Nonlinear Funct. Anal. & Appl., Vol. 11, No. 4 (2006), pp. 655–658

COMPACTNESS OF EMBEDDINGS

A. G. Ramm

Abstract. An improvement of the author's result, proved in 1961, concerning necessary and sufficient conditions for the compactness of embedding operators is given. A discussion of the necessity of the compatibility of the norms of the Banach spaces X_2 and X_3, where $X_2 \subset X_3$, is given. The injectivity of the embedding operator $J : X_2 \to X_3$ implies this compatibility.

1. Introduction

The basic result of this note is:

Theorem 1. *Let $X_1 \subset X_2 \subset X_3$ be Banach spaces, $||u||_1 \geq ||u||_2 \geq ||u||_3$, i.e., the norms in X_1, X_2 and X_3 are comparable. Assume that the norms in X_2 and X_3 are compatible, that is, if $\lim_{n\to\infty} ||u_n||_3 = 0$ and u_n is fundamental in X_2, then $\lim_{n\to\infty} ||u_n||_2 = 0$.*

Under these assumptions, the embedding operator $i : X_1 \to X_2$ is compact if and only if the following two conditions are valid:

a) The embedding operator $j : X_1 \to X_3$ is compact,

and

b) The following inequality holds:

$||u||_2 \leq s||u||_1 + c(s)||u||_3$, $\forall u \in X_1$, $\forall s \in (0,1)$, where $c(s) > 0$ is a constant.

This result is an improvement of the author's old result, originally proved in 1961 and published in [2], where X_2 was assumed to be a Hilbert space. The proof of Theorem 1 is simpler than the one in [2]. This proof is borrowed from the recent paper [3]. In addition to this proof, we give a discussion of the role of the compatibility of the norms of X_2 and X_3. This is done in Remark 1, following the proof of Theorem 1. The compatibility of the norms in X_2

Received December 13, 2005.

2000 Mathematics Subject Classification: 46B50, 46E30, 47B07.

Key words and phrases: Banach spaces, compactness, embedding operator.

and X_3 is often assumed implicitly when one writes $X_2 \subset X_3$. We discuss this point in Remark 1.

PROOF OF THEOREM 1.

1. *Proof of the sufficiency of conditions a) and b) for the compactness of the embedding operator* $i : X_1 \to X_2$.

Assuming that a) and b) hold, let us prove the compactness of the embedding operator i. Let $S = \{u : u \in X_1, ||u||_1 = 1\}$ be the unit sphere in X_1. Using assumption a), select a sequence u_n which converges in X_3. We claim that this sequence converges also in X_2. Indeed, since $||u_n||_1 = 1$, one uses assumption b) to get

$$||u_n - u_m||_2 \leq s||u_n - u_m||_1 + c(s)||u_n - u_m||_3 \leq 2s + c(s)||u_n - u_m||_3.$$

Let $\eta > 0$ be an arbitrary small given number. Choose $s > 0$ such that $2s < \frac{1}{2}\eta$, and for a fixed s choose n and m so large that $c(s)||u_n - u_m||_3 < \frac{1}{2}\eta$. This is possible because the sequence u_n converges in X_3. Consequently, $||u_n - u_m||_2 \leq \eta$ if n and m are sufficiently large. This means that the sequence u_n converges in X_2. Thus, the embedding $i : X_1 \to X_2$ is compact. The sufficiency part is proved. $\square$

In the above argument, i.e., in the proof of the sufficiency, the compatibility of the norms in X_2 and X_3 was not used.

2. *Proof of the necessity of the compactness of the embedding operator* $i : X_1 \to X_2$ *for the conditions a) and b) to hold.*

Assume now that i is compact. Let us prove that conditions a) and b) hold. In the proof of the necessity of these conditions the assumption about the compatibility of the norms of X_2 and X_3 is used essentially. This assumption is satisfied if one assumes that the embedding operator $J : X_2 \to X_3$ is linear, injective, and bounded.

If i is compact, then assumption a) holds because $||u||_2 \geq ||u||_3$.

