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SOME NEW CLASSES OF NONCONVEX FUNCTIONS

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Abstract. In this paper, we introduce and study a new class of convex functions, which is called the φ-convex function. These functions are nonconvex functions and include the convex functions as special cases. We study some properties of φ-convex functions. It is shown that the minimum of the φ-convex functions on the φ-convex set can be characterized by a class of variational inequalities, which is called the φ-directional variational inequalities. Some open problems are suggested for further research.

1. Introduction

In recent years, several extensions and generalizations have been considered for classical convexity. In this paper, we consider a new class of convex functions which is called the φ-convex functions. These new classes of functions include the convex functions as special cases. We also define the nonconvex set, which is called the φ-convex set. Several new concepts are defined and their properties have been studied. We prove that the minimum of the differentiable φ-convex functions on the φ-convex set can be characterized by a class of variational inequalities. In order to convey the flavour of these new concepts, we have tried to emphasize the basic characteristic of these new classes of nonconvex functions. Some basic properties of these nonconvex functions along with some open problems are discussed.

2. Preliminaries

Let K be a nonempty closed set in a normed space H. We denote by $\langle \cdot, \cdot \rangle$ and $\|.\|$ be the inner product and norm respectively. Let $\varphi : K \longrightarrow R$ be a continuous function.

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Definition 2.1. Let $u \in K$. The set K is said to be φ-convex set, if there exists a function φ such that

$$u + te^{i\varphi}(v-u) \in K, \quad \forall u, v \in K, \quad t \in [0,1].$$

Clearly for for $\varphi = 0$, the set K is a convex set. However, the φ-convex set K is not a convex set.

From now onward, the set K is a φ-convex set, unless otherwise specified.

Definition 2.2. The function f on K is called φ-convex function, if and only if, there exists a function φ such that

$$f(u + te^{i\varphi}(v-u)) \leq (1-t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0,1].$$

Obviously every convex function with $\varphi = 0$ is a φ-convex function, but the converse is not true. Also for $t = 1$, the φ-convex function reduces to:

$$f(u + e^{i\varphi}(v-u)) \leq f(v), \quad \forall u, v \in K.$$

Definition 2.3. The function f on K is said to be quasi φ-convex, if there exists a function φ such that

$$f(u + te^{i\varphi}(v-u)) \leq max\{f(u), f(v)\}, \quad \forall u, v \in K, \quad t \in [0,1].$$

Definition 2.4. The function f on K is said to be logarithmic φ-convex, if there exists a function φ such that

$$f(u + te^{i\varphi}(v-u)) \leq (f(u))^{1-t}(f(v))^{t}, \quad \forall u, v \in K, \quad t \in [0,1],$$

where $f(.) > 0$.

Lemma 2.1. *Let f be a φ-convex function. Then any local minimum of f on K is a global minimum.*

Proof. Let the φ-convex function f have a local minimum at $u \in K$. Assume the contrary, that is, $f(v) < f(u)$ for some $v \in K$. Since f is a φ-convex function, so

$$f(u + te^{i\varphi}(v-u)) \leq f(u) + t(f(v) - f(u)),$$

which implies that

$$f(u + te^{i\varphi}(v-u)) - f(u) < 0,$$

for arbitrary small $t > 0$, contradicting the local minimum. □

Essentially using the technique and ideas of the classical convexity, one can easily prove the following results.

Theorem 2.1. *If f is a φ-convex function on K, then the level set $L_\alpha = \{u \in K : f(u) \leq \alpha,\ \alpha \in R\}$ is a φ-convex set set with respect to φ.*

Theorem 2.2. *The function f is a φ-convex function if and only if $epi(f) = \{(u, \alpha) : u \in K, \alpha \in R,\ f(u) \leq \alpha\}$ is a φ-convex set set with respect to φ.*

Theorem 2.3. *The function f is a quasi φ-convex function if and only if the level set $L_\alpha = \{u \in K, \alpha \in R : f(u) \leq \alpha\}$ is a φ-convex set with respect to φ.*

Definition 2.5. The function f is said to be a pseudo φ-convex function with respect to φ, if there exists a strictly positive function $b(\cdot, \cdot)$ such that

$$\begin{aligned} f(v) < f(u) \Longrightarrow &f(u + te^{i\varphi}(v-u)) \\ &\leq f(u) + t(t-1)b(u,v), \quad \forall u, v \in K, t \in (0,1). \end{aligned}$$

Theorem 2.4. *If the function f is a φ-convex function with respect to φ, then f is pseudo φ-connvex function with respect to φ.*

Proof. Without loss of generality, we assume that $f(v) < f(u), \quad \forall u, v \in K$. For every $t \in [0,1]$, we have

$$\begin{aligned} F(u + te^{i\varphi}(v-u)) &\leq (1-t)f(u) + tf(v) \\ &< f(u) + t(t-1)\{f(u) - f(v)\} \\ &= f(u) + t(t-1)b(v,u), \end{aligned}$$

where $b(v,u) = f(v) - f(u) > 0$.

