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A COMMON FIXED POINT THEOREM WITH APPLICATIONS IN DYNAMIC PROGRAMMING

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Abstract. In this paper, we introduce the concept of compatible mappings and weakly compatible mappings by using pseudometrics, give a common fixed point theorem for two pairs of weakly compatible mappings, and discuss the existence and uniqueness of common solutions for several classes of system of functional equations arising in dynamic programming.

1. Introduction

As suggested in Bellman and Lee [1], the basic form of the functional equations of dynamic programming is

$$f(x) = opt_y H(x, y, f(T(x, y))),$$

where x and y represent the state and decision vectors, respectively, T represents the transformation of the process, and $f(x)$ represents the optimal return function with initial state x, the *opt* denotes the sup or inf. Bhakta and Choudhury [2], Bhakta and Mitra [3], Liu [4]-[6], Liu and Ume [7], Liu, Agarwal and Kang [8], Pathak and Fisher [9] and others established the existence and uniqueness of solutions for several classes of functional equations or system of functional equations arising in dynamic programming. In particular, Bhakta and Choudhury [2] obtained two fixed point theorems by using pseudometrics and discussed also the existence of solutions for the following functional equation:

$$f(x) = \inf_{y \in D} G(x, y, f), \ x \in S.$$

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On the other hand, Singh and Mishra [10] obtained a few common fixed point theorems for weakly compatible mappings in metric spaces.

In this paper, we introduce the concept of compatible mappings and weakly compatible mappings by using pseudometrics, prove a common fixed point theorem for two pairs of weakly compatible mappings and show the existence and uniqueness of common solutions for system of functional equations arising programming as follows:

$$\begin{aligned}
f_1(x) &= opt_{y\in D}\{u(x,y)+H_1(x,y,f_1(T(x,y)))\},\ x\in S,\\
f_2(x) &= opt_{y\in D}\{u(x,y)+H_2(x,y,f_2(T(x,y)))\},\ x\in S,\\
f_3(x) &= opt_{y\in D}\{u(x,y)+H_3(x,y,f_3(T(x,y)))\},\ x\in S,\\
f_4(x) &= opt_{y\in D}\{u(x,y)+H_4(x,y,f_4(T(x,y)))\},\ x\in S.
\end{aligned} \tag{1.1}$$

Throughout this paper, let $\mathbb{R}=(-\infty,+\infty), \mathbb{R}^+=[0,+\infty)$ and

$$\Phi=\{\varphi:\varphi:\mathbb{R}^+\to\mathbb{R}^+ \text{ is semi-continuous and satisfies } \varphi(t)<t,\ t>0\}.$$

2. A common fixed point theorem

Let X be a nonempty set and $\{d_n\}_{n\geq 1}$ be a countable family of pseudometrics on X such that for any distinct $x,y\in X, d_k(x,y)\neq 0$ for some $k\geq 1$. Define

$$d(x,y)=\sum_{k=1}^{\infty}2^{-k}\frac{d_k(x,y)}{1+d_k(x,y)},\quad x,y\in X.$$

It is clear that d is a metric on X. A sequence $\{x_n\}_{n\geq 1}\subseteq X$ converges to a point $x\in X$ if and only if $d_k(x_n,x)\to 0$ as $n\to\infty$ and $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if and only if $d_k(x_n,x_m)\to 0$ as $n,m\to\infty$ for each $k\geq 1$. A self mapping f on a metric space (X,d) is said to be continuous in X if $\lim_{n\to\infty} fx_n = fx$ whenever $\{x_n\}_{n\geq 1}\subseteq X$ such that $\{x_n\}_{n\geq 1}$ converges to $x\in X$.

Definition 2.1. *A pair of self mappings f and g on a metric space (X,d) are called compatible if $\lim_{n\to\infty} d_k(fgx_n,gfx_n)=0$ for $k\geq 1$ whenever $\{x_n\}_{n\geq 1}$ is a sequence in X such that $\{fx_n\}_{n\geq 1}$ and $\{gx_n\}_{n\geq 1}$ converge to some $t\in X$.*

Definition 2.2. *A pair of self mappings f and g on a metric space (X,d) are said to be weakly compatible if they commute at their coincidence points.*

Lemma 2.1. *Let f and g be compatible mappings from a metric space (X, d) into itself. Then*

(i) $ffa = gga = fga = gfa$ if $fa = ga$ for some $a \in X$;

(ii) $\{gfx_n\}_{n\geq 1}$ converges to fa if f is continuous and $\{fx_n\}_{n\geq 1}$ and $\{gx_n\}_{n\geq 1}$ converge to some $a \in X$.