Suppose that assumption b) fails. Then there is a sequence u_n and a number $s_0 > 0$ such that $||u_n||_1 = 1$ and

$$||u_n||_2 \geq s_0 + n||u_n||_3. \tag{1}$$

If the embedding operator i is compact and $||u_n||_1 = 1$, then one may assume that the sequence u_n converges in X_2. Its limit cannot be equal to zero, because, by (1), $||u_n||_2 \geq s_0 > 0$. The sequence u_n converges in X_3 because $||u_n - u_m||_2 \geq ||u_n - u_m||_3$ and the sequence u_n converges in X_2. *Its limit* *in X_3 is not zero, because the norms in X_3 and in X_2 are compatible.* Thus, $\lim_{n\to\infty} ||u_n||_3 > 0$. On the other hand, inequality (1) implies $||u_n||_3 = O(\frac{1}{n})$ as $n \to \infty$, so $\lim_{n\to\infty} ||u_n||_3 = 0$. This contradiction proves that condition b) holds. The necessity part is proved. This completes the proof. $\Box$

Remark 1. *In* [1], p. 35, the following claim is stated:

Claim ([1], p.35).
Let $X_1 \subset X_2 \subset X_3$ be three Banach spaces. Suppose the embedding $X_1 \to X_2$ is compact. Then given any $\epsilon > 0$, there is a $K(\epsilon) > 0$, such that $||u||_2 \leq \epsilon||u||_1 + K(\epsilon)||u||_3$ for all $u \in X_1$.

This **Claim** preassumes implicitly the compatibility of the norms of X_2 and X_3 because the inclusion $X_2 \subset X_3$ often preassumes the linearity, boundedness, and injectivity of the embedding operator $J : X_2 \to X_3$.

Namely, assume that $||u_n||_3 \to 0$ and $||u_n - z||_2 \to 0$, denote the embedding operator from X_2 into X_3 by J, and assume this operator linear, injective, and bounded. Then, using the inequality $||u||_3 \leq ||u||_2$, one gets:

$$\lim_{n\to\infty} ||J(u_n - z)||_3 \leq \lim_{n\to\infty} ||u_n - z||_2 = 0,$$

and

$$\lim_{n\to\infty} ||Ju_n||_3 = 0.$$

Therefore,

$$0 = \lim_{n\to\infty} ||J(u_n - z)||_3 = \lim_{n\to\infty} ||Ju_n - Jz||_3 = ||Jz||_3,$$

so $Jz = 0$. This implies that $z = 0$, if one assumes J injective. Thus, the two relations: $\lim_{n\to\infty} ||u_n||_3 = 0$ and $\lim_{n\to\infty} ||u_n - z||_2 = 0$, imply that $z = 0$. Therefore the norms in X_2 and X_3 are compatible.

Consider an example of non-compatible but comparable norms.

Let $L^2(0,1)$ be the Lebesgue space of square integrable functions, $X_3 = L^2(0,1)$, and X_2 be the Banach space of $L^2(0,1)$ functions with a finite value at a fixed point $y \in [0,1]$ and with the norm

$$||u||_2 := ||u||_{L^2(0,1)} + |u(y)| = ||u||_3 + |u(y)|.$$

The elements of $L^2(0,1)$ are classes of equivalence of functions whose representatives are defined almost everywhere and not at any point. One can choose a representative u, which has any desired value at a given point $y \in [0,1]$.

This does not change u as an element of $L^2(0,1)$. The space X_2 is complete because X_3 is complete and the one-dimensional space, consisting of numbers $u(y)$ with the usual norm $|u(y)|$, is complete. A function $u_0(x) = 0$ for $x \neq y$ and $u_0(y) = 1$ has the properties

$$||u_0||_3 = 0, \quad ||u_0||_2 = 1.$$

The norms in X_2 and X_3 are *comparable*, i.e., $||u||_3 \leq ||u||_2$. However, these norms are *not compatible*: there is a convergent to zero sequence $\lim_{n\to\infty} u_n = 0$ in X_3 which does not converge to zero in X_2, for example, $\lim_{n\to\infty} ||u_n||_2 = 1$ in X_2. For instance, one may take $u_n(x) = u_0(x)$ for all $n = 1, 2, \dots,$ and an arbitrary fixed $y \in [0,1]$. Then $||u_n||_2 = 1$ and $||u_n||_3 = 0$, $\lim_{n\to\infty} ||u_n||_2 = 1$ and $\lim_{n\to\infty} ||u_n||_3 = 0$. The sequence u_n converges to zero in X_3 and to a non-zero element u_0 in X_2. In this case inequality (1) holds for any fixed $s_0 \in (0,1)$ and any n, but the contradiction, which was used in the proof of the necessity in Theorem 1, can not be obtained because $||u_n||_3 = 0$ for all n.

References

1. M. Berger, *Nonlinearity and Functional Analysis*, Acad. Press, New York, 1977.
2. A. G. Ramm, *A necessary and sufficient condition for compactness of embedding*, Vestnik of Leningrad. Univ., Ser. Math., Mech., Astron. **N1** (1963), 150-151. (Russian)
3. A. G. Ramm, *Necessary and sufficient condition for compactness of the embedding operator*, Jour. of Ineq. in Pure and Appl. Math. (JIPAM) **6** (2005), no. 5, Article 130.

A. G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602, USA
E-mail address: ramm@math.ksu.edu