Thus it follows that the function f a is pseudo φ-convex function with respect to φ, the required result. □

Theorem 2.5. *Let f be a φ-convex function with respect to φ. If $\phi : L \longrightarrow R$ is a nondecreasing function, then $\phi \circ f$ is a φ-convex function with respect to the function φ.*

Proof. Since f is a φ-convex function and ϕ is decreasing, we have, $\forall u, v \in K$ and $t \in [0,1]$

$$\begin{aligned} \phi \circ f(u + te^{i\varphi}(v-u)) &\leq \phi[f(u + te^{i\varphi}(v-u))] \\ &\leq \phi[(1-t)f(u) + tf(v)] \\ &\leq (1-t)\phi \circ f(u) + \phi \circ f(v), \end{aligned}$$

from which it follows that $\phi \circ f$ is a φ-convex function with respect to φ. □

3. Main Results

Definition 3.1. We define the φ-directional derivative of f at a point $u \in K$ in the direction $v \in K$ by

$$D_{\varphi} f(u,v) := f'_{\varphi}(u;v) = f_{\varphi,u}(u;v) = \lim_{t \longrightarrow 0^+} \frac{f(u + te^{i\varphi} v) - f(u)}{t}.$$

Note that for $\varphi = 0$, the φ-directional derivative of f at u in the direction v coincides with the usual directional derivative of f at u in a direction v given by

$$Df(u:v) = f'(u;v) = \lim_{t \longrightarrow 0^+} \frac{f(u+tv) - f(u)}{t}.$$

It is well known that the function $v \longrightarrow f'_{\varphi}(u;v)$ is subadditive, positively homogeneous and $|f'_{\varphi}(u;v)| \leq \|v\|$. For the applications of the φ-directional derivatives, see [1].

Definition 3.2. The differentiable function f on K is said to be φ-invex, if

$$f(v) - f(u) \geq f'_{\varphi}(u; v-u), \quad \forall u, v \in K,$$

where $f'_{\varphi}(u;v)$ is the φ-directional derivative of f at $u \in K$ in the direction of $v \in K$.

Theorem 3.1. *Let f be a differentiable φ-convex function on K. Then the function $v \longrightarrow f'_{\varphi}(u;v)$ is positively homogeneous and φ-convex.*

Proof. It is follows from the definition of the φ-directional derivative that $f'_{\varphi}(u;\lambda v) = \lambda f'_{\varphi}(u;v)$, whenever $v \in K$ and $\lambda \geq 0$, hence the function $v \longrightarrow f'_{\varphi}(u;v)$ is positively homogeneous.

To prove the φ-convexity of the function $v \longrightarrow f'_{\varphi}(u;v)$, we consider , $\forall u, v, z \in K, t \geq 0, \quad \lambda \in (0,1)$,

$$\begin{aligned}
&\frac{1}{t}[f(u + te^{i\varphi}(\lambda v + (1-\lambda)z) - f(u)] \\
&\qquad = \frac{1}{t}\left[f(\lambda(u + te^{i\varphi}v) + (1-\lambda)(u + te^{i\varphi}z) - f(u)\right] \\
&\qquad \leq \frac{1}{t}\left[\lambda f(u + te^{\varphi}v) + (1-\lambda)(u + te^{i\varphi}z) - f(u)\right] \\
&\qquad = \lambda \frac{f(u + te^{i\varphi}v) - f(u)}{t} + (1-\lambda)\frac{f(u + te^{i\varphi}z) - f(u)}{t}.
\end{aligned} \tag{3.1}$$

Taking the limit as $t \longrightarrow 0^+$ in (3.1), we have

$$f'_\varphi(u; \lambda v + (1-\lambda)z) \leq \lambda f'_\varphi(u; v) + (1-\lambda) f'_\varphi(u; z),$$

which shows that the function $v \longrightarrow f'_\varphi(u; v)$ is φ-convex, the required result. $\square$

For $\varphi = 0$, the φ-convex function f becomes the convex function and the φ-convex set K is a convex set. Consequently Theorem 3.1 reduces to the well known result in convexity, (see [3]).

