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THE JENSEN'S EQUATION IN BANACH MODULES OVER A C^*-ALGEBRA

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Abstract. We prove the generalized Hyers-Ulam-Rassias stability of Jensen's equations in Banach modules over a unital C^*-algebra.

0. Introduction

Let E_1 and E_2 be Banach spaces, and $f : E_1 \to E_2$ a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$. Rassias [6] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E_1$.

The stability problems of functional equations have been investigated in several papers ([4], [5], [6]).

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$, $Inv(A)$ the set of invertible elements in A, $A_1 = \{a \in A \mid \ |a| = 1\}$, A^+ the set of positive elements in A, $\mathbb{R}^+$ the set of positive real numbers, and let ${}_AM_1$ and ${}_AM_2$ be left Banach A-modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively (see [3], [7]).

We are going to prove the generalized Hyers-Ulam-Rassias stability of the Jensen's equation in Banach modules over a unital C^*-algebra.

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1. The Jensen's Equation in Banach Modules over a C^*-algebra

In this section, let A have stable rank 1, which implies that $Inv(A)$ is dense in A (see [1], [3]), and $f : {}_AM_1 \to {}_AM_2$ a mapping such that, for each fixed $x \in {}_AM_1$,

$$f(ax) \text{ is continuous in } a \in A \tag{i}$$

and

$$\lim_{n\to\infty} 3^{-n} f(3^n ax) \text{ converges uniformly on } A_1. \tag{ii}$$

Theorem 1.1. *Let $\varphi : {}_AM_1 \setminus \{0\} \times {}_AM_1 \setminus \{0\} \to [0,\infty)$ be a function and let*

$$\widetilde{\varphi}(x,y) = \sum_{k=0}^{\infty} 3^{-k} \varphi(3^k x, 3^k y) < \infty \tag{iii}$$

and

$$\|2f(\frac{ax+ay}{2}) - af(x) - af(y)\| \le \varphi(x,y)$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x, y \in {}_AM_1 \setminus \{0\}$. Then there exists a unique A-linear mapping $T : {}_AM_1 \to {}_AM_2$

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{3}(\widetilde{\varphi}(x,-x) + \widetilde{\varphi}(-x,3x)) \tag{iv}$$

for all $x \in {}_AM_1 \setminus \{0\}$.

Proof. By Theorem 1 in [5], it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $T : {}_AM_1 \to {}_AM_2$ satisfying (iv). The additive mapping T given in the proof of Theorem 1 in [5] is similar to the additive mapping T given in the proof of Theorem in [6]. By the same reasoning as the proof of Theorem in [6], it follows from the assumption that $f(ax)$ is continuous in $a \in A$ for each fixed $x \in {}_AM_1$ that the additive mapping $T : {}_AM_1 \to {}_AM_2$ is $\mathbb{R}$-linear. By the assumption,

$$\|2f(3^n ax) - af(2 \cdot 3^{n-1}x) - af(4 \cdot 3^{n-1}x)\| \le \varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x)$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x \in {}_AM_1 \setminus \{0\}$. Using the fact that there exists a $K > 0$ such that, for each $a \in A$ and each $z \in {}_AM_2$,

$\|az\| \leq K|a| \cdot \|z\|$,

$$\begin{aligned}\|f(3^n ax) - af(3^n x)\| =& \|f(3^n ax) - \frac{1}{2}af(2\cdot 3^{n-1}x) - \frac{1}{2}af(4\cdot 3^{n-1}x) \\ &+ \frac{1}{2}af(2\cdot 3^{n-1}x) + \frac{1}{2}af(4\cdot 3^{n-1}x) - af(3^n x)\| \\ \leq& \frac{1}{2}\varphi(2\cdot 3^{n-1}x, 4\cdot 3^{n-1}x) \\ &+ \frac{1}{2}K|a| \cdot \|2f(3^n x) - f(2\cdot 3^{n-1}x) - f(4\cdot 3^{n-1}x)\| \\ \leq& \frac{1+K}{2}\varphi(2\cdot 3^{n-1}x, 4\cdot 3^{n-1}x)\end{aligned}$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x \in {}_AM_1 \setminus \{0\}$. So $3^{-n}\|f(3^n ax) - af(3^n x)\| \to 0$ as $n \to \infty$ for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x \in {}_AM_1 \setminus \{0\}$. Hence

$$T(ax) = \lim_{n\to\infty} 3^{-n} f(3^n ax) = \lim_{n\to\infty} 3^{-n} af(3^n x) = aT(x) \tag{1}$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x \in {}_AM_1 \setminus \{0\}$. So

$$T(ax) = aT(x)$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x \in {}_AM_1$.

