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AN EXTENSION OF LIE-TROTTER PRODUCT FORMULA

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Abstract. The paper establishes the product formula for semigroups of nonlinear operators on a real Hilbert space in the case where one of the operators is ω-m-accretive. This extends a well-known result for maximal monotone operators (the case $\omega = 0$). An example dealing with Caginalp's model is given.

1. Introduction and main result

Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$ and the induced norm $\|\cdot\|$. For a (nonlinear) operator A on H the notation $D(A)$ represents the domain of A, and $R(A)$ denotes the range of A.

Given a number $\omega \in \mathbb{R}$, a (single-valued) operator $A : D(A) \subset H \to H$ is said to be *ω-accretive* if $A + \omega I$ (where I stands for the identity of H) is *accretive*, i.e.,

$$(Ax_1 - Ax_2, x_1 - x_2) \geq -\omega\|x_1 - x_2\|^2 \quad \forall x_1, x_2 \in D(A). \tag{1.1}$$

When $\omega = 0$ the operator A is called *accretive* (or *monotone*).

An ω-accretive operator $A : D(A) \subset H \to H$ is said to be *ω-m-accretive* if $A + \omega I$ is m-accretive (or maximal monotone). If $A : D(A) \subset H \to H$ is an ω-m-accretive operator, then $-A$ generates a semigroup $\{S_A(t);\ t \geq 0\}$ which is differentiable a.e. with respect to t (on $D(A)$) and satisfies

$$\frac{d^+}{dt}(S_A(t)x) + AS_A(t)x = 0 \quad \forall t \in [0,\infty),\ \forall x \in D(A). \tag{1.2}$$

For a detailed study we refer to Barbu [1] and Brézis [2]. Different aspects related to these topics are also discussed in Section 2.

The main result of the paper is the following extension of Lie-Trotter product formula.

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Theorem 1. *Let $A : D(A) \subset H \to H$ be an m-accretive operator and let $B : D(B) \subset H \to H$ be an ω-m-accretive operator, for some $\omega \in \mathbb{R}$, such that the operator $A + B : D(A) \cap D(B) \subset H \to H$ is ω-m-accretive. If*

$$(I + \lambda A)^{-1}(\overline{D(A) \cap D(B)}) \subset \overline{D(A) \cap D(B)} \quad \forall\, \lambda > 0$$

and

$$(I + \lambda B)^{-1}(\overline{D(A) \cap D(B)}) \subset \overline{D(A) \cap D(B)} \quad \forall\, \lambda > 0 \ \ \textit{with} \ \ \lambda\omega < 1,$$

then one has the convergence

$$\left[S_B\Big(\frac{t}{n}\Big)S_A\Big(\frac{t}{n}\Big)\right]^n x \to S_{A+B}(t)x \quad \textit{strongly in} \ \ H \ \ \textit{as} \ \ n \to \infty \tag{1.3}$$

uniformly on the bounded intervals of $[0, \infty)$, for every $x \in \overline{D(A) \cap D(B)}$. Here $\{S_A(t); t \geq 0\}$, $\{S_B(t); t \geq 0\}$, $\{S_{A+B}(t); t \geq 0\}$ denote the semigroups generated by $-A$, $-B$, $-(A + B)$, respectively.

Theorem 1 extends the Lie-Trotter product formula (on the closed convex set $\overline{D(A) \cap D(B)}$) for $\omega = 0$ (see Barbu [1], Brézis [2], Brézis and Pazy [3]).

Theorem 1 does not cover the multivalued situation for ω-accretive operators. When $\omega = 0$ the corresponding set-valued result was treated in Kobayashi [5].

As an application of Theorem 1 we indicate how the Caginalp's model [5] can be decoupled in two simpler systems.

The rest of the paper is organised as follows. Section 2 contains some relevant properties of ω-m-accretive operators that are needed in the sequel. In Section 3 the proof of Theorem 1 is presented. Section 4 contains the application of Theorem 1 to Caginalp's model describing the phase-field changes.

2. Properties of ω-m-accretive operators

This Section is devoted to the extension of some results known for m-accretive operators (see, e.g., Barbu [1] and Brézis [2]) to ω-m-accretive operators, with a fixed real number ω. We point out that if $\omega \leq 0$, an ω-m-accretive operator is m-accretive. Consequently, throughout the Section we fix a number $\omega > 0$.

