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GENERALIZATIONS OF HARDY INTEGRAL INEQUALITY

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Abstract. In this paper, some new generalizations of Hardy type integral inequalities are proved.

1. Introduction

The following Inequality is due to Hardy (cf. [1, Theorem 326]): *If* $p > 1, a_n \geq 0$ *and* $A_n = a_1 + a_2 + ... + a_n$, *then*

$$\sum \Big(\frac{A_n}{n}\Big)^p < \Big(\frac{p}{p-1}\Big)^p \sum a_n^p, \tag{1.1}$$

unless all the a's are zero. The coefficient constant is the best possible.

This inequality was discovered in the course of attempts to simplify the proofs of the Hilbert's double series theorem (cf. [4, Theorem 315]). In fact Hilbert's double series theorem was completed by the above inequality. Moreover, the corresponding Hardy's inequality [1, Theorem 327] for integrals can be stated as follows: *If* $f(x) \geq 0, p > 1$ *and* $0 < \int_0^\infty f^p(x)dx < \infty$, *then*

$$\int_0^\infty \left[\frac{1}{x}\int_0^x f(t)dt\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(t)dt, \tag{1.2}$$

where $(p/(p-1))^p$ *is the best possible constant.*

The inequalities given in (1.1) and (1.2), led to great many papers dealing with alternative proofs, various generalizations, and numerous variants and applications in analysis. For the earlier development of this kind of inequality and several important applications in analysis one can see [1, Chapter IX]. It

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is obvious that, for parameters a and b such that $0 < a < b < \infty$, the following inequality is also valid,

$$\int_0^b \left[\frac{1}{x}\int_0^x f(t)dt\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f^p(t)dt, \tag{1.3}$$

where $0 < \displaystyle\int_0^\infty f^p(x)dx < \infty$.

The main purpose of this paper is to establish some new generalizations of inequality (1.2) by using a fairly elementary treatment. Our results as special cases yield some recent generalizations of the Hardy's inequality given in [5].

2. Main Results

In this section the function Q is a real valued function such that $0 < Q(s) \leq s$ for each $s > 0$ and $Q(0) = 0$. In all the theorems it is assumed, without further mention, that the integrals exist on the respective domains of their definitions. Our results are given in the following theorems:

Lemma 2.1. *Let* $0 < b \leq \infty$, $pQ(1) > 1$, $1/p + 1/q = Q(1)$, $f \geq 0$, *and let* $0 < \displaystyle\int_0^b f^{p/Q(1)}(t)dt < \infty$. *Then there exists a real number* $x_0 \in (0, b)$ *such that for any* $x \in (x_0, b)$, *the following inequality is true:*

$$\int_0^x f(t)dt < K_1 x^{(p-Q(1))/pqQ(1)} \left(\int_0^x t^{Q(1)/q} f^{pQ(1)}(t)dt\right)^{1/pQ(1)}, \tag{2.1}$$

where $K_1 = (p/(p-Q(1)))^{1/qQ(1)} = (p/(p-Q(1)))^{p/(p+q)}$.

Proof. For any $x \in (0, b)$, by Hölder's inequality, we have

$$\begin{aligned}\int_0^x f(t)dt &= \int_0^x t^{1/pq} f(t) t^{-1/pq} dt \\ &\leq \left(\int_0^x t^{Q(1)/q} f^{pQ(1)}(t)dt\right)^{1/pQ(1)} \left(\int_0^x t^{-Q(1)/p} dt\right)^{1/qQ(1)} \\ &= K_1 x^{(p-Q(1))/pqQ(1)} \left(\int_0^x t^{Q(1)/q} f^{pQ(1)}(t)dt\right)^{1/pQ(1)},\end{aligned} \tag{2.2}$$

where $K_1 = (p/(p-Q(1)))^{1/qQ(1)} = (p/(p-Q(1)))^{p/(p+q)}$. We need to show that there exists a real number $x_0 \in (0, b)$ such that for any $x \in (x_0, b)$, the equality in (2.2) does not hold. Otherwise, there exists $x = x_n \in (0, b)$, where $n = 1, 2, ...,$ with $x_n \uparrow b$ such that (2.2) becomes an equality. Then there exist c_n and d_n which are not always zero, and satisfy (Kuang [2, p. 29])

$$c_n\Big(t^{1/pq}f(t)\Big)^{p/Q(1)} = d_n\Big(t^{-1/pq}\Big)^{q/Q(1)}, \qquad \text{a.e. in } [0, x_n].$$

