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PERIODIC SOLUTIONS OF NONAUTONOMOUS DELAY RAYLEIGH EQUATIONS

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Abstract. By the Mawhin's continuation theorem and a sharp inequality for periodic functions, existence criteria of periodic solutions for non-autonomous Rayleigh equations with deviating arguments are obtained. In the special case when the Rayleigh equations are autonmomous, our results are still better than some of the recent results.

1. Introduction

Periodic solutions of the Rayleigh equations without delays have been the subject of many investigations (see, e.g. [1-12]), while those with delays are relatively scarce. The authors in [8] studied the existence of periodic solutions of the autonomous Rayleigh equations with a deviating argument of the form

$$x''(t) + f(x'(t)) + g(x(t - \tau(t))) = p(t), \tag{1}$$

where f, g, τ and p are real continuous functions defined on R such that $f(0) = 0$, τ and p are 2π-periodic and $\int_0^{2\pi} p(t)\,dt = 0$. The existence of 2π-periodic solutions of (1) is established under relatively simple conditions on f and g. Later in [6], Lu et al. improve and extend the results in [8]. The results in [6] can be rewritten as follows.

Theorem I. ([6]) *Suppose there exist constants $K > 0$, $D > 0$, $r_1 > 0$ and $r \geqslant 0$ such that*

(i) $|f(x)| \leqslant r_1 |x| + K$ for $x \in R$,

(ii) $xg(x) > 0$ and $|g(x)| > r_1 |x| + K$ for $|x| > D$, and

(iii) $\lim_{x\to-\infty} \frac{g(x)}{x} \le r$ (or $\lim_{x\to+\infty} \frac{g(x)}{x} \le r$).

Then for $4\pi[r_1 + (2\pi + 1)r] < 1$, (1) has a 2π-periodic solution.

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Theorem II. ([6]) *Suppose there exist constants* $K > 0$, $D > 0$ *and* $r \geqslant 0$ *such that*

(i) $\mid f(x) \mid \leqslant K$ *for* $x \in R$,

(ii) $xg(x) > 0$ *and* $\mid g(x) \mid > K$ *for* $|x| > D$, *and*

(iii) $\lim_{x\to-\infty} \frac{g(x)}{x} \leq r$ *(or* $\lim_{x\to+\infty} \frac{g(x)}{x} \leq r$*).*

Then for $4\pi\left[r_1 + (2\pi+1)r\right] < 1$, *(1) has a* 2π*-periodic solution.*

Although these two results are improvements, they are not the best possible. In this paper, we show this by considering the existence of 2π-periodic solutions of a more general nonautonomous Rayleigh equations with deviating arguments of the form

$$x''(t) + F(t, x'(t - \sigma(t))) + G(t, x(t - \tau(t))) = p(t), \tag{2}$$

where $F(t,x)$ and $G(t,x)$ are real continuous functions defined on R^2 with period 2π for t, $F(t,0) = 0$ for $t \in R$, σ, τ and p are real continuous functions defined on R with period 2π, and $\int_0^{2\pi} p(t)\,dt = 0$. In particular, we will see from the corollaries of our theorems of this paper that the condition "$4\pi\left[r_1 + (2\pi+1)r\right] < 1$ " in Theorem I (or Theorem II) can be replaced by the weaker condition "$2\pi\left[r_1 + (\pi+1)r\right] < 1$" (respectively "$2\pi^2 r < 1$") and the condition "$\mid g(x) \mid > r_1|x| + K \geqslant \mid f(x) \mid$ for $|x| > D$ " in Theorem I can be improved .

For this purpose, we will apply the sharp inequality for periodic functions in [4] to find a priori bounds of periodic solutions and make use of a continuation theorem of Mawhin to prove the existence of 2π-periodic solutions of (2).