Lemma 1. *An operator $A : D(A) \subset H \to H$ is ω-accretive if and only if A is in the class $\mathcal{A}(\omega)$ as introduced in* Brézis and Pazy [4], *that is,*

$$\|(x_1 + \lambda A x_1) - (x_2 + \lambda A x_2)\| \geq (1 - \lambda\omega)\|x_1 - x_2\| \qquad \forall 0 \leq \lambda < \omega^{-1}, \quad x_1,\ x_2 \in D(A).$$

Proof. See, e.g., [8, p. 352]. □

Lemma 1 allows to define the resolvent $J_\lambda^A : R(I + \lambda A) \to D(A) \subset H$ and the Yosida approximation $A_\lambda : R(I + \lambda A) \to H$ associated to an ω-accretive operator $A : D(A) \subset H \to H$ by

$$J_\lambda^A = (I + \lambda A)^{-1} \quad \text{and} \quad A_\lambda = \frac{1}{\lambda}(I - J_\lambda^A), \tag{2.1}$$

respectively, for every $0 < \lambda < \omega^{-1}$.

For an ω-m-accretive operator A, the next lemma ensures that the mappings in (2.1) are defined on H.

Lemma 2. *If $A : D(A) \subset H \to H$ is ω-m-accretive with $\omega > 0$, then*

$$R(I + \lambda A) = H \quad \forall 0 \leq \lambda < \omega^{-1}.$$

Proof. Since A is ω-m-accretive, it follows that

$$R(I + \mu(A + \omega I)) = H \quad \forall \mu > 0,$$

or, equivalently,

$$R(I + \frac{\mu}{1 + \mu\omega} A) = H \quad \forall \mu > 0.$$

For every $0 \leq \lambda < \omega^{-1}$, there exists $\mu \geq 0$ satisfying $\mu/(1 + \mu\omega) = \lambda$. This leads to the desired result. □

Now we state some basic properties of operator A_λ.

Lemma 3. *Let $A : D(A) \subset H \to H$ be an ω-m-accretive operator. Then for each $0 < \lambda < \omega^{-1}$,*

$$(A_\lambda x_1 - A_\lambda x_2, x_1 - x_2) \geq -\frac{\omega}{1 - \lambda\omega}\|x_1 - x_2\|^2 \quad \forall x_1, x_2 \in H.$$

Proof. The result follows directly from (2.1) and Lemma 2.1 (i) in [4] (see also [8, p. 355]). □

Lemma 4. *Let $A : D(A) \subset H \to H$ be an ω-m-accretive operator. Then for each $x \in D(A)$ one has that $A_\lambda x \to Ax$ strongly in H as $\lambda \downarrow 0$.*

Proof. Let us fix $x \in D(A)$. The set $\{A_\lambda x : \lambda > 0\}$ is bounded in H (see, e.g., [8, p. 355]), thus there exists $y \in H$ fulfilling $A_\lambda x \to y$ weakly in H as $\lambda \downarrow 0$ along a relabelled subsequence. It follows that

$$\|y\| \leq \liminf_{\lambda \downarrow 0} \|A_\lambda x\| \leq \lim_{\lambda \downarrow 0} (1 - \lambda\omega)^{-1} \|Ax\|$$

(cf. [8, p. 355]). Taking into account that $A_\lambda x = AJ_\lambda^A x$, $J_\lambda^A x \to x$ as $\lambda \downarrow 0$ and the operator A is demiclosed, it turns out that $y = Ax$. Then (2.2) enables us to conclude that a subsequence of $(A_\lambda x)$ can be found such that $\|A_\lambda x\| \to \|Ax\|$, consequently, $A_\lambda x \to Ax$ strongly as $\lambda \downarrow 0$. Since this occurs for every weakly convergent subsequence of $(A_\lambda x)$, the proof is complete. □

Lemma 5. *Let $A : D(A) \subset H \to H$ be an ω-m-accretive operator and let C be a nonempty closed convex subset of H with $C \subset \overline{D(A)}$. If, for every $0 < \lambda < \omega^{-1}$, one has*

$$J_\lambda^A(C) \subset C, \tag{2.2}$$

then the estimate below holds

$$\|S_A(t)x - P_C S_A(t)x\| \leq \|x - P_C x\| e^{2\omega t} \quad \forall x \in \overline{D(A)}, \quad \forall t > 0, \tag{2.3}$$

where P_C stands for the projection (in H) on C.

Proof. Firstly we show that assumption (2.2) implies

$$(A_\lambda x, y) \geq 0 \quad \forall y \in \partial I_C(x), \quad \forall x \in C, \tag{2.4}$$

where ∂I_C represents the subdifferential of the indicator function I_C of the set C. Indeed, $y \in \partial I_C(x)$ is equivalent to $(y, z - x) \leq 0$, $\forall z \in C$. According to (2.2), we may set here $z = J_\lambda^A x$. This yields (2.4).

