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IMPULSIVE PERTURBATION OF C_0-SEMIGROUPS AND EVOLUTION INCLUSIONS

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Abstract. In this paper we consider impulsive perturbation of C_0-semigroups and construct the corresponding evolution operator which is only strongly right continuous with left hand limits. This is then used in the study of existence, uniqueness and regularity properties of solutions of systems governed by semilinear evolution equations and inclusions with impulsively perturbed generator. This is also used in the study of control systems driven by vector measures.

1. Introduction

In this paper we consider a class of semi linear impulsive systems where the principal operator is the generator of a C_0-semigroup which is impulsively perturbed multiplicatively. The basic homogeneous system is described by the following equation

$$dx(t) = Ax(t)d\beta(t), \quad x(0) = \xi. \tag{1}$$

We show that under certain assumptions on the pair $(A, \beta(\cdot))$ this generates an evolution or transition operator. This is then used to study semilinear systems governed by evolution equations and inclusions of the form

$$\begin{aligned} dx &= Axd\beta + B(t,x)dt, \quad t \geq 0, \quad x(0) = \xi, \\ dx - Axd\beta &\in F(t,x)dt, \quad t \geq 0, \quad x(0) = \xi, \end{aligned} \tag{2}$$

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where B is single valued while F is a multi valued map. An associated control system driven by vector measures may be given by either of the following equations or inclusions,

$$\begin{aligned} dx &= Axd\beta + B(t,x)dt + C(t,x)d\mu, \quad t \geq 0, \quad x(0) = \xi \\ dx &\in Axd\beta + B(t,x)dt + C(t,x)d\mu, \quad t \geq 0, \quad x(0) = \xi, \end{aligned} \tag{3}$$

where in the case of the inclusion, either B or C or both may be multi valued maps. Here the operator A is the infinitesimal generator of a C_0-semigroup in a Banach space E; and B and C are nonlinear operators and β is generally a nonnegative nondecreasing (except for A generating groups) scalar valued function of bounded variation on bounded intervals of $R_0 \equiv [0, \infty)$ and μ is a suitable vector measure on the sigma algebra of Borel subsets $\mathcal{B}_0$ of R_0. Here μ may be considered as the control like in [1, 3, 4]. Thus controls may also be impulsive.

Systems of the form (1) in finite dimensional spaces were considered by Pandit and Deo in [9] about a decade ago where existence and uniqueness of solutions were given. Impulsive systems have been widely considered both in finite [7] and infinite dimensional spaces [1-4, 8]. However, to the knowledge of the author, it appears from the literature that no such result is known for infinite dimensional systems of the form (1-3). Systems of the form (1) may arise in population biology (at the molecular or environmental level) where the population may consist of multiple species of biological agents which interact with each other and migrate from one region to another. Under normal conditions, the migration coefficient is constant which may undergo abrupt changes due to sudden intrusion of toxic chemicals or foreign biological agents in the medium creating a shock. For illustration we present some physical examples arising in optical and micro wave communication.

The paper is organized as follows. In section 2, basic notations are introduced. In section 3, we construct the basic evolution operator associated with the pair $(A, \beta(\cdot))$, study its properties and construct solution of the non homogeneous Cauchy problem

$$dx(t) = Ax(t)d\beta(t) + f(t), \quad t \geq 0, \quad x(0) = \xi. \tag{4}$$

In section 4-5, we consider the questions of existence, uniqueness, and regularity properties of solutions of the evolution equations and inclusions (2) and (3). In section 6, we consider some examples before concluding the paper.

2. Some notations and terminologies

For any metrizable topological space $\mathcal{Z}$, $2^{\mathcal{Z}} \setminus \emptyset$ will denote the class of all nonempty subsets of $\mathcal{Z}$, and $c(\mathcal{Z})(cb(\mathcal{Z})$, $cc(\mathcal{Z})$, $cbc(\mathcal{Z})$, $ck(\mathcal{Z}))$, denotes the class of nonempty closed (closed bounded, closed convex, closed bounded convex, compact convex) subsets of $\mathcal{Z}$.

Let $(\Omega, \mathcal{B})$ be an arbitrary measurable space and $\mathcal{Z}$ a Polish space. A multi function $G : \Omega \longrightarrow 2^{\mathcal{Z}} \setminus \emptyset$ is said to be measurable (weakly measurable) if for every closed (open) set $C \subset \mathcal{Z}$ the set

$$G^{-1} \equiv \{\omega \in \Omega : G(\omega) \cap C \neq \emptyset\} \in \mathcal{B}.$$

Let d be any metric induced by the topology of the Polish space $\mathcal{Z}$. It is known that measurability of the multifunction G is equivalent to the measurability of the function $\omega \to d(x, G(\omega))$ for every $x \in \mathcal{Z}$. Even more, it is also equivalent to the graph measurability of G in the sense that

$$\{(x, \omega) \in \mathcal{Z} \times \Omega : x \in G(\omega)\} \in \mathcal{B}(\mathcal{Z}) \times \mathcal{B}$$

where $\mathcal{B}(\mathcal{Z})$ denotes the sigma algebra of Borel sets of $\mathcal{Z}$. Let X, Y be any two topological spaces and $G : X \longrightarrow c(Y)$ be a multi function. G is said to be upper semi continuous (USC) if for each set $C \in c(Y)$

$$G^{-1}(C) \equiv \{x \in X : G(x) \cap C \neq \emptyset\} \in c(X).$$

If Y is a metric space with metric d, we can introduce a metric d_H on $cb(Y)$, called the Hausdorff metric, as follows:

$$d_H(K, L) \equiv \max\big\{\sup\{d(k, L), k \in K\},\ \sup\{d(K, \ell), \ell \in L\}\big\}$$

where $d(x, K) \equiv \inf\{d(x, y), y \in K\}$ is the distance of x from the set K. If Y is a complete metric space then $(c(Y), d_H)$ is also a complete metric space.

Let U be a Banach space and let $\mathcal{M}_c(J, U)$ denote the space of bounded countably additive vector measures on the sigma algebra $\mathcal{B}$ of subsets of the set $J \subset R_0 \equiv [0, \infty)$ with values in the Banach space U, furnished with the strong total variation norm. That is, for each $\mu \in \mathcal{M}_c(J, U)$, we write

$$|\mu|_v \equiv |\mu|(J) \equiv \sup_{\pi} \left\{ \sum_{\sigma \in \pi} \parallel \mu(\sigma) \parallel_U \right\}$$

where the supremum is taken over all partitions π of the interval J into a finite number of disjoint members of $\mathcal{B}$. With respect to this topology, $\mathcal{M}_c(J,U)$ is a Banach space. For any $\Gamma \in \mathcal{B}$ define the variation of μ on Γ by

$$V(\mu)(\Gamma) \equiv V(\mu,\Gamma) \equiv |\mu|(\Gamma).$$

Since μ is countably additive and bounded, this defines a countably additive bounded positive measure on $\mathcal{B}$. In case $U = R$, the real line, we have the space of real valued signed measures. We denote this by simply $\mathcal{M}_c(J)$ in place of $\mathcal{M}_c(J,R)$. Clearly for $\nu \in \mathcal{M}_c(J)$, $V(\nu)$ is also a countably additive bounded positive measure. For uniformity of notation we use λ to denote the Lebesgue measure. For any Banach space X, we let X^* denote the dual. Strong convergence of a sequence $\{\xi_n\} \in X$ to an element $\xi \in X$ is denoted by $\xi_n \xrightarrow{s} \xi$ and its weak convergence by $\xi_n \xrightarrow{w} \xi$. For any pair of Banach spaces X, Y, $\mathcal{L}(X,Y)$ will denote the space of bounded linear operators from X to Y. Let $B_\infty(J,X)$ denote the space of bounded strongly measurable functions on J with values in X. Furnished with the sup norm topology,

$$\| z \|_0 \equiv \sup\{\| z(t) \|_X, t \in J\},$$

this is a Banach space.