However $f(t) \neq 0$ a.e. in $(0, b)$, thus there exists as integer N such that for $n > N, f(t) \neq 0$ a.e. in $(0, x_n)$. Hence for both $c_n = c \neq 0$ and $d_n = d \neq 0$ for $n > N$, we obtain

$$\begin{aligned}\int_0^b f^{p/Q(1)}(t)dt &= \lim_{n\to\infty}\int_0^{x_n} f^{p/Q(1)}(t)dt \\ &= \lim_{n\to\infty}\frac{d}{c}\int_0^{x_n}\frac{t^{-1/pQ(1)}}{t^{1/qQ(1)}}dt \\ &= \lim_{n\to\infty}\frac{d}{c}\int_0^{x_n} t^{-1}dt = \infty.\end{aligned}$$

This contradicts the fact that $0 < \int_0^b f^{p/Q(1)}(t)dt < \infty$. Therefore (2.1) is valid and this completes the proof of the lemma. □

Lemma 2.2. *Let $a \geq 0$, $pQ(1) > 1$, $1/p + 1/q = Q(1)$, $f \geq 0$, and let $0 < \int_a^\infty f^{p/Q(1)}(t)dt < \infty$. Then there exists a real number $x_0 \in (a, \infty)$ such that for any $x > x_0$ the following inequality is true:*

$$\int_a^x f(t)dt < K_1\Big(T_a^x\Big)^{1/qQ(1)}\Big(\int_a^x t^{Q(1)/q}f^{pQ(1)}(t)dt\Big)^{1/pQ(1)}, \tag{2.3}$$

where

$$K_1 = (p/(p - Q(1)))^{1/qQ(1)} = (p/(p - Q(1)))^{p/(p+q)}$$

and

$$T_a^x = x^{(p-Q(1))/p} - a^{(p-Q(1))/p}.$$

Proof. For any $x \in (a, \infty)$, by Hölder's inequality as in Lemma 2.1, we have

$$\begin{aligned}\int_a^x f(t)dt &\leq \Big(\int_a^x t^{Q(1)/q}f^{pQ(1)}(t)dt\Big)^{1/pQ(1)}\Big(\int_a^x t^{-Q(1)/p}dt\Big)^{1/qQ(1)} \\ &= K_1\Big(T_a^x\Big)^{1/qQ(1)}\Big(\int_a^x t^{Q(1)/q}f^{pQ(1)}(t)dt\Big)^{1/pQ(1)},\end{aligned} \tag{2.4}$$

where

$$K_1 = (p/(p-Q(1)))^{1/qQ(1)} = (p/(p-Q(1)))^{p/(p+q)}$$

and

$$T_a^x = x^{(p-Q(1))/p} - a^{(p-Q(1))/p}.$$

We shall show that there exists a real number $x_0 \in (a, \infty)$ such that (2.4) does not assume equality for any $x > x_0$. Otherwise, there exists $x = x_n \in (a, \infty)$, where $n = 1, 2, ...$, with $x_n \uparrow \infty$ such that (2.4) becomes an equality. By the same argument as in Lemma 2.1 there exist $c > 0$ and N such that for $n > N$,

$$\left(t^{1/pqQ(1)} f(t)\right)^{p/Q(1)} = c\left(t^{-1/pqQ(1)}\right)^{q/Q(1)}, \quad \text{a.e. in } [a, x_n].$$

Hence

$$\begin{aligned}\int_a^{x_n} f^{p/Q(1)}(t)dt &= c\int_a^{x_n} \frac{t^{-1/pQ(1)}}{t^{1/qQ(1)}}dt \\ &= c\int_a^{x_n} t^{-[1/pQ(1)+1/qQ(1)]}dt \\ &= c\int_a^{x_n} t^{-1}dt \quad \to \infty \quad \text{as} \quad n \to \infty.\end{aligned}$$

This contradicts the fact that $0 < \int_a^\infty f^{p/Q(1)}(t)dt < \infty$. Hence (2.3) holds true and the proof is complete. □

Remark 2.1. *If we set* $Q(s) = s$ *in Lemmas* 2.1 *and* 2.2*, then our results reduce to the corresponding Lemmas* 2.1 *and* 2.2 *obtained in* [5].

Theorem 2.3. *Let* $0 < a < b \le \infty$, $pQ(1) > 1$, $1/p + 1/q = Q(1)$, $f \ge 0$, *and let* $0 < \displaystyle\int_a^b f^{p/Q(1)}(t)\,dt < \infty$. *Then*

$$\begin{aligned}&\int_a^b \left(\frac{1}{x}\int_a^x f(t)dt\right)^{pQ(1)} dx \\ &< K_2 \frac{q}{Q(1)} \left(1 - \left(\frac{a}{b}\right)^{Q(1)/q}\right) \left(T^1_{a/b}\right)^{p/q} \int_a^b f^{pQ(1)}(t)dt,\end{aligned} \tag{2.5}$$

where $K_2 = (p/(p-Q(1)))^{p/q} = (p/(p-Q(1)))^{pQ(1)-1}$ *and* $T_a^x = x^{(p-Q(1))/p} - a^{(p-Q(1))/p}$.