Proof. Let $fa = ga$ for some $a \in X$. Take $t = fa$ and $x_n = a$ for $n \geq 1$. Since

$$\lim_{n\to\infty} d_k(fx_n, t) = \lim_{n\to\infty} d_k(gx_n, t) = d_k(fa, fa) = 0,\ k \geq 1,$$

it follows that

$$0 = \lim_{n\to\infty} d_k(fgx_n, gfx_n) = \lim_{n\to\infty} d_k(fga, gfa) = d_k(fga, gfa),\ k \geq 1.$$

Hence $gfa = fga$ and $ffa = fga = gfa = gga$.

Let f be continuous and $\{fx_n\}_{n\geq 1}$ and $\{gx_n\}_{n\geq 1}$ converge to some $a \in X$. By the compatiblity of f and g, and the continuity of f, we infer that for $k \geq 1$,

$$0 \leq d_k(gfx_n, fa) \leq d_k(gfx_n, fgx_n) + d_k(fgx_n, fa) \to 0$$

as $n \to \infty$. That is, $\lim_{n\to\infty} d_k(gfx_n, fa) = 0$ for $k \geq 1$. Therefore $\{gfx_n\}_{n\geq 1}$ converges to fa. This completes the proof.

Remark 2.1. It follows from Lemma 2.1 and the example in [10] that compatible mappings are weakly compatible, but the converse is not true.

Using the method of Singh and Mishra [10], we have the following

Theorem 2.1. *Let f, g, h and t be self mappings of a metric space (X, d) such that*

(i) $f(X) \subseteq h(X), g(X) \subseteq t(X)$;

(ii) f, h and g, t are weakly compatible;

(iii) one of f, g, h and t is continuous on X;

(iv) one of $f(X), g(X), h(X)$ and $t(X)$ is a complete subspace of X.

If there exists some $\varphi \in \Phi$ satisfying

$$\begin{aligned} d_k(fx, gy) \leq \varphi(\max\{&d_k(hx, ty), d_k(hx, fx), d_k(gy, ty), \\ &\frac{1}{2}[d_k(hx, gy), d_k(ty, fx)]\}) \end{aligned} \tag{2.1}$$

for $x, y \in X$ and $k \geq 1$, then f, g, h and t have a unique common fixed point in X.

Remark 2.2. Theorem 2.1 extends, improves and unifies Theorems 2.1 and 2.2 in [2] and Theorem 2.1 in [8].

3. Applications

In this section, let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_1)$ be real Banach spaces, $S \subseteq X$ be the state space, and $D \subseteq Y$ be the decision space. Denote by $BB(S)$ the set of all real-value mappings on S that are bounded on bounded subsets of S. It is easy to verify that $BB(S)$ is a linear space over $\mathbb{R}$ under usual definitions of addition and multiplication by scalars. For $k \geq 1$ and $a, b \in BB(S)$, let

$$d_k(a,b) = \sup\{|a(x) - b(x)| : x \in \overline{B}(0,k)\},$$
$$d(a,b) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(a,b)}{1 + d_k(a,b)},$$

where $\overline{B}(0,k) = \{x : x \in S \text{ and } \|x\| \leq k\}$. Clearly, $\{d_k\}_{k\geq 1}$ is a countable family of pseudometrics on $BB(S)$ and $(BB(S), d)$ is a complete metric space.

Theorem 3.1. *Let $u : S \times D \to S, T : S \times D \to S$ and $H_1, H_2, H_3, H_4 : S \times D \times \mathbb{R} \to \mathbb{R}$ satisfy that for given $k \geq 1$ and $a \in BB(S)$, there exist $p(k,a) > 0$ with*

$$|u(x,y)| + |H_i(x,y,a(T(x,y)))| \leq p(k,a), \ (x,y) \in \overline{B}(0,k) \times D, \ i \in \{1,2,3,4\}. \tag{3.1}$$