We use $PWC(J,X)$ to denote the class of all piece wise continuous functions with values in the Banach space X, and $PWC_r(J,X)$ $\big(PWC_\ell(J,X)\big)$ are those elements of $PWC(J,X)$ which are continuous from the right (left) having left hand (right hand) limits. The space $PWC(J,X)$, furnished with the sup norm topology, may not be a closed subspace of $B_\infty(J,X)$ and hence not a Banach space.

3. Basic evolution operator

We start with the Cauchy problem

$$dx(t) = Ax(t)d\beta(t), \quad t \geq 0, \quad x(0) = \xi. \tag{5}$$

Let D denote the collection of ordered sequence of discrete points from R_0 given by

$$D \equiv \{0 = t_0 < t_1 < t_2, \cdots t_n < t_{n+1}, \cdots n \in N_0\}$$

and let $\mathcal{S}$ denote the step function

$$\mathcal{S}(t) = \begin{cases} 1, & \text{if } \ t \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Without loss of generality we may assume that A is the infinitesimal generator of a C_0-semigroup of contractions $T(t), t \geq 0$, in a Banach space E and that the function β is given by

$$\beta(t) \equiv t + \sum_{k \geq 0} \alpha_k \mathcal{S}(t - t_k), \quad t \geq 0, \quad t_k \in D, \tag{6}$$

where generally $\alpha_k \in R \cup \{+/-)\infty\}$, with $\alpha_0 = 0$. Define the intervals $\sigma_k \equiv [t_k, t_{k+1}); k \in N_0$ and note that

$$R_0 = \bigcup_{k \geq 0} \sigma_k.$$

We demonstrate that under some reasonable assumptions the pair $(A, \beta(\cdot))$ generates an evolution operator. First we prove the following result.

Lemma 3.1. *Consider the system* (5) *and suppose A is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$, $t \geq 0$ in the Banach space E and the function β is given by the expression* (6) *where the coefficients $\{\alpha_k\}$ are nonnegative with $\alpha_0 = 0$. Then the system* (5) *has a unique mild solution $x(t)$, $t \in R_0$, which is right continuous having left hand limits.*

Proof. For any $t \in \sigma_0$, the system reduces to $dx = Axdt$ and so the solution is given by

$$x(t) = T(t)\xi, \quad t \in \sigma_0.$$

Since T is strongly continuous, the limit

$$s - \lim_{t \uparrow t_1} x(t) = x(t_1) = T(t_1)\xi$$

is well defined. At t_1, β makes a jump and equation (5) takes the form

$$x(t_1+) = x(t_1) + \alpha_1 A x(t_1).$$

Since A is an unbounded operator with $D(A) \subset E$, this expression is valid only if $x(t_1) \in D(A)$. But this is not guaranteed in general. Thus we must use the implicit difference scheme giving

$$x(t_1+) = x(t_1) + \alpha_1 A x(t_1+).$$

Letting I denote the identity operator in E, it follows from this that the value of x immediately after the jump at t_1 is given by

$$x(t_1+) = (I - \alpha_1 A)^{-1} x(t_1)$$

provided the inverse exists. Let $\rho(A)$ denote the resolvent set of A. Since A is the generator of a C_0-semigroup of contractions, $\rho(A) \supset (0, \infty)$ and by Hille-Yosida theorem, we have

$$\| R(\lambda, A) \| \equiv \| (\lambda I - A)^{-1} \| \leq \frac{1}{\lambda}, \quad \lambda \in (0, \infty).$$

By our assumption $\{\alpha_k\}$ are nonnegative. If $\alpha_1 = 0$ there is no jump at t_1 and it follows from the above expression that $x(t_1+) = x(t_1)$. For $\alpha_1 > 0$, we have

$$(I - \alpha_1 A)^{-1} = (1/\alpha_1) R(1/\alpha_1, A)$$

and it follows that $(I - \alpha_1 A)^{-1} \in \mathcal{L}(E)$ verifying the existence of the inverse. Thus $x(t_1+)$ is a well defined element of E. Starting with initial state $x(t_1+)$, the solution of equation (5) can be constructed for the next interval σ_1 giving

$$x(t) = T(t - t_1) x(t_1+), \quad t \in \sigma_1.$$

By strong continuity of the semigroup T, again we can extend the solution uniquely to the right hand boundary of the interval σ_1 giving

$$x(t_2) = T(t_2 - t_1) x(t_1+) = T(t_2 - t_1)(I - \alpha_1 A)^{-1} T(t_1) \xi.$$

It is clear from the Hille-Yosida inequality that, in general

$$\| (I - \alpha A)^{-1} \| \leq 1 \quad \forall \alpha \geq 0.$$

Taking into consideration the jump at t_2 we have

$$x(t_2+) = (I - \alpha_2 A)^{-1} T(t_2 - t_1)(I - \alpha_1 A)^{-1} T(t_1) \xi$$

and for $t \in \sigma_2$ the value of x is given by

$$x(t) = T(t - t_2)(I - \alpha_2 A)^{-1} T(t_2 - t_1)(I - \alpha_1 A)^{-1} T(t_1) \xi.$$

Thus by induction, for any $t \in \sigma_n$ the value of x is given by

$$\begin{aligned} x(t) &= T(t - t_n)(I - \alpha_n A)^{-1} T(t_n - t_{n-1})(I - \alpha_{n-1} A)^{-1} \cdots (I - \alpha_1 A)^{-1} T(t_1) \xi \\ &= T(t - t_n) \left(\prod_{k=1}^{n} (I - \alpha_k A)^{-1} T(t_k - t_{k-1}) \right) \xi, \end{aligned}$$

where, in general, the product is ordered as in the first line. Since $R(\lambda, A)$ commutes with $R(\gamma, A)$ for all $\lambda, \gamma \in \rho(A)$ and they also commute with the semigroup $T(t)$ for any $t \in R_0$, the order in the product is immaterial and the solution can be written as

$$x(t) = \left(\prod_{k=0}^{n}(I - \alpha_k A)^{-1}\right)T(t)\xi, \quad \text{for} \quad t \in \sigma_n,\ n \in N_0, \tag{7}$$

where we have used the fact that $\alpha_0 = 0$. Thus a unique mild solution can be constructed piece by piece for all $t \geq 0$. From the construction it is evident that $t \to x(t)$ is continuous from the right and that it has limit from the left and hence $x \in PWC_r(R_0, E)$. This completes the proof. □

Remark. *Note that for any $\xi \in E$ the solution on the first interval σ_0 belongs to E while for all $t > t_1$ the solution $x(t) \in D(A)$ even though it is discontinuous in $t \geq t_1$.*

Now we can construct the evolution operator corresponding to the pair (A, β). While doing this we can also prove the validity of Dhumels formula. Towards this goal, we consider the nonhomogeneous Cauchy problem

$$dx = Axd\beta + f, \quad t \geq 0, \quad x(0) = \xi. \tag{8}$$

We prove the following result.

Theorem 3.2. *Let the pair (A, β) satisfy the assumptions of lemma 3.1. Then for every $\xi \in E$ and $f \in L_1^{\ell oc}(R_0, E)$, the non homogeneous Cauchy problem (8) has a unique mild solution $x \in PWC_r(R_0, E)$ and this solution has the expression*

$$x(t) = U_\beta(t, 0)\xi + \int_0^t U_\beta(t, s)f(s)ds, \quad t \in R_0. \tag{9}$$

where $U_\beta(t, s), 0 \leq s < t < \infty$ is a family of transition or evolution operators in E which, for each fixed $s \in R_0$, is right continuous in t on (s, ∞) in the strong operator topology and have left limits.