Proof. Using (2.3), we obtain

$$\begin{aligned}
&\int_a^b \left(\frac{1}{x}\int_a^x f(t)dt\right)^{pQ(1)} dx \\
&< K_2 \int_a^b \frac{1}{x^{pQ(1)}} \left(T_a^x\right)^{p/q} \int_a^x t^{Q(1)/q} f^{pQ(1)}(t)\,dt\,dx \\
&= K_2 \int_a^b \left\{\int_t^b x^{-pQ(1)+(p-Q(1))/q} \left[T^1_{a/x}\right]^{p/q} dx\right\} t^{Q(1)/q} f^{pQ(1)}(t)\,dt \\
&< K_2 \int_a^b \left\{\int_t^b x^{-pQ(1)+(p-Q(1))/q} \left[T^1_{a/b}\right]^{p/q} dx\right\} t^{Q(1)/q} f^{pQ(1)}(t)\,dt \\
&= K_2 \left[T^1_{a/b}\right]^{p/q} \int_a^b \left\{\int_t^b x^{-1-Q(1)/q}\, dx\right\} t^{Q(1)/q} f^{pQ(1)}(t)\,dt \\
&= K_2 \frac{q}{Q(1)} \left[T^1_{a/b}\right]^{p/q} \int_a^b \left[1-\left(\frac{t}{b}\right)^{Q(1)/q}\right] f^{pQ(1)}(t)\,dt \\
&< K_2 \frac{q}{Q(1)} \left[T^1_{a/b}\right]^{p/q} \int_a^b \left[1-\left(\frac{a}{b}\right)^{Q(1)/q}\right] f^{pQ(1)}(t)\,dt \\
&= K_2 \frac{q}{Q(1)} \left[1-\left(\frac{a}{b}\right)^{Q(1)/q}\right] \left[T^1_{a/b}\right]^{p/q} \int_a^b f^{pQ(1)}(t)\,dt,
\end{aligned}$$

where $K_2 = (p/(p-Q(1)))^{p/q} = (p/(p-Q(1)))^{pQ(1)-1}$ and $T_a^x = x^{(p-Q(1))/p} - a^{(p-Q(1))/p}$. This completes the proof of our theorem. □

Remark 2.2. *If we set $Q(s) = s$ in Theorem* 2.3*, then our results reduce to the following inequality*

$$\int_a^b \left(\frac{1}{x}\int_a^x f(t)dt\right)^p dx < q^p \left(1-\left(\frac{a}{b}\right)^{1/q}\right)^p \int_a^b f^p(t)\,dt$$

which was obtained in [5]*. This shows that inequality* (2.5) *is a generalization of inequality* (1.2)*. Also inequality* (2.5) *is an improvement of* (1.3)*.*

Theorem 2.4. *Let $a > 0$, $pQ(1) > 1$, $1/p + 1/q = Q(1)$, $f \geq 0$, and let $0 < \int_a^\infty f^{p/Q(1)}(t)\,dt < \infty$. Then*

$$\int_a^\infty \left(\frac{1}{x^\alpha}\int_a^x f(t)\,dt\right)^{pQ(1)} dx < K_3 \int_a^\infty \Big[1-\theta_p(t)\Big] f^{pQ(1)}(t)\,dt, \tag{2.6}$$

where $\alpha = 2 - Q(1)/p - 1/qQ(1)$, $K_3 = (p/(p - Q(1)))^{(p+q)/q} = (p/(p - Q(1)))^{pQ(1)}$ *and*

$$\theta_p(t) = 1 + \frac{1}{pQ(1)}\left\{\sum_{k=1}^{\infty}\binom{pQ(1)}{k}(-1)^k\left(\frac{a}{t}\right)^{k(p-Q(1))/p}\right\}a^{(Q(1)-p)/p}t^{Q(1)/q}$$
$$> 1 - t^{Q^2(1)-1}$$

with $\theta_p(a) = 1 - a^{Q^2(1)-1}/pQ(1)$, *for* $t > a$ *and* $q > 0$.