Let X, Y be real Banach spaces, let $L : \mathrm{Dom}L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \mathrm{Ker}L = \mathrm{codim}\,\mathrm{Im}L < +\infty$ and $\mathrm{Im}L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$, and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I-Q)$, then the restriction L_P of L to $\mathrm{Dom}L \cap \mathrm{Ker}P : (I-P)X \to \mathrm{Im}L$ is invertible. Denote the inverse of L_P by K_P. If Ω is an open bounded subset of X, the mapping N will be called $L-$compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Theorem A. (Mawhin's continuation theorem [2]) *Let L be a Fredholm mapping of index zero, and let N be L-compact on* $\bar{\Omega}$*.Suppose*

(i) for each $\lambda \in (0,1)$, $x \in \partial\Omega, Lx \neq \lambda Nx$; *and*

(ii) for each $x \in \partial\Omega \cap KerL, QNx \neq 0$ *and* $\deg(JQN, \Omega \cap KerL, 0) \neq 0$.

Then the equation $Lx = Nx$ *has at least one solution in* $\bar{\Omega} \cap domL$.

2. Existence criteria

Our main results of this paper are as follows.

Theorem 1. *Suppose there exist constants* $K > 0$, $D > 0$, $r_1 > 0$, $r_2 > 0$ *and* $r \geqslant 0$ *such that*

(i) $|\, F(t,x) \,| \leqslant r_1 |x| + K$ *for* $(t,x) \in R^2$,

(ii) $xG(t,x) > 0$ *and* $|\, G(t,x) \,| \geqslant r_2 |x|$ *for* $t \in R$ *and* $|x| > D$, *and*

(iii) $\lim_{x\to-\infty} \max_{0\le t\le 2\pi} \frac{G(t,x)}{x} \le r$ *(or* $\lim_{x\to+\infty} \max_{0\le t\le 2\pi} \frac{G(t,x)}{x} \le r$*).*

Then for $2\pi \left[r_1 + r \left(\frac{r_1}{r_2} + \pi \right) \right] < 1$, *equation (2) has a* 2π*-periodic solution.*

Theorem 2. *Suppose there exist* $K > 0$, $D > 0$ *and* $r \geqslant 0$ *such that*

(i) $|\, F(t,x) \,| \leqslant K$ *for* $(t,x) \in R^2$,

(ii) $xG(t,x) > 0$ *and* $|\, G(t,x) \,| > K$ *for* $t \in R$ *and* $|x| > D$, *and*

(iii) $\lim_{x\to-\infty} \max_{0\le t\le 2\pi} \frac{G(t,x)}{x} \le r < \frac{1}{2\pi^2}$ *(or* $\lim_{x\to+\infty} \max_{0\le t\le 2\pi} \frac{G(t,x)}{x} \le r < \frac{1}{2\pi^2}$*).*

Then (2) has a 2π*-periodic solution.*

Corollary 1. *Suppose there exist constants* $K > 0$, $D > 0$, $r_1 > 0$, $r_2 > 0$ *and* $r \geqslant 0$ *such that*

(i) $|\, f(x) \,| \leqslant r_1 |x| + K$ *for* $x \in R$,

(ii) $xg(x) > 0$ *and* $|\, g(x) \,| \geqslant r_2 |x|$ *for* $|x| > D$,

(iii) $\lim_{x\to-\infty} \frac{g(x)}{x} \le r$ *(or* $\lim_{x\to+\infty} \frac{g(x)}{x} \le r$*).*

Then for $2\pi \left[r_1 + r \left(\frac{r_1}{r_2} + \pi \right) \right] < 1$, *(1) has a* 2π*-periodic solution.*

Corollary 2. *Suppose there exist* $K > 0$, $D > 0$ *and* $r \geqslant 0$ *such that*

(i) $|\, f(x) \,| \leqslant K$ *for* $x \in R$,

(ii) $xg(x) > 0$ *and* $|\, g(x) \,| > K$ *for* $|x| > D$,

(iii) $\lim_{x\to-\infty} \frac{g(x)}{x} \le r < \frac{1}{2\pi^2}$ *(or* $\lim_{x\to+\infty} \frac{g(x)}{x} \le r < \frac{1}{2\pi^2}$*).*