Proof. We follow the procedure used for Lemma 3.1. Consider the first interval $\sigma_0 \equiv [0, t_1)$. On this interval the system reduces to the standard differential equation and hence the solution is given by

$$x(t) = T(t)\xi + \int_0^t T(t - r)f(r)dr, \quad t \in \sigma_0. \tag{10}$$

To reduce notational burden we shall use f and its various restrictions on $\{\sigma_k\}$ by f itself. For example, for $f|_{\sigma_k} \in L_1(\sigma_k, E)$ we simply write $f \in L_1(\sigma_k, E)$. Since $f \in L_1(\sigma_0, E)$, it follows from the above expression that $x \in C(\sigma_0, E)$. By strong continuity of the semigroup T, $x(t_1)$ is well defined and it belongs to E. Since f is Lebesgue-Bochner integrable and T is strongly continuous, the discontinuity of the state process x is a contribution of the measure β alone. Hence we have

$$x(t_1+) = x(t_1) + \alpha_1 A x(t_1+)$$

and consequently

$$x(t_1+) = (I - \alpha_1 A)^{-1} x(t_1).$$

Substituting the expression for $x(t_1)$ into the above equation and recalling that the resolvent commutes with the semigroup, it is easy to see that

$$\begin{aligned} x(t) &= T(t-t_1)x(t_1+) + \int_{t_1}^{t} T(t-\theta) f(\theta) d\theta, \\ &= (I-\alpha_1 A)^{-1} T(t)\xi + (I-\alpha_1 A)^{-1} \int_0^{t_1} T(t-r) f(r) dr + \int_{t_1}^{t} T(t-r) f(r) dr. \end{aligned}$$

for $t \in \sigma_1 \equiv [t_1, t_2)$. Similarly for $t \in \sigma_2$ we have

$$\begin{aligned} x(t) = {} & (I-\alpha_2 A)^{-1}(I-\alpha_1 A)^{-1} T(t)\xi + (I-\alpha_2 A)^{-1}(I-\alpha_1 A)^{-1} \int_0^{t_1} T(t-r) f(r) dr \\ & + (I-\alpha_2 A)^{-1} \int_{t_1}^{t_2} T(t-r) f(r) dr + \int_{t_2}^{t} T(t-r) f(r) dr. \end{aligned}$$

In general, by induction, one can easily verify that for any positive integer $n \in N_0$ and any $t \in \sigma_n \equiv [t_n, t_{n+1})$, $x(t)$ is given by

$$\begin{aligned} x(t) = {} & \prod_{k=1}^{n} (I-\alpha_k A)^{-1} T(t)\xi + \int_0^{t_1} \prod_{k=1}^{n} (I-\alpha_k A)^{-1} T(t-r) f(r) dr \\ & + \int_{t_1}^{t_2} \prod_{k=2}^{n} (I-\alpha_k A)^{-1} T(t-r) f(r) dr \\ & + \cdots + \int_{t_{n-1}}^{t_n} (I-\alpha_n A)^{-1} T(t-r) f(r) dr + \int_{t_n}^{t} T(t-r) f(r) dr. \end{aligned} \tag{11}$$

Since $n \in N_0$ is arbitrary, this shows that, corresponding to any initial data $\xi \in E$ and $f \in L_1^{\ell oc}(R_0, E)$, we can construct a unique solution x for the evolution equation (8) on any bounded subinterval of the set R_0 and that such a solution is right continuous having left limits. In other words, we have $x \in PWC_r(R_0, E)$. The expression (11) can be written compactly as follows

$$x(t) \tag{12}$$
$$= \prod_{k=1}^{n}(I-\alpha_k A)^{-1}T(t)\xi + \int_0^t \sum_{\ell=0}^{n}\biggl(\prod_{k=\ell+1}^{n}(I-\alpha_k A)^{-1}\biggr)\chi_{\sigma_\ell}(r)T(t-r)f(r)dr.$$

Clearly it follows from this expression that we can define a transition operator U_β as follows:

$$U_\beta(t,r) \equiv \sum_{\ell=0}^{n}\biggl(\prod_{k=\ell+1}^{n}(I-\alpha_k A)^{-1}\biggr)chi_{\sigma_\ell}(r)T(t-r), \tag{13}$$

for $t \in \sigma_n$ and $0 \leq r < t$.
For arbitrary $t \in R_0$, we define the following integer valued function

$$i(t) \equiv k, \quad \text{for} \ \ t \in \sigma_k, \ k \in N_0.$$

Using this we rewrite (13) in the most general form

$$U_\beta(t,r) \equiv \sum_{\ell=0}^{i(t)}\biggl(\prod_{k=\ell+1}^{i(t)}(I-\alpha_k A)^{-1}\biggr)\chi_{\sigma_\ell}(r)T(t-r), \tag{14}$$

for any $t \in R_0$ and $0 \leq r < t$.
From this expression it is clear that for $r = 0$ all the terms except the one with $\ell = 0$ vanish and hence

$$U_\beta(t,0) = \biggl(\prod_{k=1}^{i(t)}(I-\alpha_k A)^{-1}\biggr)T(t) \tag{15}$$

which coincides with the first term of equation (12). Hence the (mild) solution of equation (8) is given by

$$x(t) = U_\beta(t,0)\xi + \int_0^t U_\beta(t,r)f(r)dr, \quad \text{for } t \in R_0. \tag{16}$$

Since the semigroup T is strongly continuous in t, again it is clear from the construction that the solution x is right continuous with left hand limits. This implies that for any fixed $r \in R_0$, the evolution operator $U_\beta(t,r)$ is strongly right continuous for $t \geq r$ having left hand limits. This completes the proof. $\square$

Remark. *Note that for any $s \in R_0$, the solution of the problem* (8) *with f locally Lebesgue-Bochner integrable satisfying $f(r) = 0$ for almost all $r \geq s$ is given by*

$$x(t) = U_\beta(t,s)x(s+), \quad t \geq s.$$

We summarize the properties of the evolution operator $U_\beta(t,s)$ in the following corollary.

Corollary 3.3. *Under the assumptions of Lemma* 3.1, *the evolution operator U_β satisfies the following properties*:

- (P1): *$t \to U_\beta(t,r)$, $t > r$ is continuous from the right in the strong operator topology in E; that is, $s - \lim_{t \downarrow r} U_\beta(t,r)\xi = \xi$, $\xi \in E$.*
- (P2): *$s - \lim_{t \uparrow \tau > r} U_\beta(t,r)\xi$ exists $\forall \xi \in E$ and $\tau > r$.*
- (P3): *$\| U_\beta(t,s)\xi \| \leq \| \xi \|$, $\forall \xi \in E$.*
- (P4): *$r \to g_t(r) \equiv U_\beta(t,r)\xi, 0 \leq r \leq t$, is piecewise continuous having simple discontinuities at $r \in \{t_k, t_k < t, k \geq 1\}$.*
- (P5): *$U_\beta(t,s)U_\beta(s,r) = U_\beta(t,r)$ $\forall 0 \leq r < s < t < \infty$.*

Proof. Properties (P1)-(P2) follow from the analysis given in the proof of the previous Lemmas. For (P3) note that

$$\| (I - \alpha A)^{-1} \|_{\mathcal{L}(E)} \leq 1, \quad \forall \alpha > 0.$$

This combined with the assumption that T is a contraction semigroup in E, the assertion follows from the expression (14). For the property (P4), we note that

$$g_t(r) = \sum_{m=0}^{i(t)} \left(\prod_{k=m+1}^{i(t)} (I - \alpha_k A)^{-1} \right) \chi_{\sigma_m}(r) T(t-r)\xi.$$

Clearly this is a measurable function of $r \in [0,t]$ and piecewise continuous with simple discontinuities at $\{t_k, k \geq 1\}$. Indeed, for $r \in \sigma_\ell$, and $t \in \cup_{k \geq \ell} \sigma_k$, one can easily verify that

$$g_t(t_\ell -) - g_t(t_\ell +) = \alpha_\ell A \prod_{k=\ell}^{i(t)} (I - \alpha_k A)^{-1} T(t - t_\ell)\xi, \quad \ell = 1, 2 \cdots i(t).$$

This gives the size of the jumps at the points indicated. The last property (evolution or semigroup property) follows from uniqueness of solution in the sense of Lemma 3.1. □

Similar results can be proved for C_0-groups.