The next step in the proof is to show that

$$(A_\lambda x, (\partial I_C)_\mu(x)) \geq -\frac{\omega}{\mu(1 - \lambda\omega)} \|x - P_C x\|^2 \tag{2.5}$$

for all $x \in H$, $0 < \lambda < \omega^{-1}$, $\mu > 0$. Since $(\partial I_C)_\mu(x) = \mu^{-1}(x - P_C x)$ (see Brézis [2], p. 46), by Lemma 3 we can write

$$(A_\lambda x, (\partial I_C)_\mu(x)) \geq \frac{1}{\mu} \left[-\frac{\omega}{1 - \lambda\omega} \|x - P_C x\|^2 + (A_\lambda P_C x, x - P_C x) \right]. \tag{2.6}$$

Using the properties $x - P_C x \in \partial I_C(P_C x)$, (2.4) (for $P_C x$ in place of x) and (2.6), we arrive at (2.5).

Lemma 4 allows to pass to the limit as $\lambda \downarrow 0$ in (2.5), which leads to

$$(Ax, (\partial I_C)_\mu(x)) \geq -\frac{\omega}{\mu}\|x - P_C x\|^2 \quad \forall\, x \in D(A), \quad \forall \mu > 0. \tag{2.7}$$

We check now that the convex regularization $(I_C)_\mu$ of I_C (see, e.g., [2, p. 46] or [8, p. 404]) satisfies

$$(I_C)_\mu(S_A(t)x) \leq (I_C)_\mu(x) + \frac{\omega}{\mu}\int_0^t \|S_A(s)x - P_C S_A(s)x\|^2 ds \tag{2.8}$$
$$\forall\, x \in \overline{D(A)}, \quad \forall \mu > 0, \quad \forall t > 0.$$

Clearly, to establish (2.8) it is sufficient to consider the case where $x \in D(A)$. By (1.2) and (2.7) (with $S_A(t)x$ in place of x) we derive

$$\begin{aligned}\frac{d}{dt}(I_C)_\mu(S_A(t)x) &= (I_C)'_\mu(S_A(t)x)\frac{d}{dt}(S_A(t)x)\\ &= -(A(S_A(t)x), (\partial I_C)_\mu(S_A(t)x))\\ &\leq \frac{\omega}{\mu}\|S_A(t)x - P_C S_A(t)x\|^2\\ &\qquad \text{a.e. } t \in (0, +\infty), \quad \forall \mu > 0.\end{aligned} \tag{2.9}$$

Since the map $t \mapsto (I_C)_\mu(S_A(t)x)$ is Lipschitz continuous on bounded intervals, by integrating (2.9) over $[0, t]$ we get (2.8).

Substituting the expression of $(I_C)_\mu(S_A(t)x)$ (see, e.g., [1, p. 46]) in (2.8) we infer that

$$\begin{aligned}&\frac{1}{2\mu}\|S_A(t)x - P_C S_A(t)x\|^2\\ &\leq \frac{1}{2\mu}\|x - P_C x\|^2 + \frac{\omega}{\mu}\int_0^t \|S_A(s)x - P_C S_A(s)x\|^2 ds \quad \forall t > 0.\end{aligned}$$

Then Gronwall inequality implies (2.3) which completes the proof. □

3. Proof of Theorem 1

If $\omega \leq 0$, the operator B is m-accretive. As A is also an m-accretive operator with $A + B$ m-accretive, we may conclude by applying the result known for m-accretive operators. It remains to treat the case $\omega > 0$.

We claim that

$$\lim_{t\to 0}\frac{1}{t}\Big(x - S_B(t)S_A(t)x\Big) = Ax + Bx \quad \forall x \in D(A)\cap D(B). \tag{3.1}$$

To this end, for a fixed $x \in D(A)\cap D(B)$, one sees that

$$\frac{1}{t}\|S_B(t)x - S_B(t)S_A(t)x\| \le e^{\omega t}\|Ax\| \quad \forall t \ge 0. \tag{3.2}$$

Indeed, it is known from Lemma 1 that the operator B belongs to the class $\mathcal{A}(\omega)$. Then the following estimate holds

$$\|S_B(t)y - S_B(t)z\| \le e^{\omega t}\|y - z\| \quad \forall t \ge 0, \quad y, z \in \overline{D(B)} \tag{3.3}$$

(see relation (1.5) in [4]). Using (3.3) and (1.2) we find that

$$\begin{aligned} \frac{1}{t}\|S_B(t)x - S_B(t)S_A(t)x\| &\le e^{\omega t}\frac{1}{t}\|x - S_A(t)x\| \\ &= e^{\omega t}\frac{1}{t}\int_0^t \|AS_A(\tau)x\| d\tau. \end{aligned} \tag{3.4}$$

Part (6) of Theorem 3.1 in Brézis [2, p. 54], ensures that the mapping $\tau \mapsto \|AS_A(\tau)x\|$ is nonincreasing. Hence (3.4) yields (3.2).