Theorem 3.2. *Let K be a φ-convex set. If function $f : K \longrightarrow R$ is a differentiable φ-convex function, then*
(i). *f is φ-invex.*
(ii). *φ-directional derivative $f'_\varphi(\cdot;\cdot)$ of f is monotone, that is,*

$$f'_\varphi(u : v - u) + f'_\varphi(v; u - v) \leq 0, \quad \forall u, v \in K.$$

Proof. We first prove (i).
Let f be a φ-convex function. Then

$$f(u + te^{i\varphi}(v - u)) \leq f(u) + t\{f(v) - f(u)\}, \quad \forall u, v \in K, t \in [0, 1],$$

which can be written as

$$f(v) - f(u) \geq \frac{f(u + te^{i\varphi}(v - u)) - f(u)}{t}. \tag{3.2}$$

Taking the limit as $t \longrightarrow 0^+$ in (3.2), we have

$$f(v) - f(u) \geq f'_\varphi(u; v - u), \quad \forall u, v \in K, \tag{3.3}$$

showing that the φ-convex function f is a φ-invex function.
Now we prove (ii).
Changing the role of u and v in (3.3), we have

$$f(u) - f(v) \geq f'_\varphi(v; u - v), \quad \forall u, v \in K, \tag{3.4}$$

Adding (3.3) and (3.4), we have

$$f'_\varphi(u : v - u) + f'_\varphi(v; u - v) \leq 0, \quad \forall u, v \in K,$$

which shows that the φ-directional derivative $f'_\varphi(\cdot;\cdot)$ of f is monotone. $\square$

Remark 3.1. The converse of Theorem 3.1 is not true. It is an interesting open problem. Note that the φ-directional derivative of the φ-convex function is positively homogeneous.

Definition 3.3. A differentiable function f on K is said to be a pseudo φ-invex function, if

$$f'_{\varphi}(u; v-u) \geq 0, \quad \Longrightarrow \quad f(v) - f(u) \geq 0, \quad \forall u, v \in K.$$

Definition 3.4. A differentiable function f on K is said to be quasi φ-invex, if and only if,

$$f(v) \leq f(u) \quad \Longrightarrow \quad f'_{\varphi}(u; v-u) \leq 0, \quad \forall u, v \in K.$$

Theorem 3.3. *Let f be a differentiable φ-convex function on K. Then the minimum $u \in K$ of f on K can be characterized by the inequality:*

$$f'_{\varphi}(u; v-u) \geq 0, \quad \forall v \in K. \tag{3.5}$$

Proof. Let $u \in K$ be a minimum of the φ-convex function f. Then

$$f(u) \leq f(v), \quad \forall v \in K. \tag{3.6}$$

Since K is a φ-convex set, so, $\forall u, v, \varphi \in K, t \in [0,1], \quad v_t = u + te^{i\varphi}(v-u) \in K$. Taking $v = v_t$ in (3.6), we have

$$f(u) \leq f(v_t) = f(u + te^{i\varphi}(v-u)),$$

which implies that

$$\frac{f(u + te^{i\varphi}(v-u)) - f(u)}{t} \geq 0.$$

Taking the limit as $t \longrightarrow 0^+$ in the above inequality, we have

$$f'_{\varphi}(u; v-u) \geq 0, \quad \forall v \in K,$$

the required (3.5).

Conversely, let $u \in K$ be a solution of (3.5). Since f is a φ-convex function, it follows, using (3.5), that

$$f(v) - f(u) \geq f'_{\varphi}(u; v-u) \geq 0,$$

which implies that

$$f(u) \leq f(v), \quad \forall v \in K,$$

showing that $u \in K$ is the minimum of the φ-convex function f, the required result. □

The inequality of the type (3.5) is called the φ-directional variational inequality. For $\varphi = 0$, problem (3.5) reduces to the directional variational inequalities. It is worth mentioning that even the directional variational inequalities have not been studied in the literature.

Remark 3.2. If the φ-directional derivative of f is pseudomonotone, then the φ-directional variational inequality is equivalent to finding $u \in K$ such that

$$-f'_{\varphi}(v; u-v) \geq 0, \quad \forall v \in K, \tag{3.7}$$

which is known as the Minty φ-directional variational inequality. Note that (3.5) $\Longrightarrow$ (3.7), but not conversely even $f'_{\varphi}(\cdot;\cdot)$ is hemicontinuous, since $f'_{\varphi}(\cdot;\cdot)$ is positively homogeneous. See also Noor [2].

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