Theorem 3.4. *Consider the system* (5) *and suppose A is the infinitesimal generator of a C_0-group of contractions $T(t)$, $t \in R$, in the Banach space E and the function β is given by the expression* (6) *where the coefficients $\{\alpha_k\} \in R$ with $\alpha_0 = 0$. Then for each $\xi \in E$, the system* (5) *has a unique mild solution $x(t)$, $t \in R$, which is right continuous having left hand limits. The pair (A, β) generates a non expansive evolution operator $\{V_\beta(t,r), r, t \in R\}$ on E. Further, if $f \in L_1^{\ell oc}(R, E)$, then the Cauchy problem* (8) *has a unique mild solution $x \in PWC(R, E)$ given by*

$$x(t) = V_\beta(t,s)\xi + \int_s^t V_\beta(t,r)f(r)dr, \quad s \le t, \quad s,t \in R.$$

Proof. Since under the present assumptions, A generates a C_0-group of contractions, it follows from Hille-Yosida theory that $(-\infty, 0) \cup (0, +\infty) \subset \rho(A)$ and that

$$\| (\lambda I - A)^{-1} \| \le (1/|\lambda|) \quad \forall \lambda \in R \setminus \{0\}.$$

In other words if A is the generator of a group, the steps given in the proof of Lemma 3.1 hold for all $\{\alpha_k \in R\}$. Thus the proof is identical to those of Lemma 3.1 and Theorem 3.2. □

Remark. *Clearly if E is a Hilbert space and A is the infinitesimal generator of a unitary group, the conclusions of Theorem* 3.4 *hold.*

4. Semilinear evolution equations

Now we consider the semilinear problem

$$dx = Axd\beta + B(t,x)dt, \quad t \in J_a \equiv [0,a], \quad a < \infty, \quad x(0) = \xi. \tag{17}$$

We can prove the following result.

Theorem 4.1. *Suppose the pair (A, β) satisfy the assumptions of Lemma* 3.1. *Let B be measurable in t on J_a and locally Lipschitz in x on E and that it satisfies the following growth condition*

$$\| B(t,x) \|_E \le K(t)(1 + \| x \|)$$

for some $K \in L_1^+(J_a)$. Then, for every $\xi \in E$, equation (17) *has a unique mild solution $x \in PWC_r(J_a, E)$ having left hand limits.*

Proof. Since we are going to give a detailed proof for the measure driven system (3) which includes this system as special case we present only a brief outline. However the proof given here is direct. For this, we first establish an a priori bound which then guarantees the existence of a ball $B_\gamma \subset E$ of finite radius $\gamma > 0$ such that any solution, if one exists, lies in B_γ for all $t \in J_a$. Then we use Banach fixed point theorem to show that on each of the subintervals $\sigma_k, k \in N_0$, starting with the state $x(t_k+)$, the integral equation

$$x(t) = U_\beta(t, t_k)x(t_k+) + \int_{t_k}^{t} U_\beta(t, s)B(s, x(s))ds, \quad t \in \sigma_k \tag{18}$$

has a unique solution in $PWC_r(\sigma_k, E)$. This is continued till the interval J_a is covered. This completes our brief outline. $\square$

Next we consider the system (3) given by

$$dx = Axd\beta + B(t, x)dt + C(t, x)d\mu(t), \ \ t \in J_a \equiv [0, a], \ a < \infty, \ x(0) = \xi. \tag{19}$$

We prove the following existence result.

Theorem 4.2. *Suppose the pair (A, β) and the operator B satisfy the assumptions of Theorem 4.1, and $\mu \in \mathcal{M}_c(J_a, U)$ with 0 not an atom of μ. Let $C : J_a \times E \longrightarrow \mathcal{L}(U, E)$ be uniformly measurable in t on J_a, locally Lipschitz in x on E, and satisfies the following growth condition*

$$\| C(t, x) \|_{\mathcal{L}(U,E)} \leq L(t)(1+ \| x \|)$$

for some $L \in L_1^+(J_a, |\mu|)$. Then, for every $\xi \in E$, equation (19) has a unique mild solution $x \in B_\infty(J_a, E)$.

Proof. First we prove an a priori bound. Define the operator G on $B_\infty(J_a, E)$ as follows:

$$(Gy)(t) \tag{20}$$
$$\equiv U_\beta(t, 0)\xi + \int_0^t U_\beta(t, s)B(s, y(s))ds + \int_0^t U_\beta(t, s)C(s, y(s))d\mu(s), \quad t \in J_a,$$

for $y \in B_\infty(J_a, E)$. Let $x \in B_\infty(J_a, E)$ be a (mild) solution of the evolution equation (19). Then, clearly

$$x(t) = (Gx)(t), \quad \forall t \in J_a.$$

Then using the growth properties of B and C, and the non expansive property of the evolution operator U_β, it is easy to verify that

$$\begin{aligned} &\| x(t) \| \\ &\leq \| \xi \| + \int_0^t K(s)\{1+ \| x(s) \|\}ds + \int_0^t L(s)\{1+ \| x(s) \|\}|\mu|(ds), \quad t \in J_a. \end{aligned} \tag{21}$$

Define the measure

$$\nu(\sigma) \equiv \int_\sigma K(s)ds + \int_\sigma L(s)|\mu|(ds), \quad \forall \sigma \in \mathcal{B}_a, \tag{22}$$

where $\mathcal{B}_a$ denotes the sigma algebra of Borel subsets of J_a. Using this measure, (21) can be written compactly as

$$\| x(t) \| \leq \| \xi \| + \int_0^t \{1+ \| x(s) \|\}\nu(ds), \quad t \in J_a. \tag{23}$$

Since μ is a countably additive vector measure of bounded total variation, the measure $|\mu|$ is also a countably additive finite positive measure on $\mathcal{B}_a$. By our hypothesis, $K \in L_1^+(J_a)$, $L \in L_1^+(J_a, |\mu|)$ and thus the measure ν is also a countably additive bounded positive measure. Thus, due to the fact that $x \in B_\infty(J_a, E)$ and so clearly measurable, the integrals in (21) are well defined. By virtue of a generalized Gronwall inequality [2], it follows from this that

$$\sup\{\| x(t) \|, t \in J_a\} \leq \Big(\| \xi \| + \nu(J_a)\Big) \exp \nu(J_a) \equiv b < \infty. \tag{24}$$

Thus if x is any solution of (19), it must be bounded in norm by b, that is, $x(t) \in B_b$, for all $t \in J_a$. This, combined with the fact that U_β is piecewise continuous and that μ is a vector measure of bounded total variation on J_a, implies that G maps bounded subsets of $B_\infty(J_a, E)$ into bounded subsets of $B_\infty(J_a, E)$. Define

$$\Sigma \equiv \{x \in B_\infty(J_a, E) : x(0) = \xi, x(t) \in B_b, \quad \forall t \in J_a\} \tag{25}$$

where B_b is the closed ball in E of radius b as given by (24). For $x, y \in \Sigma$, define

$$d_t(x, y) \equiv \sup_{0 \leq \theta \leq t} \| x(\theta) - y(\theta) \|_E \quad \text{and} \quad d(x, y) = d_a(x, y).$$