We note that

$$\begin{aligned} &\Big(\frac{\xi - S_B(t)\xi}{t} + \frac{x - S_A(t)x}{t} - \frac{x - S_B(t)x}{t} - \frac{1}{t}\Big(S_B(t)x - S_B(t)S_A(t)x\Big), \xi - S_A(t)x\Big) \\ &= \frac{1}{t}\Big(\xi - S_B(t)\xi - (I - S_B(t))S_A(t)x, \xi - S_A(t)x\Big) \\ &= \frac{1}{t}\Big[\|\xi - S_A(t)x\|^2 - \Big(S_B(t)\xi - S_B(t)S_A(t)x, \xi - S_A(t)x\Big)\Big] \\ &\ge \frac{1}{t}\Big[\|\xi - S_A(t)x\|^2 - \|S_B(t)\xi - S_B(t)S_A(t)x\|\|\xi - S_A(t)x\|\Big] \\ &\qquad \forall \xi \in D(B), \quad \forall x \in D(A)\cap D(B). \end{aligned}$$

By (3.3) this leads to

$$\begin{aligned} &\Big(\frac{\xi - S_B(t)\xi}{t} + \frac{x - S_A(t)x}{t} - \frac{x - S_B(t)x}{t} - \frac{1}{t}\Big(S_B(t)x - S_B(t)S_A(t)x\Big), \xi - S_A(t)x\Big) \\ &\ge \frac{1}{t}\left[\|\xi - S_A(t)x\|^2 - e^{\omega t}\|\xi - S_A(t)x\|^2\right] \\ &= \frac{1}{t}(1 - e^{\omega t})\|\xi - S_A(t)x\|^2 \qquad \forall \xi \in D(B), \quad \forall x \in D(A)\cap D(B). \end{aligned} \tag{3.5}$$

Since the left-hand side of (3.4) is bounded as $t \to 0$, there exists the weak limit (in H)

$$\frac{1}{t}(S_B(t)x - S_B(t)S_A(t)x) \to y \quad \text{weakly as } t \to 0. \tag{3.6}$$

Letting $t \to 0$ in (3.5), on the basis of definition for the generator of a semi-group of type ω (see [1, pp. 102-104]) and (3.6) we find

$$(B\xi + Ax - Bx - y, \xi - x) \geq -\omega\|\xi - x\|^2.$$

Equivalently, this is expressed as follows

$$((B + \omega I)\xi - (Bx - Ax + y + \omega x), \xi - x) \geq 0 \tag{3.7}$$

for all $\xi \in D(B)$ and $x \in D(A) \cap D(B)$. Because B is ω-m-accretive we derive from (3.7) that $(B + \omega I)x = Bx - Ax + y + \omega x$, that is

$$y = Ax. \tag{3.8}$$

By (3.2) and (3.8) we get

$$\limsup_{t \to 0} \frac{1}{t}\|S_B(t)x - S_B(t)S_A(t)x\| \leq \|Ax\| = \|y\|.$$

This in conjunction with the weak convergence in (3.6) implies

$$\frac{1}{t}\Big(S_B(t)x - S_B(t)S_A(t)x\Big) \to Ax \quad \text{strongly as } t \to 0. \tag{3.9}$$

Then, in view of (3.9), it is seen that the claim in (3.1) is proved.

By Lemma 1 we know

$$A + B \in \mathcal{A}(\omega), \tag{3.10}$$

while Lemma 2 insures that

$$R(I + \lambda(A + B)) = H \quad \forall 0 < \lambda < \frac{1}{\omega}. \tag{3.11}$$

For proving the convergence in (1.3) we proceed by applying Theorem 3.2 in Brézis and Pazy [4] for the family of mappings

$$T(t) = S_B(t)S_A(t) : \overline{D(A) \cap D(B)} \to \overline{D(A) \cap D(B)} \quad \forall t \geq 0. \tag{3.12}$$

The mapping $T(t)$ as introduced in (3.12) is from $\overline{D(A)\cap D(B)}$ to itself due to Lemma 5 used with $C=\overline{D(A)\cap D(B)}$ and $A+B$ in place of A. Let us remark that the set $\overline{D(A)\cap D(B)}$ is convex because the operator $A+B+\omega I$ is maximal monotone.