Suppose that there exists $\varphi \in \Phi$ such that

$$\begin{aligned} &|H_1(x,y,a(t)) - H_2(x,y,b(t))| \\ &\leq \varphi(\max\{d_k(f_3a, f_4b), d_k(f_1a, f_3a), d_k(f_2b, f_4b), \\ &\qquad \frac{1}{2}[d_k(f_3a, f_2b) + d_k(f_1a, f_4b)]\}) \end{aligned} \tag{3.2}$$

for $k \geq 1, (x,y,t) \in \overline{B}(0,k) \times D \times S$ and $a, b \in BB(S)$, where f_1, f_2, f_3 and f_4 are defined as follows:

$$\begin{aligned} f_1a(x) &= \inf_{y\in D}\{u(x,y) + H_1(x,y,a(T(x,y)))\}, \\ f_2a(x) &= \inf_{y\in D}\{u(x,y) + H_2(x,y,a(T(x,y)))\}, \\ f_3a(x) &= \inf_{y\in D}\{u(x,y) + H_3(x,y,a(T(x,y)))\}, \\ f_4a(x) &= \inf_{y\in D}\{u(x,y) + H_4(x,y,a(T(x,y)))\} \end{aligned} \tag{3.3}$$

for $x \in S$ and $a \in BB(S)$. Suppose that
(i) for any $\{a_n\}_{n\geq 1} \subset BB(S), k \geq 1$ and $x \in \overline{B}(0,k)$, if there exist $a, b \in$

$BB(S)$ such that

$$\lim_{n\to\infty}\sup_{x\in S}|f_1a_n(x)-a(x)| = \lim_{n\to\infty}\sup_{x\in S}|f_3a_n(x)-a(x)| = 0,$$
$$\lim_{n\to\infty}\sup_{x\in S}|f_2a_n(x)-b(x)| = \lim_{n\to\infty}\sup_{x\in S}|f_4a_n(x)-b(x)| = 0,$$

then

$$\lim_{n\to\infty}\sup_{x\in S}|f_if_{i+2}a_n(x)-f_{i+2}f_ia_n(x)| = 0,\ i=1,2;$$

(ii) one of $f_1(BB(S)), f_2(BB(S)), f_3(BB(S))$ and $f_4(BB(S))$ is a complete subspace of $BB(S)$;
(iii) there exists some $f_i \in \{f_1,f_2,f_3,f_4\}$ such that for any sequence $\{a_n\}_{n\geq 1} \subset BB(S)$ and $a \in BB(S)$,

$$\lim_{n\to\infty} d_k(a_n,a)=0 \Rightarrow \lim_{n\to\infty} d_k(f_ia_n,fa)=0,\ k\geq 1;$$

(iv) $f_1(BB(S)) \subseteq f_3(BB(S)), f_2(BB(S)) \subseteq f_4(BB(S))$.
Then the system of functional equations

$$\begin{aligned} f_1(x) &= \inf_{y\in D}\{u(x,y)+H_1(x,y,f_1(T(x,y)))\},\ x\in S,\\ f_2(x) &= \inf_{y\in D}\{u(x,y)+H_2(x,y,f_2(T(x,y)))\},\ x\in S,\\ f_3(x) &= \inf_{y\in D}\{u(x,y)+H_3(x,y,f_3(T(x,y)))\},\ x\in S,\\ f_4(x) &= \inf_{y\in D}\{u(x,y)+H_4(x,y,f_4(T(x,y)))\},\ x\in S \end{aligned} \tag{3.4}$$

possesses a unique common solution in $BB(S)$.

Proof. Given $k \geq 1$ and $a \in BB(S)$. It follows from (3.1) that there exists $p(k,a) > 0$ with

$$\begin{aligned} |f_ia(x)| &\leq |u(x,y)| + |H_i(x,y,a(T(x,y)))|\\ &\leq p(k,a), \quad (x,y)\in \overline{B}(0,k)\times D,\ i\in\{1,2,3,4\}, \end{aligned}$$

which implies that f_1, f_2, f_3 and f_4 map $BB(S)$ into itself. Clearly, (i) means that f_1, f_3 and f_2, f_4 are compatible, and (iii) means that f_i is continuous.