Let $b \in (A_1 \cap A^+) \setminus Inv(A)$. Since $Inv(A)$ is dense in A, there exists a sequence $\{b_m\}$ in $Inv(A)$ such that $b_m \to b$ as $m \to \infty$. Put $c_m = \frac{1}{|b_m|}b_m$, then $c_m \to \frac{1}{|b|}b = b$ as $m \to \infty$ and $c_m \in Inv(A) \cap A_1 \cap A^+$. Put $a_m = \sqrt{{c_m}^* c_m}$, then $a_m \to \sqrt{b^* b} = b$ as $m \to \infty$ and $a_m \in Inv(A) \cap A_1 \cap A^+$. Thus there exists a sequence $\{a_m\}$ in $Inv(A) \cap A_1 \cap A^+$ such that $a_m \to b$ as $m \to \infty$, and so

$$\begin{aligned}\lim_{m\to\infty} T(a_m x) &= \lim_{m\to\infty}\lim_{n\to\infty} 3^{-n} f(3^n a_m x) \\ &= \lim_{n\to\infty}\lim_{m\to\infty} 3^{-n} f(3^n a_m x) \text{ by (ii)} \\ &= \lim_{n\to\infty} [3^{-n} f(3^n \lim_{m\to\infty} a_m x)] \text{ by (i)} \\ &= \lim_{n\to\infty} 3^{-n} f(3^n bx) \\ &= T(bx)\end{aligned} \tag{2}$$

for all $x \in {}_AM_1$. By (1),

$$\|T(a_m x) - bT(x)\| = \|a_m T(x) - bT(x)\| \to \|bT(x) - bT(x)\| = 0 \tag{3}$$

as $m \to \infty$. By (2),

$$\|3^{-n}f(3^n a_m x) - T(a_m x)\| \to \|3^{-n}f(3^n bx) - T(bx)\| \tag{4}$$

as $m \to \infty$. By (3) and (4),

$$\begin{aligned}\|T(bx) - bT(x)\| \le & \|T(bx) - 3^{-n}f(3^n bx)\| + \|3^{-n}f(3^n bx) - 3^{-n}f(3^n a_m x)\| \\ & + \|3^{-n}f(3^n a_m x) - T(a_m x)\| + \|T(a_m x) - bT(x)\| \\ \to & \|T(bx) - 3^{-n}f(3^n bx)\| + \|3^{-n}f(3^n bx) - T(bx)\| \text{ as } m \to \infty \\ \to & 0 \quad \text{as} \quad n \to \infty\end{aligned} \tag{5}$$

for all $x \in {}_AM_1$. By (1) and (5),

$$T(ax) = aT(x)$$

for all $a \in (A_1 \cap A^+) \cup \{i\}$ and all $x \in {}_AM_1$.

Thus

$$T(ax) = |a| \cdot T(\frac{a}{|a|}x) = aT(x)$$

for all $a \in (A^+ \setminus \{0\}) \cup \{i\}$ and all $x \in {}_AM_1$.

For any element $a \in A$, $a = a_1 + ia_2$, where $a_1 = \frac{a+a^*}{2}$ and $a_2 = \frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = a_1{}^+ - a_1{}^- + ia_2{}^+ - ia_2{}^-$, where $a_1{}^+$, $a_1{}^-$, $a_2{}^+$, and $a_2{}^-$ are positive elements (see Lemma 38.8 in [2]). So

$$\begin{aligned}T(ax) =& T(a_1{}^+x - a_1{}^-x + ia_2{}^+x - ia_2{}^-x) \\ =& (a_1{}^+ - a_1{}^- + ia_2{}^+ - ia_2{}^-)T(x) \\ =& aT(x)\end{aligned}$$

for all $a \in A$ and all $x \in {}_AM_1$.

Therefore, there exists a unique A-linear mapping $T : {}_AM_1 \to {}_AM_2$, as desired. □

Corollary 1.2. *Let $p < 1$ and*

$$\|2f(\frac{ax+ay}{2}) - af(x) - af(y)\| \le \|x\|^p + \|y\|^p$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x, y \in {}_AM_1 \setminus \{0\}$. Then there exists a unique A-linear mapping $T : {}_AM_1 \to {}_AM_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{3+3^p}{3-3^p}\|x\|^p$$

for all $x \in {}_AM_1 \setminus \{0\}$.

Proof. Define $\varphi : {}_AM_1 \setminus \{0\} \times {}_AM_1 \setminus \{0\} \to [0,\infty)$ by $\varphi(x,y) = \|x\|^p + \|y\|^p$ and apply Theorem 1.1. $\square$

Theorem 1.3. *Let $\varphi : {}_AM_1 \setminus \{0\} \times {}_AM_1 \setminus \{0\} \to [0,\infty)$ be a function such that*

$$\widetilde{\varphi}(x,y) = \sum_{k=0}^{\infty} 3^k \varphi(3^{-k}x, 3^{-k}y) < \infty \tag{v}$$

and

$$\|2f(\frac{ax+ay}{2}) - af(x) - af(y)\| \le \varphi(x,y)$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x, y \in {}_AM_1 \setminus \{0\}$. Then there exists a unique A-linear mapping $T : {}_AM_1 \to {}_AM_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \widetilde{\varphi}(\frac{x}{3}, \frac{-x}{3}) + \widetilde{\varphi}(\frac{-x}{3}, x) \tag{vi}$$

for all $x \in {}_AM_1 \setminus \{0\}$.