Furnished with the metric d_a, Σ is a complete metric space which we denote by Σ_a. Since both B and C are locally Lipschitz on E, there exist $K_b \in L_1^+(J_a)$ and $L_b \in L_1^+(J_a, |\mu|)$ so that

$$\begin{aligned} \| B(t,\eta) - B(t,\zeta) \|_E &\leq K_b(t) \| \eta - \zeta \|_E, \quad \forall \eta,\ \zeta \in B_b \\ \| C(t,\eta) - C(t,\zeta) \|_E &\leq L_b(t) \| \eta - \zeta \|_E, \quad \forall \eta,\ \zeta \in B_b. \end{aligned} \tag{26}$$

As in (22), define the measure ν_b as

$$\nu_b(\sigma) \equiv \int_\sigma K_b(t)dt + \int_\sigma L_b(t)|\mu|(dt).$$

This is also a countably additive bounded positive measure. Define $v_b(t) \equiv \nu_b([0,t]), t \in J_a$. Clearly, this is a nondecreasing function of bounded variation on J_a with $v_b(a) < \infty$, and since 0 is not an atom of μ we have $v_b(0) = 0$. For $x, y \in \Sigma_a$, using the expression for G given by (20), one can verify that

$$d_t(Gx, Gy) \leq \int_0^t d_s(x,y) dv_b(s). \tag{27}$$

At the second iteration we have,

$$d_t(G^2x, G^2y) \leq v_b(t) \int_0^t d_s(x,y) dv_b(s).$$

Continuing with this iteration we find that

$$d_t(G^n x, G^n y) \leq \big(v_b^n(t)/\Gamma(n)\big) d_t(x,y), \quad \forall t \in J_a,$$

where Γ is the standard gamma function. From this it is clear that

$$d_a(G^n x, G^n y) \leq \big(v_b^n(a)/\Gamma(n)\big) d_a(x,y). \tag{28}$$

This shows that for n sufficiently large, the n-th iterate of G given by G^n is a contraction in the metric space Σ_a. Thus by Banach fixed point theorem both G^n and G have a unique fixed point in Σ_a which is the unique solution of the evolution equation (19). This completes the proof. $\square$

Remark. *Note that the assumption that* 0 *is not an atom of the measure* μ *does not restrict the generality of the result since this can be passed on to the initial state of the system* (19).

Remark. *The solutions of the evolution equation* (19) *generally belongs to* $B_\infty(J_a, E)$. *However, if the vector measure* μ *is purely atomic or has Radnon-Nikodym derivative with respect to Lebesgue measure then the solutions may posses* PWC *regularity.*

5. Differential inclusions

Often, systems governed by parabolic (and hyperbolic) variational inequalities, uncertain parameters, systems with discontinuous vector fields, and control systems can be modeled as differential inclusions. The results given in the previous section can be extended to systems governed by differential inclusions. Let $F : J_a \times E \longrightarrow 2^E \setminus \emptyset$ be a multivalued map and consider the evolution inclusion

$$dx - Axd\beta \in F(t, x(t))dt, \quad t \in J_a;\ x(0) = \xi. \tag{29}$$

We wish to prove existence of solutions of the Cauchy problem (29). By a solution of the problem (29), it is understood that there exists a function $x \in B_\infty(J_a, E)$ and an $f \in L_1(J_a, E)$ such that x is the mild solution of equation

$$dx = Axd\beta + fdt, \quad t \in I;\ x(0) = \xi; \tag{30}$$

and that

$$f(t) \in F(t, x(t))\ a.e\ t \in J_a. \tag{31}$$

We know from Theorem 3.2 that for every $f \in L_1(J_a, E)$ and $\xi \in E$ equation (30) has a unique mild solution $x \in B_\infty(J_a, E)$ given by

$$x(t) = U_\beta(t.0)\xi + \int_0^t U_\beta(t, r)f(r)dr, \quad t \in J_a.$$

Define the affine linear map L_β from $L_1(J_a, E)$ to $B_\infty(J_a, E)$ by

$$(L_\beta f)(t) = U_\beta(t, 0)\xi + \int_0^t U_\beta(t, r)f(r)dr, \quad t \in J_a.$$

Here we need some more notation. Let $(\Omega, \mathcal{B}, \nu)$ be a complete measure space and E a Banach space and $L_1(\Omega, E)$ the space of measurable functions which are Bochner integrable with respect to the measure ν. For a measurable multi function $G : \Omega \longrightarrow 2^E \setminus \emptyset$, we let S_G^1 denote the collection of Bochner integrable selections of G, that is,

$$S_G^1 \equiv \{f \in L_1(\Omega, E) : f(\omega) \in G(\omega) - \nu\ a.e\}.$$

Now we introduce the multifunction

$$\begin{aligned} \hat{F}(f) &\equiv \{g \in L_1(J_a, E) : g(t) \in F(t, (L_\beta f)(t))\ a.e\ t \in J_a\} \\ &\equiv S^1_{F(\cdot, (L_\beta f)(\cdot))}, \end{aligned} \tag{32}$$

defined on the function space $L_1(J_a, E)$. This could very well be an empty set. So in the process of existence study we must prove nonemptyness of this set. Given that it is nonempty, it is clear that the multi function

$$\hat{F} : L_1(J_a, E) \longrightarrow 2^{L_1(J_a,E)} \setminus \emptyset.$$

Now, we note that the primary question of existence of a solution of the evolution inclusion (29) is equivalent to the question of existence of a fixed point of the multi function $\hat{F}$. Indeed, suppose $h \in L_1(J_a, E)$ is a fixed point of $\hat{F}$; then by definition

$$h(t) \in F(t, (L_\beta h)(t)) \ a.e \ t \in J_a.$$

Now define the function y as

$$y(t) \equiv (L_\beta h)(t) \quad \forall t \in J_a.$$

Clearly the pair $\{y, h\}$ satisfies the equation (30) and the inclusion (31) and hence y is a solution of the evolution inclusion (29). The reverse implication (other side of the equivalence) follows trivially from the definition of solutions. We prove the following result.

Theorem 5.1. *Suppose the pair (A, β) satisfy the assumptions of Lemma* 3.1 *and the multi function F satisfy the following assumptions*:

(F1): $F : J_a \times E \longrightarrow c(E)$, *measurable in t on J_a for each fixed $x \in E$, and, for almost all $t \in J_a$, it is (USC) upper semi continuous on E,*

(F2): *for every $r > 0$, finite, there exists an $\ell_r \in L_1^+(J_a)$ such that*

$$\inf\{\| v \|, v \in F(t, e)\} \leq \ell_r(t), \quad \forall e \in B_r, \quad t \in J_a,$$

(F3): *there exists an $\ell \in L_1^+(J_a)$ such that*

$$d_H(F(t, x), F(t, y)) \leq \ell(t) \| x - y \|, \quad \forall x, \ y \in E, \quad t \in J_a.$$

Then for each $\xi \in E$, the Evolution Inclusion (29) *has at least one solution $x \in B_\infty(J_a, E)$.*