Then (1) has a 2π*-periodic solution.*

Let Y be the Banach space of all real 2π-periodic continuous functions of the form $y = y(t)$ which is defined on R and endowed with the usual linear structure as well as the norm $\|y\|_0 = \max_{0\le t\le 2\pi} |y(t)|$. Let X be the Banach space of all real 2π-periodic continuous differentiable functions of the form $x = x(t)$ which is defined on R and endowed with the usual linear structure as well as the norm $\|x\|_1 = \max\{\|x\|_0, \|x'\|_0\}$. Define the mappings $L : X \cap C^{(2)}(R^n, R) \to Y$ and $N : X \to Y$ respectively by

$$Lx(t) = x''(t), \ t \in R. \tag{3}$$

and

$$Nx(t) = -F(t, x'(t - \sigma(t))) - G(t, x(t - \tau(t))) + p(t). \tag{4}$$

Clearly,

$$\text{Ker}L = R \text{ and } \text{Im}L = \left\{ y \in Y \mid \int_0^{2\pi} y(t)\, dt = 0 \right\} \tag{5}$$

Let us define $P : X \to X$ and $Q : Y \to Y/\text{Im}L$ respectively by

$$Px(t) = \frac{1}{2\pi} \int_0^{2\pi} x(t)\, dt, \quad t \in R, \tag{6}$$

for $x = x(t) \in X$ and

$$Qy(t) = \frac{1}{2\pi} \int_0^{2\pi} y(t)\, dt, \quad t \in R. \tag{7}$$

for $y = y(t) \in Y$. It is easy to see that

$$\text{Im}P = \text{Ker}L \text{ and } \text{Im}L = \text{Ker}Q = \text{Im}(I - Q). \tag{8}$$

It follows that $L_{|\text{Dom}L \cap \text{Ker}P} : (I - P)X \to \text{Im}L$ has an inverse which will be denoted by K_P. Furthermore, we let

$$x''(t) = -\lambda F(t, x'(t - \sigma(t))) - \lambda G(t, x(t - \tau(t))) + \lambda p(t), \tag{9}$$

where $\lambda \in (0, 1)$.

Lemma 1. *The mapping L defined by (3) is a Fredholm mapping of index zero.*

Indeed, from (5), (7), (8) and the definition of Y, $\dim \text{Ker}L = \text{codim}\, \text{Im}L = 1 < +\infty$, and

$$\text{Im}L = \left\{ y \in Y \mid \int_0^{2\pi} y(t)\, dt = 0 \right\}$$

is closed in Y. Hence L is a Fredholm mapping of index zero.

Lemma 2. *Let L and N be defined by (3) and (4) respectively. Suppose Ω is an open and bounded subset of X. Then N is L-compact on $\overline{\Omega}$.*

Proof. We can conclude for any $x \in \overline{\Omega}$ that

$$QNx(t) = \frac{1}{2\pi} \int_0^{2\pi} Nx(s)\, ds, \tag{10}$$

and

$$K_P N x(t) = \int_0^t ds \int_0^s Nx(v)\,dv - \frac{1}{2\pi}\int_0^{2\pi} dt \int_0^t ds \int_0^s Nx(v)\,dv - \left(\frac{t}{2\pi} - \frac{1}{2}\right)\int_0^{2\pi} ds \int_0^s Nx(v)\,dv. \tag{11}$$

By (4) and (10), we see that $QN\left(\bar{\Omega}\right)$ is bounded, and that it is relatively compact. Furthermore, noting that K_P is continuous, we know $K_P QN\left(\bar{\Omega}\right)$ is relatively compact. On the other hand, from (11) and the Arzela-Ascoli theorem, we may see that $K_P N\left(\bar{\Omega}\right)$ is relatively compact, which leads to the fact that $\overline{K_P(I-Q)N\left(\bar{\Omega}\right)}$ is compact. Thus N is L-compact on $\overline{\Omega}$.The proof is complete. □

