$\Omega_x = \cap_{n=1}^{\infty} \cup \{U(A, x, k) : \ A \in S(X), \ k \geq n\}. \tag{4.5}$$

It is easy to see that $\Omega_x \times \{x\} \subset \Omega$ for all $x \in X$, Ω_x is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X)$ for all $x \in X$, and Ω is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X) \times X$.

Completion of the proof of Theorem 2.2. Let $(A, \tilde{x}) \in \Omega$. We will show that $(A, \tilde{x})$ has property (C2). By the definition of Ω (see (4.4)), for each integer $n \geq 1$ there exist an integer $k_n \geq n$ and a pair $(A_n, x_n) \in S(X) \times X$ such that

$$(A, \tilde{x}) \in V(A_n, x_n, k_n). \tag{4.6}$$

Let $\{z_i\}_{i=1}^{\infty} \subset A$ be such that

$$\lim_{i \to \infty} f(\tilde{x} - z_i) = \rho_f(\tilde{x}, A). \tag{4.7}$$

Fix an integer $n \geq 1$. It follows from (4.6), (4.2) and property (P1) that for all large enough integers i, $f(\tilde{x} - z_i) < \rho_f(\tilde{x}, A) + \delta(x_n, A_n, k_n)$ and $||z_i - \bar{x}_n(A_n, k_n)|| \leq 2^{-n}$. Since $n \geq 1$ is arbitrary, we conclude that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to some $\tilde{y} \in A$. Clearly $f(\tilde{x} - \tilde{y}) = \rho_f(\tilde{x}, A)$. If the minimizer $\tilde{y}$ were not unique we would be able to construct a nonconvergent minimizing sequence $\{z_i\}_{i=1}^{\infty}$. Thus $\tilde{y}$ is the unique solution to problem (P) (with $x = \tilde{x}$).

Let $\epsilon > 0$. Choose an integer $n > 4/\min\{1, \epsilon\}$. By property (P1), (4.6) and (4.2),

$$||\tilde{y} - \bar{x}_n(A_n, k_n)|| \leq 2^{-n}. \tag{4.8}$$

Assume that $z \in X$ satisfies $||z - \tilde{x}|| \leq \delta(x_n, A_n, k_n)$, $B \in S(X)$ satisfies $h(A, B) \leq \delta(x_n, A_n, k_n)$, and $y \in B$ satisfies $f(z - y) \leq \rho_f(z, B) + \delta(x_n, A_n, k_n)$. Then

$$h(B, A_n(x_n, k_n)) \leq 2\delta(x_n, A_n, k_n) \text{ and } ||z - \bar{x}_n(A_n, k_n))| \leq 2\delta(x_n, A_n, k_n)$$

by (4.6) and (4.2). Now it follows from property (P1) that $||y - \bar{x}_n(A_n, k_n)|| \leq 2^{-n}$. When combined with (4.8), this implies that $||y - \tilde{y}|| \leq 2^{1-n} < \epsilon$. The proof of Theorem 2.2 is complete. $\square$

Theorem 2.1 follows from Theorem 2.2 and the inclusion $\Omega_{\tilde{x}} \times \{\tilde{x}\} \subset \Omega$.

Although a variant of Theorem 2.3 also follows from Theorem 2.2 by a classical result of Kuratowski and Ulam [11], the following direct proof may also be of interest.

Proof of Theorem 2.3. Let the sequence $\{x_i\}_{i=1}^{\infty} \subset X_0$ be everywhere dense in X_0. Set $\mathcal{F} = \cap_{p=1}^{\infty} \Omega_{x_p}$. Clearly $\mathcal{F}$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $S(X)$.

Let $A \in \mathcal{F}$ and let $p, n \geq 1$ be integers. Clearly $A \in \Omega_{x_p}$ and by (4.5) and (4.3), there exist $A_n \in S(X)$ and an integer $k_n \geq n$ such that

$$h(A, A_n(x_p, k_n)) < \delta(x_p, A_n, k_n) \text{ with } A \in S(X). \tag{4.9}$$

It follows from this inequality and property (P1) that the following property holds:

(P2) For each $y \in X$ satisfying $||y - x_p|| \leq \delta(x_p, A_n, k_n)$ and each $z \in A$ satisfying $f(y - z) \leq \rho_f(y, A) + 2\delta(x_p, A_n, k_n)$, the inequality $||z - \bar{x}_p(A_n, k_n)|| \leq 2^{-n}$ holds.

Set $W(p, n) = \{z \in X_0 : \ ||z - x_p|| < \delta(x_p, A_n, k_n)\}$ and

$$F = \cap_{n=1}^{\infty} \cup \{W(p, n) : \ p = 1, 2, \dots\}.$$

Clearly F is a countable intersection of open everywhere dense subsets of X_0.

Let $x \in F$. Consider a sequence $\{z_i\}_{i=1}^{\infty} \subset A$ such that

$$\lim_{i \to \infty} f(x - z_i) = \rho_f(x, A). \tag{4.10}$$

Let $\epsilon > 0$. Choose an integer $n > 8 / \min\{1, \epsilon\}$. There exists an integer $p \geq 1$ such that $x \in W(p, n)$. By the definition of $W(p, n)$, $||x - x_p|| < \delta(x_p, A_n, k_n)$. It follows from this inequality, (4.10) and property (P2) that for all sufficiently large integers i, $f(x - z_i) \leq \rho_f(x, A) + \delta(x_p, A_n, k_n)$ and $||z_i - \bar{x}_p(A_n, k_n)|| \leq 2^{-n} < \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to $\tilde{y} \in A$. Clearly $\tilde{y}$ is the unique minimizer of the minimization problem $z \to f(x - z)$, $z \in A$. Note that we have shown that any sequence $\{z_i\}_{i=1}^{\infty} \subset A$ satisfying (4.10) converges to $\tilde{y}$. This completes the proof of Theorem 2.3. $\square$

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