Proof. As discussed above, it suffices to prove that the multi function $\hat{F}$, has fixed points in $L_1(J_a, E)$. For this we must show that $\hat{F}$, as given by (32), is nonempty with closed values in $L_1(J_a, E)$ and that it is also Lipschitz with respect to a suitable Hausdorff metric on $L_1(J_a, E)$ with Lipschitz constant less than 1. Then we use the generalized Banach fixed point theorem for multi valued maps. Let $g \in L_1(J_a, E)$ and $x = L_\beta g$. Since $x \in B_\infty(J_a, E)$, it is clearly measurable; and since F is USC in $e \in E$, and measurable in $t \in J_a$, by (F1), the multi function

$$t \longrightarrow F(t, x(t)) = F(t, (L_\beta g)(t))$$

is nonempty, measurable, and has closed values. Thus by the well known Yankov-Von Neumann-Auman selection theorem, see [6, Theorem 2.14, p154], it has measurable selections. We must show that it has $L_1(J_a, E)$ selections. Since $\sup\{\| x(t) \|, t \in J_a\}$ is finite, there exists an $r > 0$ finite, such that $x(t) \in B_r, \forall t \in J_a$ and hence, by virtue of assumption (F2), it follows from a well known result [6, Lemma 3.2, p175] that it has L_1 selections, that is $\hat{F}(g) \equiv S^1_{F(\cdot, x(\cdot))} \neq \emptyset$. Further, since F is closed valued, $\hat{F}$ is also closed valued. That is, for each $g \in L_1(J_a, E)$, $\hat{F}(g) \in c(L_1(J_a, E))$. Now we must show that, with respect to a suitable Hausdorff metric, it is Lipschitz on $L_1(J_a, E)$ with Lipschitz constant less than 1. Define the function γ

$$\gamma(t) \equiv \int_0^t \ell(r) dr, \quad t \in J_a.$$

At this point we can follow a similar procedure as given in Bian [5, Theorem 3.3]. Let X_0 denote the Lebesgue-Bochner space $L_1(J_a, E)$ with the standard norm, and, for $\delta > 0$, X_δ the vector space,

$$X_\delta \equiv \{z \in L_0(J_a, E) : \int_{J_a} \| z(t) \| e^{-2\delta\gamma(t)} dt < \infty\},$$

where $L_0(J_a, E)$ denotes the linear space of strongly measurable functions with values in E. Furnished with the norm topology,

$$\| z \|_\delta \equiv \int_{J_a} \| z(t) \|_E \, e^{-2\delta\gamma(t)} dt,$$

X_δ is a Banach space. Let $g_1, g_2 \in L_1(J_a, E)$ and let $x_1, x_2 \in B_\infty(J_a, E)$ denote, respectively, the corresponding solutions of (30). Then it follows from the non expansive property (P3) of Corollary 3.3 that

$$\| x_1(t) - x_2(t) \| \leq \int_0^t \| g_1(s) - g_2(s) \| ds, \quad t \in J_a.$$

Further, it follows from the proceeding arguments that

$$\hat{F}(g_i) \equiv S^1_{F(\cdot,L_\beta g_i)} = S^1_{F(\cdot,x_i(\cdot))} \neq \emptyset, \quad i = 1,2.$$

Let $\varepsilon > 0$, and $z_1 \in \hat{F}(g_1)$, that is $z_1(t) \in F(t, x_1(t)),\ a.e$, be given. Since F is Lipschitz in the second variable with respect to the Hausdorff metric d_H, there exists $z_2 \in \hat{F}(g_2)$, that is, $z_2(t) \in F(t, x_2(t))\ a.e$, such that

$$\| z_1(t) - z_2(t) \| \leq \ell(t) \| x_1(t) - x_2(t) \| + \varepsilon, \quad t \in J_a. \tag{33}$$

Since $\| U_\beta(t,r) \|_{\mathcal{L}(E)} \leq 1, 0 \leq r \leq t$, we have

$$\| x_1(t) - x_2(t) \|_E \leq \int_0^t \| g_1(r) - g_2(r) \|_E \, dr, \quad t \in J_a.$$

For convenience, let us denote by h the integral

$$h(t) \equiv \int_0^t \| g_1(r) - g_2(r) \|_E \, dr, \quad t \in J_a,$$

and write (33) as

$$\| z_1(t) - z_2(t) \| \leq \ell(t)h(t) + \varepsilon, \quad t \in J_a. \tag{34}$$

Multiplying (34) by $e^{-2\delta\gamma(t)}$ and integrating over the interval J_a we obtain

$$\int_{J_a} e^{-2\delta\gamma(t)} \| z_1(t) - z_2(t) \|_E \, dt \leq \int_{J_a} e^{-2\delta\gamma(t)} h(t) d\gamma(t) + \varepsilon a. \tag{35}$$

Integrating by parts, one can readily verify that

$$\int_{J_a} e^{-2\delta\gamma(t)} h(t) d\gamma(t) \leq (1/2\delta) \int_{J_a} e^{-2\delta\gamma(t)} \| g_1(t) - g_2(t) \|_E \, dt. \tag{36}$$

Thus it follows from (35) and (36) that

$$\| z_1 - z_2 \|_\delta \leq (1/2\delta) \| g_1 - g_2 \|_\delta + \varepsilon a. \tag{37}$$

Letting $D_{H,\delta}$ denote the Hausdorff distance on $c(X_\delta)$, it follows from (37) that

$$D_{H,\delta}(\hat{F}(g_1), \hat{F}(g_2)) \leq (1/2\delta) \| g_1 - g_2 \|_\delta + \varepsilon a. \tag{38}$$

Since $\varepsilon > 0$ is arbitrary and $a \in (0, \infty)$, it follows from this inequality that $\hat{F}$ is Lipschitz with respect to the Hausdorff metric $D_{H,\delta}$ on $c(X_\delta)$. Hence for any $\delta > (1/2)$, $\hat{F}$ is a multivalued contraction map from X_δ to the metric space $c(X_\delta)$. Thus by the generalized Banach fixed point theorem for multivalued maps [10, Theorem 9A, p449], $\hat{F}$ has a fixed point $g^* \in X_\delta$. Letting $\| \cdot \|_0$ denote the usual norm topology of $X_0 \equiv L_1(J_a, E)$, one can easily verify that

$$\| z \|_\delta \leq \| z \|_0 \leq c \| z \|_\delta, \quad \text{with } c = e^{2\delta\gamma(a)}.$$

Thus the two norms are equivalent, that is $X_0 \cong X_\delta$, and consequently g^* is also a fixed point of $\hat{F}$ in the original space X_0. Hence it follows from our previous discussion, that x^*, given by $x^* \equiv L_\beta g^* \in B_\infty(J_a, E)$, is a solution of the evolution inclusion. This completes the proof. □

The result of theorem 5.1 can be easily extended to the evolution inclusion

$$dx - Ax d\beta - C(t, x) d\mu \in F(t, x) dt, \quad t \geq 0, \quad x(0) = \xi, \tag{39}$$

where F is a multivalued map. This is an extension of the model (3) given at the introduction.

In the following two theorems we assume, without loss of generality, that 0 is not an atom of the measure μ.

Theorem 5.2. *Suppose* 0 *is not an atom of* μ *and the pair* (A, β) *and* C *satisfy the assumptions of Theorem* 4.2 *and the multi function* F *satisfy the assumptions of Theorem* 5.1. *Then for each* $\xi \in E$, *the evolution inclusion* (39) *has at least one solution* $x \in B_\infty(J_a, E)$.

Proof. The proof is based on a combination of arguments similar to those of Theorem 4.2 and Theorem 5.1. □

Remark. *This result can be further extended to the system governed by the following evolution inclusion*

$$dx - Ax d\beta - B(t, x) dt \in C(t, x) d\mu, \quad t \geq 0, \quad x(0) = \xi, \tag{40}$$

where B *is single valued as in equation* (19) *and* C *is a multivalued map from* $J_a \times E$ *to* $2^{\mathcal{L}(U,E)} \setminus \emptyset$.

Let $\mathcal{L}(U, E)$ denote the space of bounded linear operators from the Banach space U to the Banach space E furnished with the uniform operator topology. Clearly with respect to the uniform operator topology this is a Banach space.