Lemma 3. ([4]) *Let $C_{2\pi}$ be the set of all real continuous 2π-periodic functions of the form $x = x(t)$. Then for any $x = x(t) \in C^{(1)}(R,R) \cap C_{2\pi}$ and any $\zeta \in [0, 2\pi]$, we have*

$$\|x\|_0 \leq |x(\zeta)| + \frac{1}{2}\int_0^{2\pi} |x'(s)|\,ds, \tag{12}$$

where the constant factor $1/2$ is the best possible.

Now we are in a position to prove the main theorems.

Proof of Theorem 1. We only give the proof in case

$$\lim_{x\to -\infty} \max_{0\leq t\leq 2\pi} \frac{G(t,x)}{x} \leq r,$$

since the other case can be treated in similar manners. Let L, N, P and Q be defined by (3), (4), (6) and (7) respectively. Let $x(t)$ be a 2π-periodic solution of (9). By (9), we have

$$\int_0^{2\pi} \left(F\left(t, x'(t-\sigma(t))\right) + G\left(t, x(t-\tau(t))\right)\right) dt = 0. \tag{13}$$

By means of the integral mean value theorem, there is a point $\xi \in [0, 2\pi]$ such that

$$F\left(\xi, x'(\xi - \sigma(\xi))\right) = -G\left(\xi, x(\xi - \tau(\xi))\right). \tag{14}$$

We will prove that

$$|x(\xi-\tau(\xi))| \leq \frac{r_1}{r_2}\|x'\|_0 + D + \frac{K}{r_2}. \tag{15}$$

Indeed, if $|x(\xi-\tau(\xi))| \leq D$, then (15) holds. If $|x(\xi-\tau(\xi))| > D$, then from (i), (ii) and (14), we have

$$\begin{aligned} r_2|x(\xi-\tau(\xi))| &\leq |G(\xi, x(\xi-\tau(\xi)))| \\ &= |F(\xi, x'(\xi-\sigma(\xi)))| \\ &\leq r_1|x'(\xi-\sigma(\xi))| + K. \end{aligned} \tag{16}$$

This shows that (15) holds.

Since $\xi-\tau(\xi) \in R$ and $x(t)$ has period 2π, there is a $t_1 \in [0, 2\pi]$ such that $x(t_1) = x(\xi-\tau(\xi))$. From (15), we have

$$|x(t_1)| \leq \frac{r_1}{r_2}\|x'\|_0 + D + \frac{K}{r_2}. \tag{17}$$

Furthermore, by Lemma 3 we have

$$\|x\|_0 \leq |x(t_1)| + \frac{1}{2}\int_0^{2\pi} |x'(s)|\,ds \leq \left(\frac{r_1}{r_2}+\pi\right)\|x'\|_0 + D + \frac{K}{r_2}. \tag{18}$$

By the condition $2\pi\left[r_1 + r\left(\frac{r_1}{r_2}+\pi\right)\right] < 1$, we know there is a positive number ε such that

$$\eta_1 = 2\pi\left[r_1 + (r+\varepsilon)\left(\frac{r_1}{r_2}+\pi\right)\right] < 1. \tag{19}$$