Let $a, b \in BB(S),\ k \geq 1, x \in \overline{B}(0,k)$ and $\varepsilon > 0$. Using (3.3), we deduce that there exist $y, z \in D$ such that

$$f_1a(x) > u(x,y) + H_1(x,y,a(T(x,y))) - \varepsilon, \tag{3.5}$$
$$f_2b(x) > u(x,z) + H_2(x,z,b(T(x,z))) - \varepsilon. \tag{3.6}$$

It is easy to see that

$$f_1a(x) \le u(x,z) + H_1(x,z,a(T(x,z))), \tag{3.7}$$
$$f_2b(x) \le u(x,y) + H_2(x,y,b(T(x,y))). \tag{3.8}$$

By virtue of (3.6), (3.7) and (3.2), we infer that

$$\begin{aligned} f_1a(x) - f_2b(x) &< H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z))) + \varepsilon \\ &\le |H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))| + \varepsilon \\ &\le \varepsilon + \varphi(\max\{d_k(f_3a, f_4b), d_k(f_1a, f_3a), d_k(f_2b, f_4b), \\ &\qquad \frac{1}{2}[d_k(f_3a, f_2b) + d_k(f_1a, f_4b)]\}). \end{aligned} \tag{3.9}$$

From (3.5), (3.8) and (3.2) we conclude that

$$\begin{aligned} f_1a(x) - f_2b(x) &> H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y))) - \varepsilon \\ &\ge -|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))| - \varepsilon \\ &\ge -\varepsilon - \varphi(\max\{d_k(f_3a, f_4b), d_k(f_1a, f_3a), d_k(f_2b, f_4b), \\ &\qquad \frac{1}{2}[d_k(f_3a, f_2b) + d_k(f_1a, f_4b)]\}). \end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned} |f_1a(x) - f_2b(x)| \le \varepsilon + \varphi(\max\{&d_k(f_3a, f_4b), d_k(f_1a, f_3a), d_k(f_2b, f_4b), \\ &\frac{1}{2}[d_k(f_3a, f_2b) + d_k(f_1a, f_4b)]\}), \end{aligned}$$

which gives that

$$\begin{aligned} d_k(f_1a, f_2b) \le \varepsilon + \varphi(\max\{&d_k(f_3a, f_4b), d_k(f_1a, f_3a), d_k(f_2b, f_4b), \\ &\frac{1}{2}[d_k(f_3a, f_2b) + d_k(f_1a, f_4b)]\}), \end{aligned}$$

letting $\varepsilon \to 0$ in the above inequality, we have

$$\begin{aligned} d_k(f_1a, f_2b) \le \varphi(\max\{&d_k(f_3a, f_4b), d_k(f_1a, f_3a), d_k(f_2b, f_4b), \\ &\frac{1}{2}[d_k(f_3a, f_2b) + d_k(f_1a, f_4b)]\}). \end{aligned}$$

Theorem 2.1 and Lemma 2.1 ensure that f_1, f_2, f_3 and f_4 have a unique common fixed point $w \in BB(S)$. That is, $w(x)$ is a unique common solution of the system of functional equations (3.4). This completes the proof.

Remark 3.1. Theorems 3.1-3.4 in [2] are special cases of Theorem 3.1 in this paper.

Theorem 3.2. *Let* u, T, H_1, H_2, H_3 *and* H_4 *satisfy* (3.1), (3.2) *and* $(i)-(iv)$, *where* f_1, f_2, f_3 *and* f_4 *are defined as follows:*

$$\begin{aligned}
f_1a(x) &= \sup_{y\in D}\{u(x,y) + H_1(x,y,a(T(x,y)))\},\\
f_2a(x) &= \sup_{y\in D}\{u(x,y) + H_2(x,y,a(T(x,y)))\},\\
f_3a(x) &= \sup_{y\in D}\{u(x,y) + H_3(x,y,a(T(x,y)))\},\\
f_4a(x) &= \sup_{y\in D}\{u(x,y) + H_4(x,y,a(T(x,y)))\}
\end{aligned} \tag{3.11}$$

for $x \in S$ *and* $a \in BB(S)$. *Then the system of functional equations*

$$\begin{aligned}
f_1(x) &= \sup_{y\in D}\{u(x,y) + H_1(x,y,f_1(T(x,y)))\},\ x \in S,\\
f_2(x) &= \sup_{y\in D}\{u(x,y) + H_2(x,y,f_2(T(x,y)))\},\ x \in S,\\
f_3(x) &= \sup_{y\in D}\{u(x,y) + H_3(x,y,f_3(T(x,y)))\},\ x \in S,\\
f_4(x) &= \sup_{y\in D}\{u(x,y) + H_4(x,y,f_4(T(x,y)))\},\ x \in S
\end{aligned} \tag{3.12}$$

possesses a unique common solution in $BB(S)$.