Assumption (i) of Theorem 3.2 in Brézis and Pazy [2] holds with $M(t)=e^{\omega t}$, $t>0$, since

$$\begin{aligned}\|T(t)x-T(t)y\| &= \|S_B(t)S_A(t)x-S_B(t)S_A(t)y\| \\ &\le e^{\omega t}\|S_A(t)x-S_A(t)y\|\le e^{\omega t}\|x-y\|.\end{aligned}\tag{3.13}$$

Relation (3.13) shows that the mapping $T(t)$ is Lipschitz continuous with the Lipschitz constant $e^{\omega t}>1$. Then we may apply Lemma 2.2 part (i) in Brézis and Pazy [4]. Consequently, there exists on $\overline{D(A)\cap D(B)}$ the mapping

$$\Big(I+\frac{\lambda}{t}(I-T(t))\Big)^{-1}\quad \text{if } 0<\frac{\lambda}{t}<(e^{\omega t}-1)^{-1}.$$

We deduce that there exists the mapping from $\overline{D(A)\cap D(B)}$ into itself

$$\Big(I+\frac{\lambda}{t}(I-T(t))\Big)^{-1}\qquad \text{if } 0<\lambda<\frac{1}{2\omega} \text{ and } 0<t<t_0 \tag{3.14}$$

for some $t_0>0$.

Let us check now that assumption (ii) of Theorem 3.2 in Brézis and Pazy [4] is fulfilled for $T(t)$ given by (3.12) and $\lambda_0=\frac{1}{2\omega}$. To justify this claim we have to show that

$$\left(I+\frac{\lambda}{t}\Big(I-T(t)\Big)\right)^{-1}x\to\Big(I+\lambda(A+B)\Big)^{-1}x\quad \text{in } H \text{ as } t\to 0 \tag{3.15}$$

for every $x\in\overline{D(A)\cap D(B)}$ and $0<\lambda<\frac{1}{2\omega}$.

Towards this, taking into account (3.12) and (3.13) we can write

$$\begin{aligned}&\bigg(\frac{1}{t}\Big(\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-T(t)\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x\Big)-\frac{1}{t}\big(\xi-T(t)\xi\big),\\ &\qquad\qquad \big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-\xi\bigg)\\ &\ge\frac{1}{t}\left\|\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-\xi\right\|^2 \\ &\quad-\frac{1}{t}\left\|T(t)\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-T(t)\xi\right\|\left\|\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-\xi\right\|\\ &\ge\frac{1}{t}(1-e^{\omega t})\left\|\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-\xi\right\|^2,\end{aligned}\tag{3.16}$$

for all $0 < t < t_0$, $0 < \lambda < \frac{1}{2\omega}$, $\xi, x \in \overline{D(A) \cap D(B)}$. Since

$$\frac{1}{t}\Big(\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-T(t)\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x\Big)=\frac{1}{\lambda}\Big(x-\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x\Big),$$

relation (3.16) implies

$$\begin{aligned}&\Big(\frac{1}{\lambda}(x-\xi),\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-\xi\Big)-\Big(\frac{1}{t}(\xi-T(t)\xi),\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-\xi\Big)\\&\geq\Big(\frac{1}{\lambda}-\frac{1}{t}(e^{\omega t}-1)\Big)\Big\|\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-\xi\Big\|^2,\end{aligned}\tag{3.17}$$

for all $0 < t < t_0$, $0 < \lambda < \frac{1}{2\omega}$, $\xi, x \in \overline{D(A) \cap D(B)}$. Using the Cauchy-Schwarz inequality we derive

$$\begin{aligned}&\frac{1}{\lambda}\|x-\xi\|+\frac{1}{t}\|\xi-T(t)\xi\|\\&\geq\Big(\frac{1}{\lambda}-\frac{1}{t}(e^{\omega t}-1)\Big)\Big\|\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-\xi\Big\|,\end{aligned}\tag{3.18}$$

for all $0 < t < t_0$, $0 < \lambda < \frac{1}{2\omega}$, $\xi, x \in \overline{D(A) \cap D(B)}$. Let us fix $0 < \lambda < \frac{1}{2\omega}$ and $\xi \in D(A) \cap D(B)$ in (3.18). By the choice of $t_0 > 0$ it is true that

$$\frac{1}{\lambda}-\frac{1}{t}(e^{\omega t}-1)>\frac{1}{\lambda}-2\omega>0 \qquad \text{whenever } \ 0<t<t_0.$$