Proof. By Theorem 6 in [5], it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $T : {}_AM_1 \to {}_AM_2$ satisfying (vi). The additive mapping T given in the proof of Theorem 6 in [5] is similar to the additive mapping T given in the proof of Theorem in [6]. By the same reasoning as the proof of Theorem in [6], it follows from the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_AM_1$ that the additive mapping $T : {}_AM_1 \to {}_AM_2$ is $\mathbb{R}$-linear.

By the assumption,

$$\|2f(3^{-n}ax) - af(2\cdot 3^{-n-1}x) - af(4\cdot 3^{-n-1}x)\| \le \varphi(2\cdot 3^{-n-1}x, 4\cdot 3^{-n-1}x)$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x \in {}_A M_1 \setminus \{0\}$. Using the fact that for each $a \in A$ and each $z \in {}_A M_2$, $\|az\| \le K|a| \cdot \|z\|$ for some $K > 0$,

$$\begin{aligned}\|f(3^{-n}ax) - af(3^{-n}x)\| &= \|f(3^{-n}ax) - \frac{1}{2}af(2 \cdot 3^{-n-1}x) - \frac{1}{2}af(4 \cdot 3^{-n-1}x) \\ &\quad + \frac{1}{2}af(2 \cdot 3^{-n-1}x) + \frac{1}{2}af(4 \cdot 3^{-n-1}x) - af(3^{-n}x)\| \\ &\le \frac{1}{2}\varphi(2 \cdot 3^{-n-1}x, 4 \cdot 3^{-n-1}x) \\ &\quad + \frac{1}{2}K|a| \cdot \|2f(3^{-n}x) - f(2 \cdot 3^{-n-1}x) - f(4 \cdot 3^{-n-1}x)\| \\ &\le \frac{1+K}{2}\varphi(2 \cdot 3^{-n-1}x, 4 \cdot 3^{-n-1}x)\end{aligned}$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x \in {}_A M_1 \setminus \{0\}$. Thus $3^n\|f(3^{-n}ax) - af(3^{-n}x)\| \to 0$ as $n \to \infty$ for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x \in {}_A M_1 \setminus \{0\}$. Hence

$$T(ax) = \lim_{n\to\infty} 3^n f(3^{-n}ax) = \lim_{n\to\infty} 3^n af(3^{-n}x) = aT(x)$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x \in {}_A M_1 \setminus \{0\}$.

By a similar method to the proof of Theorem 1.1, one can show that

$$T(ax) = aT(x)$$

for all $a \in A$ and all $x \in {}_A M_1$.

Therefore, there exists a unique A-linear mapping $T : {}_A M_1 \to {}_A M_2$ satisfying (vi). □

Corollary 1.4. *Let $p > 1$ and*

$$\|2f(\frac{ax+ay}{2}) - af(x) - af(y)\| \le \|x\|^p + \|y\|^p$$

for all $a \in [Inv(A) \cap A_1 \cap A^+] \cup \{i\}$ and all $x, y \in {}_A M_1 \setminus \{0\}$. Then there exists a unique A-linear mapping $T : {}_A M_1 \to {}_A M_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{3^p + 3}{3^p - 3}\|x\|^p$$

for all $x \in {}_A M_1$.

Proof. Define $\varphi : {}_A M_1 \setminus \{0\} \times {}_A M_1 \setminus \{0\} \to [0, \infty)$ by $\varphi(x, y) = \|x\|^p + \|y\|^p$ and apply Theorem 1.3. □

Remark 1.1.

(1) *When the inequalities*

$$\|2f(\frac{ax+ay}{2}) - af(x) - af(y)\| \le \varphi(x,y)$$

in the statements are replaced by

$$\|2af(\frac{x+y}{2}) - f(ax) - f(ay)\| \le \varphi(x,y),$$

the results do also hold.

(2) *When the inequalities*

$$\|2f(\frac{ax+ay}{2}) - af(x) - af(y)\| \le \varphi(x,y) \qquad \text{or}$$
$$\|2af(\frac{x+y}{2}) - f(ax) - f(ay)\| \le \varphi(x,y)$$

in the statements are replaced by

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \le \varphi(x,y),$$
$$\|f(ax) - af(x)\| \le \varphi(x,x),$$

the results do also hold.

(3) *If the inequalities*

$$\|2f(\frac{ax+ay}{2}) - af(x) - af(y)\| \le \varphi(x,y)$$

in the statements are replaced by

$$\|2f(\frac{ax+y}{2}) - af(x) - f(y)\| \le \varphi(x,y),$$

then

$$\|2f(\frac{ax+ay}{2}) - af(x) - f(ay)\| \le \varphi(x,ay),$$
$$\|2f(\frac{ax+ay}{2}) - f(ax) - af(y)\| \le \varphi(y,ax),$$
$$\|2f(\frac{ax+ay}{2}) - f(ax) - f(ay)\| \le \varphi(ax,ay).$$

So

$$\|2f(\frac{ax+ay}{2}) - af(x) - af(y)\| \le \varphi(x,ay) + \varphi(y,ax) + \varphi(ax,ay),$$

hence the results do also hold.

Remark 1.2. *The A-linear mappings $T : {}_AM_1 \to {}_AM_2$, constructed above, are continuous A-linear.*

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