Proof. Applying inequalities (2.3) and (2.4), we have

$$\begin{aligned}
&\int_a^\infty \left(\frac{1}{x^\alpha}\int_a^x f(t)\,dt\right)^{pQ(1)} dx \\
&< K_2\int_a^\infty x^{-\alpha pQ(1)}\left(T_a^x\right)^{p/q}\int_a^x t^{Q(1)/q}f^{pQ(1)}(t)\,dt\,dx \\
&= K_2\int_a^\infty\left\{\int_t^\infty x^{-2+Q(1)/p}\left(T^1_{a/x}\right)^{p/q}dx\right\}t^{Q(1)/q}f^{pQ(1)}(t)\,dt \\
&= K_3\int_a^\infty\left\{\int_t^\infty \left(T^1_{a/x}\right)^{p/q}d\left(T^1_{a/x}\right)\right\}a^{(Q(1)-p)/p}t^{Q(1)/q}f^{pQ(1)}(t)\,dt \\
&= K_3\int_a^\infty \frac{q}{p+q}\left\{1 - \left(T^1_{a/t}\right)^{pQ(1)}\right\}a^{(Q(1)-p)/p}t^{Q(1)/q}f^{pQ(1)}(t)\,dt \\
&= K_3\int_a^\infty \left[1 - \theta_p(t)\right]f^{pQ(1)}(t)\,dt,
\end{aligned}$$

where $\alpha = 2 - Q(1)/p - 1/qQ(1)$, $K_2 = (p/(p - Q(1)))^{p/q}$, $K_3 = (p/(p - Q(1)))^{(p+q)/q} = (p/(p - Q(1)))^{pQ(1)}$, $T_a^x = x^{(p-Q(1))/p} - a^{(p-Q(1))/p}$, and

$$\theta_p(t) = 1 - \frac{q}{p+q}\left\{1 - \left(T^1_{a/t}\right)^{pQ(1)}\right\}a^{(Q(1)-p)/p}t^{Q(1)/q},$$

since

$$\left(T^1_{a/t}\right)^{pQ(1)} = \sum_{k=0}^{\infty}\binom{pQ(1)}{k}(-1)^k\left(\frac{a}{t}\right)^{k(p-Q(1))/p} \quad (t > a > 0).$$

From this we easily obtain

$$\theta_p(a) = 1 - \frac{1}{pQ(1)}a^{Q^2(1)-1},$$
$$\theta_p(t) = 1 + \frac{1}{pQ(1)}\left\{\sum_{k=1}^{\infty}\binom{pQ(1)}{k}(-1)^k\left(\frac{a}{t}\right)^{k(p-Q(1))/p}\right\}a^{(Q(1)-p)/p}t^{Q(1)/q}$$

for $t > a > 0$. Applying Bernoulli's inequality (see [3, p. 65]), we obtain

$$1 - pQ(1)\Big(\frac{a}{t}\Big)^{(p-Q(1))/p} < \bigg[1 - \Big(\frac{a}{t}\Big)^{(p-Q(1))/p}\bigg]^{pQ(1)},$$

and therefore one has

$$\begin{aligned}
\theta_p(t) &> 1 - \frac{1}{pQ(1)}\bigg\{1 - \bigg[1 - pQ(1)\Big(\frac{a}{t}\Big)^{(p-Q(1))/p}\bigg]\bigg\} a^{(Q(1)-p)/p} t^{Q(1)/q} \\
&= 1 - \frac{1}{pQ(1)}\bigg[pQ(1)\Big(\frac{a}{t}\Big)^{(p-Q(1))/p}\bigg] a^{(Q(1)-p)/p} t^{Q(1)/q} \\
&= 1 - t^{Q^2(1)-1}
\end{aligned}$$

Hence, the proof is complete. $\square$

Theorem 2.5. *Let $0 < b \le \infty$, $r \ge pQ(1) > 1$, $1/p + 1/q = Q(1)$, $f \ge 0$, and let $0 < \int_0^b x^{-r+pQ(1)} f^{p/Q(1)}(x)\,dx < \infty$. Then*

(i) *For $b \in (0, \infty)$, we have*

$$\begin{aligned}
&\int_0^b x^{-r}\Big(\int_0^x f(t)\,dt\Big)^{pQ(1)} dx \\
&< \beta K_2 \int_0^b \bigg[1 - \Big(\frac{t}{b}\Big)^{r-pQ(1)+Q(1)/q}\bigg] t^{-r+pQ(1)} f^{pQ(1)}(t)\,dt,
\end{aligned} \tag{2.7}$$

where $1/\beta = r - pQ(1) + Q(1)/q$ and $K_2 = (p/(p-Q(1)))^{p/q} = (p/(p-Q(1)))^{pQ(1)-1}$. In particular, when $r = pQ(1)$ we obtain

$$\int_0^b \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^{pQ(1)} dx < K_2 \frac{q}{Q(1)} \int_0^b \bigg[1 - \Big(\frac{t}{b}\Big)^{Q(1)/q}\bigg] f^{pQ(1)}(t)\,dt. \tag{2.8}$$