According to (3.1) and (3.12) we know that

$$\frac{1}{t}(\xi-T(t)\xi) \qquad \text{is bounded in } H, \ \ 0<t<t_0 \tag{3.19}$$

for a possibly smaller t_0. Then, combining (3.18) and (3.19) we deduce that for any $x \in \overline{D(A) \cap D(B)}$ and $0 < \lambda < \frac{1}{2\omega}$,

$$\Big(I+\frac{\lambda}{t}(I-T(t))\Big)^{-1}x \qquad \text{is bounded in } H, \ \ 0<t<t_0. \tag{3.20}$$

Let

$$\Big(I+\frac{\lambda}{t}(I-T(t))\Big)^{-1}x \to z \quad \text{weakly in } H \ \text{ as } t\to 0$$

for some $z \in H$. The existence of a weak limit point z is guaranteed by (3.20). Passing to the limit as $t \to 0$ in (3.17) and making use of (3.1), one finds that

$$\begin{aligned}\left(\frac{1}{\lambda}-\omega\right)\|z-\xi\|^2 &\le \left(\frac{1}{\lambda}-\omega\right)\liminf_{t\to 0}\|(I+\frac{\lambda}{t}(I-T(t)))^{-1}x-\xi\|^2\\ &\le \limsup_{t\to 0}\|(I+\frac{\lambda}{t}(I-T(t)))^{-1}x-\xi\|^2\\ &\le \limsup_{t\to 0}\Big[\Big(\frac{1}{\lambda}(x-\xi),\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-\xi\Big)\\ &\quad -\frac{1}{t}\Big(\xi-T(t)\xi,\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-\xi\Big)\Big] \qquad (3.21)\\ &\le \left(\frac{1}{\lambda}(x-\xi),z-\xi\right)-(A\xi+B\xi,z-\xi).\end{aligned}$$

It turns out that

$$\left((A+B+\omega I)\xi+\frac{1}{\lambda}(z-x)-\omega z,\xi-z\right)\ge 0,$$

for $0<\lambda<\frac{1}{2\omega}$, $\xi\in D(A)\cap D(B)$, $x\in H$. In view of m-ω-accretiveness of operator $A+B$, i.e., $A+B+\omega I$ is maximal monotone, it follows that $z\in D(A)\cap D(B)$ and

$$Az+Bz+\omega z=\frac{1}{\lambda}(x-z)+\omega z.$$

On the basis of (3.13) the formula above can be expressed as follows

$$z=\big(I+\lambda(A+B)\big)^{-1}x. \tag{3.22}$$

Since z is an arbitrary weak limit in (3.20) we infer from (3.22) that

$$(I+\frac{\lambda}{t}(I-T(t)))^{-1}x\to(I+\lambda(A+B))^{-1}x \quad \text{weakly in } H \quad \text{as } t\to 0 \tag{3.23}$$

for every $0<\lambda<\frac{1}{2\omega}$ and $x\in H$ (see (3.14)).

Choosing $\xi=(I+\lambda(A+B))^{-1}x$ in (3.21), property (3.21) shows that

$$\limsup_{t\to 0}\left\|\big(I+\frac{\lambda}{t}(I-T(t))\big)^{-1}x-(I+\lambda(A+B))^{-1}x\right\|^2\le 0. \tag{3.24}$$

Combining (3.23) and (3.24) it results that the claim in (3.15) is true.

All the hypotheses of Theorem 3.2 of Brézis-Pazy [2] are verified. Therefore we are in a position to conclude that the convergence result stated in (1.3) is valid. This completes the proof of Theorem 1. $\square$

4. Example

As application of Theorem 1 we treat the Caginalp's model, namely:

$$\begin{cases} u_t + \dfrac{\ell}{2}\varphi_t = k\Delta u & \text{in } Q = (0,T)\times\Omega, \\ \varphi_t = \dfrac{\xi^2}{\tau}\Delta\varphi + \dfrac{1}{2a\tau}(\varphi - \varphi^3) + \dfrac{2}{\tau}u & \text{in } Q, \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \Sigma = (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x), \quad \varphi(0,x) = \varphi_0(x) & x\in\Omega, \end{cases} \tag{4.1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ whose boundary $\partial\Omega$ is sufficiently smooth (for instance of class C^2), u is the reduced temperature, φ represent the phase function, and the positive parameters τ,ξ,ℓ,k,a are physical constants (see [5] for details).