Let Y be a closed, linear, separable subspace of $\mathcal{L}(U,E)$ with the relative norm topology denoted by $\| \cdot \|_Y$. Since a closed subspace of a Banach space is also a Banach space, Y is a separable Banach space. Let μ be a fixed countably additive bounded vector measure on the sigma algebra $\mathcal{B}_a$ of subsets of the set J_a taking values in U and let $|\mu|$ denote the countably additive positive measure induced by the variation of μ. Let $(J_a, \mathcal{B}_a, |\mu|)$ denote a complete measure space completed by including all $|\mu|$ null sets. We introduce the vector space $L_1(J_a, |\mu|; Y)$ which consists of all uniformly $|\mu|$ measurable functions defined on J_a and taking values from Y having finite Lebesgue integrable norms (with respect to the positive measure $|\mu|$), that is for each $L \in L_1(J_a, |\mu|; Y)$ we have,

$$\int_{J_a} \| L(t) \|_Y \, d|\mu| < \infty.$$

Let $cb(Y)$ denote the class of nonempty closed bounded subsets of Y and d_H denote the Hausdorff metric on $cb(Y)$. It is easy to verify that $cb(Y)$, furnished with this metric, is a complete separable metric space and hence a Polish space.

Theorem 5.3. *Suppose* 0 *is not atom of* μ *and the pair* (A, β) *satisfy the assumptions of Lemma* 3.1 *and* B *satisfy the assumption,*

(B): *there exists* $K \in L_1^+(J_a)$ *such that*

$$\| B(t,\zeta) \| \leq K(t)(1 + \| \zeta \|), \quad \| B(t,\zeta) - B(t,\xi) \| \leq K(t)(\| \zeta - \xi \|), \ \xi, \zeta \in E,$$

and the multi function C *satisfy the following assumptions*:

(C1): $C : J_a \times E \longrightarrow cb(Y)$, *measurable in* t *on* J_a *for each fixed* $x \in E$, *and, for almost all* $t \in J_a$, *it is* (*USC*) *upper semi continuous on* E,

(C2): *for every* $r > 0$, *finite, there exists an* $\ell_r \in L_1^+(J_a, |\mu|)$ *such that*

$$\inf\{\| L \|, \quad L \in C(t,e)\} \leq \ell_r(t), \quad \forall e \in B_r, \quad t \in J_a,$$

(C3): *there exists an* $\ell \in L_1^+(J_a, |\mu|)$ *such that*

$$d_H(C(t,x), C(t,y)) \leq \ell(t) \| x - y \|, \quad \forall x, \ y \in E, \quad t \in J_a.$$

Then for each $\xi \in E$, *the evolution inclusion* (40) *has at least one solution* $x \in B_\infty(J_a, E)$.

Proof. In essence the proof is similar to that of Theorem 5.1 and so we give a brief outline. Since assumption (B) implies those of Theorem 4.1, it follows from theorem 4.2 that, for each $L \in L_1(J_a, |\mu|; Y)$, the evolution equation,

$$dx = Axd\beta + B(t,x)dt + L(t)d\mu, \quad x(0) = \xi, \tag{41}$$

has a unique (mild) solution $x_L(\cdot) \equiv x(L) \in B_\infty(J_a, E)$. Let N_β denote the map $L \longrightarrow x(L)$ from $L_1(J_a, |\mu|; Y)$ to $B_\infty(J_a, E)$. By the same argument as in Theorem 5.1, it follows from hypothesis (C1) that the multi function

$$t \longrightarrow C(t, x(L)(t)) = C(t, N_\beta(L)(t))$$

is $|\mu|$ measurable, nonempty having closed values. Hence by the well known Yankov-Von Neumann-Auman selection theorem, see [6, Theorem 2.14, p154], it has $|\mu|$-measurable selections. Since the multi function as defined above is measurable, it is graph measurable and by virtue of assumption (C2), it follows from [6, Lemma 3.2, p175] that it has $L_1(J_a, |\mu|; Y)$ selections. That is,

$$\begin{aligned} \hat{C}(L) &\equiv \{\Gamma \in L_1(J_a, |\mu|; Y) : \Gamma(t) \in C(t, N_\beta(L)(t)) \ |\mu| - a.e\} \\ &\equiv S^1_{C(\cdot, N_\beta(L)(\cdot))} \neq \emptyset. \end{aligned} \tag{42}$$

Since $C(t, x(L)(t)) \in cb(Y)$, $\hat{C}(L)$ is also closed and bounded in

$$L_1(J_a, |\mu|; Y).$$

In other words

$$\hat{C} : L_1(J_a, |\mu|; Y) \longrightarrow cb(L_1(J_a, |\mu|; Y)).$$

We must show that $\hat{C}$ has a fixed point in $L_1(J_a, |\mu|; Y)$. Again, we prove this by showing that there is a Hausdorff metric on $cb(L_1(J_a, |\mu|; Y))$ with respect to which $\hat{C}$ is Lipschitz with Lipschitz constant less than one and then use the Banach fixed point theorem for multi valued maps. Define

$$\gamma(t) \equiv \int_0^t \ell(s) d|\mu|, \quad t \in J_a. \tag{43}$$

Let X_0 denote the Banach space $L_1(J_a, |\mu|; Y)$ and define, for $\delta > 0$, the normed vector space

$$X_\delta \equiv \{\Gamma \in L_0(J_a, |\mu|; Y) : \parallel \Gamma \parallel_\delta \equiv \int_{J_a} \parallel \Gamma(t) \parallel_Y e^{-2\delta\gamma(t)} d|\mu| < \infty\}. \tag{44}$$

With respect to the norm topology, as defined above, X_δ is also a Banach space and one can easily verify that the two spaces X_0 and X_δ are topologically equivalent. Let $L_1, L_2 \in X_0$ with the corresponding solutions of equation (41) given by $x_1 = x(L_1) = N_\beta(L_1)$ and $x_2 = x(L_2) = N_\beta(L_2)$ respectively. Using Gronwall inequality, one can easily verify that

$$\| x_1(t) - x_2(t) \| \le \tilde{K} h(t), \quad t \in J_a, \tag{45}$$

where

$$h(t) \equiv \int_0^t \| L_1(s) - L_2(s) \|_Y \, d|\mu|, \quad t \in J_a \tag{46}$$

and

$$\tilde{K} = (1 + \overline{K} \exp \overline{K}), \overline{K} = \int_{J_a} K(t) dt.$$

Let $\varepsilon > 0$ and $\Gamma_1 \in \hat{C}(L_1)$, that is, $\Gamma_1(t) \in C(t, x_1(t))$ $|\mu| - a.e.$ Since the multi function C is Lipschitz in the second argument with respect to the Hausdorff metric d_H on $cb(Y)$, there exists a $\Gamma_2 \in \hat{C}(L_2)$, that is, $\Gamma_2(t) \in C(t, x_2(t))$ $|\mu| - a.e$ such that

$$\| \Gamma_1(t) - \Gamma_2(t) \|_Y \le \ell(t) \| x_1(t) - x_2(t) \| + \varepsilon \le \tilde{K} \ell(t) h(t) + \varepsilon, \quad t \in J_a. \tag{47}$$

Multiplying (47) by $\exp\{-2\delta\gamma(t)\}$ and integrating with respect to the measure $|\mu|$ and then using integration by parts we find that

$$\begin{aligned} &\int_{J_a} \| \Gamma_1(t) - \Gamma_2(t) \|_Y \exp\{-2\delta\gamma(t)\} d|\mu| \\ &\le (\tilde{K}/2\delta) \int_{J_a} \| L_1(t) - L_2(t) \|_Y \exp\{-2\delta\gamma(t)\} d|\mu| + \varepsilon |\mu|(J_a). \end{aligned} \tag{48}$$

Again denoting the Hausdorff metric on $cb(X_\delta)$ by $D_{H,\delta}$, it follows from (48) that

$$D_{H,\delta}(\hat{C}(L_1), \hat{C}(L_2)) \le (\tilde{K}/2\delta) \| L_1 - L_2 \|_\delta + \varepsilon |\mu|(J_a). \tag{49}$$

Since $\varepsilon > 0$ is arbitrary and variation of μ on J_a is finite it follows from the above expression that, for $\delta > (\tilde{K}/2)$, the multi function $\hat{C}$ is Lipschitz on X_δ and hence by Banach fixed point theorem it has at least one fixed point $\Gamma^o \in X_\delta$. Since $X_\delta \cong X_0$, this is also a fixed point of $\hat{C}$ in X_0. Hence $x^o \equiv N_\beta(\Gamma^o)$ is a (mild) solution of the evolution inclusion (40). This completes the proof. □

Remark. *In Theorem* 5.3 *we have imposed the requirement that the Range* $\{C(t, e)\}$ *is contained in a closed linear separable subspace* Y *of* $\mathcal{L}(U, E)$. *This is done in order to assure that* Y *is a complete separable metric space and hence a Polish space. Probably this assumption can be relaxed.*

Remark. *It is the author's conjecture that the conclusion of theorem* 5.3 *may also hold under local Lipschitz condition for* B *as in Theorem* 5.2.