From condition (iii), we see that there is an $\rho > D$ such that for $t \in R$ and $x < -\rho$,

$$\frac{G(t,x)}{x} < r + \varepsilon. \tag{20}$$

Let

$$E_1 = \{t \mid t \in [0, 2\pi], x(t-\tau(t)) < -\rho\}, \tag{21}$$

$$E_2 = \{t \mid t \in [0, 2\pi], |x(t-\tau(t))| \leq \rho\}, \tag{22}$$

$$E_3 = [0, 2\pi] \setminus (E_1 \cup E_2), \tag{23}$$

and

$$M_0 = \max_{0\le t\le 2\pi, |x|\le\rho} |G(t,x)| . \tag{24}$$

By (18), (20) and (21), we have

$$\begin{aligned}\int_{E_1} |G(t, x(t-\tau(t)))|\,dt &\le \int_{E_1} (r+\varepsilon)\,|x(t-\tau(t))|\,dt \\ &\le 2\pi(r+\varepsilon)\max_{0\le t\le 2\pi}|x(t)| = 2\pi(r_3+\varepsilon)\|x\|_0 \\ &\le 2\pi(r+\varepsilon)\left[\left(\frac{r_1}{r_2}+\pi\right)\|x'\|_0 + D + \frac{K}{r_2}\right].\end{aligned} \tag{25}$$

From (22) and (24), we have

$$\int_{E_2} |G(t, x(t-\tau(t)))|\,dt \le 2\pi M_0. \tag{26}$$

It follows from condition (i) that

$$\int_0^{2\pi} |F(t, x'(t-\sigma(t)))|\,dt \leqslant 2\pi(r_1\|x'\|_0 + K). \tag{27}$$

In view of (ii), (13), (24), (25), (26) and (27), we get

$$\begin{aligned}&\int_{E_3} |G(t, x(t-\tau(t)))|\,dt \\ &= \int_{E_3} G(t, x(t-\tau(t)))\,dt \\ &= -\int_0^{2\pi} F(t, x'(t-\sigma(t)))\,dt - \int_{E_1} G(t, x(t-\tau(t)))\,dt \\ &\quad - \int_{E_2} G(t, x(t-\tau(t)))\,dt \\ &\le \int_0^{2\pi} |F(t, x'(t-\sigma(t)))|\,dt + \int_{E_1} |G(t, x(t-\tau(t)))|\,dt \\ &\quad + \int_{E_2} |G(t, x(t-\tau(t)))|\,dt \\ &\le 2\pi(r_1\|x'\|_0 + K) + 2\pi M_0 + 2\pi(r+\varepsilon)\left[\left(\frac{r_1}{r_2}+\pi\right)\|x'\|_0 + D + \frac{K}{r_2}\right] \\ &\le 2\pi\left[r_1 + (r+\varepsilon)\left(\frac{r_1}{r_2}+\pi\right)\right]\|x'\|_0 + M_1,\end{aligned} \tag{28}$$

for some positive number M_1. It follows from (9), (25), (26), (27) and (28) that

$$\begin{aligned}
&\int_0^{2\pi} |x''(t)|\, dt \\
&\leq \int_0^{2\pi} |F(t, x'(t-\sigma(t)))|\, dt + \int_{E_1} |G(t, x(t-\tau(t)))|\, dt \\
&\quad + \int_{E_2} |G(t, x(t-\tau(t)))|\, dt + \int_{E_3} |G(t, x(t-\tau(t)))|\, dt + 2\pi \|p\|_0 \\
&\leq 2\pi \left(r_1 \|x'\|_0 + K\right) + 2\pi (r+\varepsilon) \left[\left(\frac{r_1}{r_2} + \pi\right) \|x'\|_0 + D + \frac{K}{r_2}\right] \\
&\quad + 2\pi M_0 + 2\pi \left[r_1 + (r+\varepsilon)\left(\frac{r_1}{r_2} + \pi\right)\right] \|x'\|_0 + M_1 + 2\pi \|p\|_0 \\
&= 2\eta_1 \|x'\|_0 + M_2,
\end{aligned} \tag{29}$$

for some positive number M_2. Note that $x(0) = x(2\pi)$, thus there is a $t_2 \in [0, 2\pi]$ such that $x'(t_2) = 0$. Hence, by Lemma 3, we have