Proof. As in the proof of Theorem 3.1, we easily conclude that f_1, f_2, f_3 and f_4 map $BB(S)$ into itself, f_1, f_3 and f_2, f_4 are compatible, and f_i is continuous.

Let $a, b \in BB(S), k \geq 1, x \in \overline{B}(0,k)$ and $\varepsilon > 0$. By virtue of (3.11), we know that there exist $y, z \in D$ such that

$$\begin{aligned}
f_1a(x) &< u(x,y) + H_1(x,y,a(T(x,y))) + \varepsilon,\\
f_2b(x) &< u(x,z) + H_2(x,z,b(T(x,z))) + \varepsilon,\\
f_1a(x) &\geq u(x,z) + H_1(x,z,a(T(x,z))),\\
f_2b(x) &\geq u(x,y) + H_2(x,y,b(T(x,y))).
\end{aligned} \tag{3.13}$$

It follows from (3.2) and (3.13) that

$$\begin{aligned}
|f_1a(x) - f_2b(x)| &\leq \varepsilon + \max\{|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))|,\\
&\qquad |H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))|\}\\
&\leq \varepsilon + \varphi(\max\{d_k(f_3a, f_4b), d_k(f_1a, f_3a), d_k(f_2b, f_4b),\\
&\qquad \frac{1}{2}[d_k(f_3a, f_2b) + d_k(f_1a, f_4b)]\}).
\end{aligned}$$

The rest of the proof follows precisely as in the proof of Theorem 3.1. This completes the proof.

From Theorems 3.1 and 3.2, we have

Theorem 3.3. *Let* u, T, H_1, H_2, H_3 *and* H_4 *satisfy* (3.1), (3.2) *and* $(i) - (iv)$, *where* f_1, f_2, f_3 *and* f_4 *are defined as follows:*

$$\begin{aligned} f_1a(x) &= opt_{y\in D}\{u(x,y) + H_1(x,y,a(T(x,y)))\}, \\ f_2a(x) &= opt_{y\in D}\{u(x,y) + H_2(x,y,a(T(x,y)))\}, \\ f_3a(x) &= opt_{y\in D}\{u(x,y) + H_3(x,y,a(T(x,y)))\}, \\ f_4a(x) &= opt_{y\in D}\{u(x,y) + H_4(x,y,a(T(x,y)))\} \end{aligned} \tag{3.14}$$

for $x \in S$ *and* $a \in BB(S)$. *Then the system of functional equations* (1.1) *possesses a unique common solution in* $BB(S)$.

Taking $f_3 = f_4 =$the identity mapping in Theorem 3.3, we have

Theorem 3.4. *Let* $u : S\times D \to S, T : S\times D \to S$ *and* $H_1, H_2 : S\times D\times\mathbb{R} \to \mathbb{R}$ *satisfy that for given* $k \geq 1$,

$$|u(x,y)| + |H_i(x,y,a(T(x,y)))| \leq p(k,a),\ \ (x,y) \in \overline{B}(0,k) \times D, i \in \{1,2\} \tag{3.15}$$

Suppose that there exists $\varphi \in \Phi$ *such that*

$$\begin{aligned} &|H_1(x,y,a(t) - H_2(x,y,b(t)| \\ &\leq \varphi(\max\{d_k(a,b), d_k(f_1a,a), d_k(f_2b,b), \frac{1}{2}[d_k(a,f_2b) + d_k(f_1a,b)]\}) \end{aligned} \tag{3.16}$$

for $k \geq 1, (x,y,t) \in \overline{B}(0,k) \times D \times S$ *and* $a, b \in BB(S)$, *where* f_1 *and* f_2 *are defined as follows:*

$$\begin{aligned} f_1a(x) &= opt_{y\in D}\{u(x,y) + H_1(x,y,a(T(x,y)))\}, \\ f_2a(x) &= opt_{y\in D}\{u(x,y) + H_2(x,y,a(T(x,y)))\} \end{aligned} \tag{3.17}$$

for $x \in S$ *and* $a \in BB(S)$.

Then the system of functional equations

$$\begin{aligned} f_1(x) &= opt_{y\in D}\{u(x,y) + H_1(x,y,f_1(T(x,y)))\},\ \ x \in S, \\ f_2(x) &= opt_{y\in D}\{u(x,y) + H_2(x,y,f_2(T(x,y)))\},\ \ x \in S \end{aligned} \tag{3.18}$$

possesses a unique common solution in $BB(S)$.

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