(ii) *For $b = \infty$, we have*

$$\int_0^\infty x^{-r}\Big(\int_0^x f(t)\,dt\Big)^{pQ(1)} dx < \beta K_2 \int_0^\infty t^{-r+pQ(1)} f^{pQ(1)}(t)\,dt, \tag{2.9}$$

where $1/\beta = r - pQ(1) + Q(1)/q$ and $K_2 = (p/(p-Q(1)))^{p/q} = (p/(p-Q(1)))^{pQ(1)-1}$.

Proof. For case (i), $b \in (0,\infty)$, we use (2.1) to obtain

$$\begin{aligned}
&\int_0^b x^{-r}\left(\int_0^x f(t)\,dt\right)^{pQ(1)} dx \\
&< K_2 \int_0^b x^{-r+(p-Q(1))/q} \int_0^x t^{Q(1)/q} f^{pQ(1)}(t)\,dt\,dx \\
&= K_2 \int_0^b \left(\int_t^b x^{-r+(p-Q(1))/q}\,dx\right) t^{Q(1)/q} f^{pQ(1)}(t)\,dt \\
&= -\beta K_2 \int_0^b (b^{-r+1+(p-Q(1))/q} - t^{-r+1+(p-Q(1))/q}) t^{Q(1)/q} f^{pQ(1)}(t)\,dt \\
&= \beta K_2 \int_0^b \left[1-\left(\frac{t}{b}\right)^{r-pQ(1)+Q(1)/q}\right] t^{-r+pQ(1)} f^{pQ(1)}(t)\,dt,
\end{aligned}$$

where $1/\beta = r - pQ(1) + Q(1)/q$ and $K_2 = (p/(p-Q(1)))^{p/q} = (p/(p-Q(1)))^{pQ(1)-1}$. This proves (2.7).

For case (ii), $b=\infty$, we use (2.1) to find,

$$\begin{aligned}
\int_0^\infty x^{-r}\left(\int_0^x f(t)\,dt\right)^{pQ(1)} dx
&< K_2 \int_0^\infty x^{-r+(p-Q(1))/q} \int_0^x t^{Q(1)/q} f^{pQ(1)}(t)\,dt\,dx \\
&= K_2 \int_0^\infty \left(\int_t^\infty x^{-r+(p-Q(1))/q}\,dx\right) t^{Q(1)/q} f^{pQ(1)}(t)\,dt \\
&= \beta K_2 \int_0^\infty t^{-r+pQ(1)} f^{pQ(1)}(t)\,dt,
\end{aligned}$$

where $1/\beta = r - pQ(1) + Q(1)/q$ and $K_2 = (p/(p-Q(1)))^{p/q} = (p/(p-Q(1)))^{pQ(1)-1}$. This proves (2.9) and the proof of the theorem is complete. □

Remark 2.3. *In the limits as $a \to 0$ and $b \to \infty$, and the function as $Q(s)=s$,* (2.6) *reduces to* (1.2). *This shows that inequality* (2.6) *is a generalization of inequality* (1.2). *When $r=p$ and $Q(s)=s$, inequality* (2.9) *reduces to* (1.2). *This means* (2.9) *and* (2.7) *provide also a generalization of inequality* (1.2).

Remark 2.4. *If we let* $Q(s) = s - 1/c(c > 1)$, *in* (2.9), *then we shall obtain the following inequality:*

$$\begin{aligned} &\int_0^\infty x^{-r} \left(\int_0^x f(t)\,dt \right)^{p(1-1/c)} dx \\ &< K_* \left(\frac{1}{r - (1-1/c)(p-1/q)} \right) \int_0^\infty t^{-r+p(1-1/c)} f^{p(1-1/c)}(t)\,dt, \end{aligned} \tag{2.10}$$

where $K_* = (p/(p-(1-1/c)))^{p(1-1/c)-1}$. *If we let* $c \to \infty$ *in* (2.10), *then one obtains the following inequality:*

$$\int_0^\infty x^{-r} \left(\int_0^x f(t)\,dt \right)^p dx < \frac{q^p}{(r-p)q+1} \int_0^\infty t^{-r+p} f^p(t)\,dt. \tag{2.11}$$

If we set $r = p$ *in* (2.11) *we shall get inequality* (1.2).

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