$$\|x'\|_0 \leq \frac{1}{2} \int_0^{2\pi} |x''(t)|\, dt. \tag{30}$$

By (29) and (30), we see that

$$\|x'\|_0 \leq \eta_1 \|x'\|_0 + \frac{M_2}{2}. \tag{31}$$

It follows that

$$\|x'\|_0 \leq D_1, \tag{32}$$

where $D_1 = M_2/2(1-\eta_1)$. From (18) and (32), we get

$$\|x\|_0 \leq D_0 \tag{33}$$

where $D_0 = \left(\frac{r_1}{r_2} + \pi\right) D_1 + D + \frac{K}{r_2}$. Take a positive number $\overline{D}$ which is greater than $\max\{D_0, D_1\} + D$ and let

$$\Omega = \left\{x \in X \mid \|x\|_1 < \overline{D}\right\}. \tag{34}$$

From Lemma 1 and Lemma 2, we know that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$. In terms of the a priori bounds found above, we see that for any $\lambda \in (0,1)$ and any $x \in \partial\Omega$, $Lx \neq \lambda Nx$. Since for any $x \in \partial\Omega \cap \mathrm{Ker}L$, $x = \overline{D}$ $(> D)$ or $x = -\overline{D}$, then in view of (ii), (7) and $\int_0^{2\pi} p\,(t)\,dt = 0$, we have

$$\begin{aligned} QNx &= \frac{1}{2\pi}\int_0^{2\pi} \left(-F\left(s, x'\left(s-\sigma\left(s\right)\right)\right) - G\left(s, x\left(s-\tau\left(s\right)\right)\right) + p\left(s\right)\right) ds \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(-F\left(s, 0\right) - G\left(s, x\left(s-\tau\left(s\right)\right)\right)\right) ds \\ &= \frac{-1}{2\pi}\int_0^{2\pi} G\left(s, x\right) ds \neq 0. \end{aligned}$$

In particular, we see that

$$\frac{-1}{2\pi}\int_0^{2\pi} G\left(s, -\overline{D}\right) ds > 0 \text{ and } \frac{-1}{2\pi}\int_0^{2\pi} G\left(s, \overline{D}\right) ds < 0.$$

This show that $\deg\left(JQN, \Omega \cap \mathrm{Ker}L, 0\right) \neq 0$. In view of Theorem A, there exists a 2π-periodic solution of (2). The proof is complete. □

Proof of Theorem 2. We only give the proof in case

$$\lim_{x\to-\infty} \max_{0\leq t\leq 2\pi} \frac{G\left(t, x\right)}{x} \leq r < \frac{1}{2\pi^2},$$

since the other case can be treated in similar manners. Let $x\left(t\right)$ be a ω-periodic solution of (9). Then (13) and (14) hold. We will prove that there are positive numbers D_2 and D_3 such that

$$\left\|x\right\|_0 \leq D_2 \text{ and } \left\|x'\right\|_0 \leq D_3. \tag{35}$$

From (i) and (14), we see that

$$\left|G\left(\xi, x\left(\xi-\tau\left(\xi\right)\right)\right)\right| = \left|F\left(\xi, x'\left(\xi-\sigma\left(\xi\right)\right)\right)\right| \leq K. \tag{36}$$

It follows from (ii) and (36) that

$$\left|x\left(\xi-\tau\left(\xi\right)\right)\right| \leq D. \tag{37}$$

Since $\xi - \tau(\xi) \in R$ and $x\left(t\right)$ is 2π-periodic, thus there is a $t_3 \in [0, 2\pi]$ such that $x\left(t_3\right) = x(\xi - \tau(\xi))$, and so

$$\left|x\left(t_3\right)\right| \leq D. \tag{38}$$

Furthermore, by Lemma 3, we have

$$\|x\|_0 \leq |x(t_3)| + \frac{1}{2}\int_0^{2\pi} |x'(s)|\, ds \leq D + \pi \|x'\|_0 . \tag{39}$$