Setting $y = u + \dfrac{\ell}{2}\varphi$, the system (4.1) takes the form

$$\begin{cases} y_t - k\Delta y + \dfrac{k\ell}{2}\Delta\varphi = 0 & \text{in } Q, \\ \varphi_t - \dfrac{\xi^2}{\tau}\Delta\varphi + \dfrac{1}{\tau}\left(\ell - \dfrac{1}{2a}\right)\varphi + \dfrac{1}{2a\tau}\varphi^3 - \dfrac{2}{\tau}y = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial\nu} = \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \Sigma, \\ y(0,x) = u_0(x) + \dfrac{\ell}{2}\varphi_0(x), \quad \varphi(0,x) = \varphi_0(x) & x\in\Omega. \end{cases} \tag{4.2}$$

Next, let us put (4.2) in an abstract framework. To this end we consider the space $H = L^2(\Omega)\times L^2(\Omega)$ endowed with the norm $\|\cdot\|$ defined by

$$\left\| \begin{pmatrix} y \\ \varphi \end{pmatrix} \right\|_H = \|y\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}.$$

We define the operator $A : D(A) = \{(y,\varphi)\in H^2(\Omega)^2 :\ \dfrac{\partial y}{\partial\nu} = \dfrac{\partial\varphi}{\partial\nu} = 0\} \subset H \to H$ by

$$A\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} -k\Delta y + \dfrac{k\ell}{2}\Delta\varphi \\ -\dfrac{\xi^2}{\tau}\Delta\varphi \end{pmatrix},$$

and the operator $B : D(B) = L^2(\Omega) \times L^6(\Omega) \subset H \to H$ by

$$B\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{1}{\tau}\left(\ell - \dfrac{1}{2a}\right)\varphi + \dfrac{1}{2a\tau}\varphi^3 - \dfrac{2}{\tau}y \end{pmatrix}.$$

Thus the system (4.2) becomes

$$\frac{d}{dt}\begin{pmatrix} y \\ \varphi \end{pmatrix} + A\begin{pmatrix} y \\ \varphi \end{pmatrix} + B\begin{pmatrix} y \\ \varphi \end{pmatrix} = 0. \tag{4.3}$$

Lemma 4.1. *If $k\ell^2 < 16\xi^2/\tau$ then the operator A is m-accretive.*

Proof. Using Green formula and Cauchy-Schwarz's inequality, we get

$$\begin{aligned} &\left\langle \begin{pmatrix} -k\Delta y + \frac{k\ell}{2}\Delta\varphi \\ -\frac{\xi^2}{\tau}\Delta\varphi \end{pmatrix}, \begin{pmatrix} y \\ \varphi \end{pmatrix} \right\rangle_H \\ &= k\|\nabla y\|^2_{L^2(\Omega)} - \frac{k\ell}{2}\langle \nabla y, \nabla\varphi\rangle_{L^2(\Omega;\mathbb{R}^n)} + \frac{\xi^2}{\tau}\|\nabla\varphi\|^2_{L^2(\Omega)} \\ &\geq k\|\nabla y\|^2_{L^2(\Omega)} - \frac{k\ell}{2}\|\nabla y\|_{L^2(\Omega)} \cdot \|\nabla\varphi\|_{L^2(\Omega)} + \frac{\xi^2}{\tau}\|\nabla\varphi\|^2_{L^2(\Omega)}. \end{aligned}$$

Since $k\ell^2 < 16\xi^2/\tau$ we have

$$k\|\nabla y\|^2_{L^2(\Omega)} - \frac{k\ell}{2}\|\nabla y\|_{L^2(\Omega)} \cdot \|\nabla\varphi\|_{L^2(\Omega)} + \frac{\xi^2}{\tau}\|\nabla\varphi\|^2_{L^2(\Omega)} \geq 0,$$

thus

$$\left\langle A\begin{pmatrix} y \\ \varphi \end{pmatrix}, \begin{pmatrix} y \\ \varphi \end{pmatrix} \right\rangle_H \geq 0.$$

Hence A is accretive.