6. Some examples

For illustration, here we present few examples of physical systems in which the differential generator may be subjected to impulsive perturbation. The reader may discover such examples in other fields.

E1: At low frequencies, with negligible leakage conductance, the voltage distribution along an electric cable is governed by the following system of equations (non homogeneous initial boundary value problems):

$$\begin{aligned} &\partial v/\partial t = (1/RC)\partial^2 v/\partial x^2, \quad t \geq 0, \quad x \in (0,\ell), \\ &v(t,0) = E_0(t), \quad v(t,\ell) = E_1(t) \\ &v(0,x) = e_0(x), \end{aligned} \tag{50}$$

where $R(>0)$ and $C(>0)$ are the line resistance and shunt capacitance respectively, ℓ the length of the cable, E_0, E_1 are the terminal voltages, e_0 is the initial voltage distribution. By simple change of variables we can rewrite this system as a homogeneous boundary value problem. For this we define

$$e(t,x) = v(t,x) - (1 - x/\ell)E_0(t) - (x/\ell)E_1(t).$$

This reduces the system to

$$\begin{aligned} &\partial e/\partial t = (1/RC)\partial^2 e/\partial x^2 + f(t,x), \quad t \geq 0, \quad x \in (0,\ell), \\ &e(0,x) = e_0(x), \end{aligned} \tag{51}$$

where

$$f(t,x) \equiv -(1 - x/\ell)\dot{E}_0 - (x/\ell)\dot{E}_1.$$

The voltage distribution along the line may change substantially as a result of momentary short circuit or due to momentary change of the line to ground capacitance. This later change may occur due to change of the dielectric property of the medium (separating the cable and the ground) due to fast passing objects. The capacitance may also be changed impulsively by switching additional control capacitors in and out to control the voltage distribution along the cable. Under any one of these situations, system (51) can be reformulated as system (8). Let $\gamma \equiv (1/RC)$ be given by sum of two terms

$$\gamma \equiv \gamma_0 + \sum_{i=1} \gamma_i \delta(t - t_i), \quad t \geq 0 \tag{52}$$

where γ_i denotes the size of the impulsive jump at time t_i from its normal value γ_0. Define

$$\beta(t) \equiv \gamma_0\{t + \sum_{i=1}(\gamma_i/\gamma_0)\mathcal{S}(t-t_i)\}. \tag{53}$$

Using this beta, we can rewrite system (51) as an abstract differential equation

$$dy = Ayd\beta(t) + f(t)dt, \quad y(0) = e_0 \tag{54}$$

on the Hilbert space $H \equiv L_2(0,\ell)$ where the operator A is given by

$$D(A) \equiv \{\phi \in H : \phi(0) = \phi(\ell) = 0, \text{and} \triangle\phi \in H\}, \;\; A\phi \equiv \gamma_0 \triangle\phi \;\text{ for } \phi \in D(A).$$

Note that $D(A) = H_0^1 \cap H^2$ where $H^k, k \in R$, denotes the standard sobolev spaces. It follows from semigroup theory that A generates a strongly continuous contraction semi group on H. Thus Lemma 3.1 applies to the pair (A,β) and hence it generates a non expansive evolution operator on H. The function f is given by the expression following equation (51). If both E_0 and E_1 posses locally square integrable (first) derivatives, then it is clear that $f \in L_2^{\ell oc}(R_0, H)$. Under this assumptions, Theorem 3.1 holds and thus equation (54) and hence (50) has a unique solution in $PWC_r(J_a, H)$ for each $a > 0$ finite, and $e_0 \in H$.

E2: An exactly similar model arises in population biology, as discussed in the introduction, which includes migration (diffusion), interaction and even transport terms giving rise to the semilinear model

$$dy = Ayd\beta + f(t,y)dt.$$

Here, f is nonlinear; it includes the interaction and the transport terms and β denotes the migration coefficient which may undergo abrupt changes from its usual value due to biological or bio-chemical shock.

E3: In optical communication, optical signals are modulated by applying electric field, on off, on optical wave guides. The electric field changes the permitivity of the constituent materials of the wave guide thereby changing the refractive index and producing impulsive changes in the dispersion operator. Similar situations occur in the study of propagation of micro waves through air space where the permitivity of the medium may change abruptly due to sudden on set of rain or electrical storms in the medium. In free space the electric field is governed by the Maxwell's equation

$$(\partial^2/\partial t^2)e(t,\xi) - (1/\mu\varepsilon)\triangle e(t,\xi) = f(t,\xi), \; div\; e = 0, \; t \geq 0, \; \xi \in \Omega \subseteq R^3, \tag{55}$$

where e denotes the three dimensional electric field and f is an external force field. In case of abrupt changes of the dielectric properties of the medium, for example, the permitivity ε, we have impulsive perturbation of the Laplacian. Let $L_2 \equiv L_2(\Omega, R^3)$ denote the usual Hilbert space and

$$H \equiv \overline{c\ell}^{L_2}\{\varphi \in C_0^\infty(\Omega, R^3) : div\varphi = 0\}$$

a closed subspace consisting of divergence free vector fields. Clearly, this is also a Hilbert space with the same inner products. Define the operator A as follows

$$\begin{aligned} D(A) &\equiv \{\phi \in H : \triangle\phi \in L_2\} \\ A\phi &\equiv \triangle\phi \quad \text{for} \quad \phi \in D(A). \end{aligned} \tag{56}$$

Using this Hilbert space H as the state space and the operator A one can write equation (55) as a second order abstract differential equation (ODE)on H given by,

$$(d^2/dt^2)y = (1/\mu\varepsilon)Ay + f, \quad t \geq 0. \tag{57}$$

It is not difficult to verify that the operator A is unbounded self adjoint negative on the Hilbert space H as defined above. Now using standard approach, setting

$$x_1 \equiv (1/\sqrt{\mu}\varepsilon)y, \quad \text{and} \quad x_2 \equiv \dot{y}$$

this can be reduced to a first order evolution equation

$$(dx/dt) = \gamma\mathcal{A}x + \tilde{f}, \quad t \geq 0, \quad \gamma \equiv (1/\sqrt{\mu}\varepsilon), \tag{58}$$

where the operator $\mathcal{A}$ and the function $\tilde{f}$ are given by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \quad \tilde{f} \equiv \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

The domain of $\mathcal{A}$ is given by $D(\mathcal{A}) = D(A) \times D(\sqrt{-}A)$, and the appropriate state space for the system (58) is given by the physical energy space

$$E \equiv D(\sqrt{-}A) \times H,$$

which is also a Hilbert space with respect to the scalar product

$$(\varphi, \psi)_E = (\sqrt{-}A\varphi_1, \sqrt{-}A\psi_1)_{L_2} + (\varphi_2, \psi_2)_{L_2}.$$

In case of impulsive changes in the parameter γ around a normal value γ_0 as in (52), again we arrive at an equation like (54),

$$dx = \mathcal{A}x d\beta + \tilde{f}dt.$$

Of course, for the solution of this equation in E, one must provide the initial data in E.

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