It is well-known that for every $\binom{f}{g} \in H = \overline{D(A)}$ the system

$$\begin{cases} y - \lambda k\Delta y = f - \lambda\dfrac{k\ell}{2}\Delta\varphi \in L^2(\Omega), \\ \varphi - \lambda\dfrac{\xi^2}{\tau}\Delta\varphi = g \in L^2(\Omega), \end{cases}$$

has a unique solution $\binom{y}{\varphi} \in D(A)$ for every $\lambda > 0$ (see Barbu [1] p. 80), so the operator A is m-accretive. $\square$

Lemma 4.2. *The operator B is m-ω-accretive.*

Proof. Let us choose

$$\omega \geq \max\left\{\frac{1}{\tau},\ -\frac{1}{\tau}\left(\ell - \frac{1}{2a} - 1\right)\right\}. \tag{4.4}$$

We have

$$\begin{aligned}
&\left\langle (B+\omega I)\begin{pmatrix} y_1 \\ \varphi_1 \end{pmatrix} - (B+\omega I)\begin{pmatrix} y_2 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \varphi_1 \end{pmatrix} - \begin{pmatrix} y_2 \\ \varphi_2 \end{pmatrix}\right\rangle_H \\
&= \omega\|y_1 - y_2\|^2_{L^2(\Omega)} + \left(\frac{1}{\tau}\left(\ell - \frac{1}{2a}\right) + \omega\right)\|\varphi_1 - \varphi_2\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2a\tau}\langle \varphi_1^3 - \varphi_2^3, \varphi_1 - \varphi_2\rangle_{L^2(\Omega)} - \frac{2}{\tau}\langle y_1 - y_2, \varphi_1 - \varphi_2\rangle_{L^2(\Omega)} \\
&\geq \left(\omega - \frac{1}{\tau}\right)\|y_1 - y_2\|^2_{L^2(\Omega)} + \left(\frac{1}{\tau}\left(\ell - \frac{1}{2a} - 1\right) + \omega\right)\|\varphi_1 - \varphi_2\|^2_{L^2(\Omega)} \geq 0,
\end{aligned}$$

where (4.4) has been used.

It remains to check that $R(B+(\lambda+\omega)I) = H$, $\forall \lambda > 0$. Given $\alpha, \beta \in L^2(\Omega)$, the system

$$\begin{cases} (\lambda+\omega)y = \alpha \\ \dfrac{1}{\tau}\left(\ell - \dfrac{1}{2a}\right)\varphi + \dfrac{1}{2a\tau}\varphi^3 - \dfrac{2}{\tau}y + (\lambda+\omega)\varphi = \beta \end{cases}$$

has a unique solution $(y,\varphi) \in L^2(\Omega) \times L^6(\Omega)$ for every $\lambda > 0$. This expresses that the operator $B : D(B) \subset H \to H$ is m-ω-accretive. $\square$

The lemmas above show that the conditions of Theorem 1 are satisfied. Theorem 1 suggests the following approximating scheme for solving system (4.2)

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} y_\varepsilon \\ \varphi_\varepsilon \end{pmatrix} + A\begin{pmatrix} y_\varepsilon \\ \varphi_\varepsilon \end{pmatrix} = 0, \quad \text{in } \ [i\varepsilon, (i+1)\varepsilon], \\ y_\varepsilon(i\varepsilon) = \psi_\varepsilon((i+1)\varepsilon), \\ \varphi_\varepsilon(i\varepsilon) = z_\varepsilon((i+1)\varepsilon), \qquad i = 0, 1, \ldots, M-1, \end{cases}$$

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} \psi_\varepsilon \\ z_\varepsilon \end{pmatrix} + B\begin{pmatrix} \psi_\varepsilon \\ z_\varepsilon \end{pmatrix} = 0, \qquad\qquad \text{in } \ [i\varepsilon, (i+1)\varepsilon], \\ \psi_\varepsilon(i\varepsilon) = y_\varepsilon^+(i\varepsilon), \quad y_\varepsilon^+(0) = y(0,x), \\ z_\varepsilon(i\varepsilon) = \varphi_\varepsilon^+(i\varepsilon), \quad \varphi_\varepsilon^+(0) = \varphi_0(x), \quad i = 0, 1, \ldots, M-1, \end{cases}$$

where $0 < \varepsilon < \cdots < M\varepsilon = T$ is a partition of the time–interval $[0, T]$, and $y_\varepsilon^+(i\varepsilon)$, $\varphi_\varepsilon^+(i\varepsilon)$ are the right limits of y_ε, φ_ε, respectively, at $i\varepsilon$. Theorem 1 justifies the convergence of this approximating scheme to the unique solution of system (4.2) (or (4.1)). For an abstract result regarding a model more general than (4.1) we refer to Moroşanu and Motreanu [7].

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