From condition (iii), we can take a positive number ε_1 such that

$$\eta_2 = 2\pi^2 (r + \varepsilon_1) < 1.$$

Furthermore we see that there is a $\rho_1 > D$ such that for $t \in R$ and $x < -\rho_1$,

$$\frac{G(t,x)}{x} < r + \varepsilon_1. \tag{40}$$

Let

$$E_1' = \{t \mid t \in [0,2\pi], x(t-\tau(t)) < -\rho_1\}, \tag{41}$$

$$E_2' = \{t \mid t \in [0,2\pi], |x(t-\tau(t))| \leq \rho_1\}, \tag{42}$$

$$E_3' = [0,2\pi] \setminus (E_1' \cup E_2'), \tag{43}$$

and

$$M_3 = \max_{0 \leq t \leq 2\pi, |x| \leq \rho_1} |G(t,x)|. \tag{44}$$

By (39), (40) and (41), we have

$$\begin{aligned} \int_{E_1'} |G(t, x(t-\tau(t)))|\, dt &\leq \int_{E_1'} (r+\varepsilon)\, |x(t-\tau(t))|\, dt \\ &\leq 2\pi (r+\varepsilon) \max_{0 \leq t \leq 2\pi} |x(t)| \\ &= 2\pi (r+\varepsilon) \|x\|_0 \\ &\leq 2\pi (r+\varepsilon) [D + \pi \|x'\|_0]. \end{aligned} \tag{45}$$

From (42) and (44), we have

$$\int_{E_2'} |G(t, x(t-\tau(t)))|\, dt \leq 2\pi M_3. \tag{46}$$

It follows from condition (i) that

$$\int_0^{2\pi} |F(t, x'(t-\sigma(t)))|\, dt \leqslant 2\pi K. \tag{47}$$

In view of (ii), (13), (43), (45), (46) and (47), we get

$$\begin{aligned}
&\int_{E_3'} |G(t, x(t-\tau(t)))|\, dt \\
&= \int_{E_3'} G(t, x(t-\tau(t)))\, dt \\
&= -\int_0^{2\pi} F(t, x'(t-\sigma(t)))\, dt - \int_{E_1} G(t, x(t-\tau(t)))\, dt \\
&\quad - \int_{E_2} G(t, x(t-\tau(t)))\, dt \\
&\leq \int_0^{2\pi} |F(t, x'(t-\sigma(t)))|\, dt + \int_{E_1} |G(t, x(t-\tau(t)))|\, dt \\
&\quad + \int_{E_2} |G(t, x(t-\tau(t)))|\, dt \\
&\leq 2\pi K + 2\pi M_3 + 2\pi (r+\varepsilon_1) [D + \pi \|x'\|_0] \\
&\leq 2\pi^2 (r+\varepsilon_1) \|x'\|_0 + M_4,
\end{aligned} \tag{48}$$

for some positive number M_4. It follows from (9), (45), (46), (47) and (48) that

$$\begin{aligned}
&\int_0^{2\pi} |x''(t)|\, dt \\
&\leq \int_0^{2\pi} |F(t, x'(t-\sigma(t)))|\, dt + \int_{E_1'} |G(t, x(t-\tau(t)))|\, dt \\
&\quad + \int_{E_2'} |G(t, x(t-\tau(t)))|\, dt + \int_{E_3'} |G(t, x(t-\tau(t)))|\, dt + 2\pi \|p\|_0 \\
&\leq 2\pi K + 2\pi (r+\varepsilon_1) [D + \pi \|x'\|_0] + 2\pi M_3 + 2\pi^2 (r+\varepsilon_1) \|x'\|_0 \\
&\quad + M_4 + 2\pi \|p\|_0 \\
&= 2\eta_2 \|x'\|_0 + M_5,
\end{aligned} \tag{49}$$

for some positive number M_5. Since $x(0) = x(2\pi)$, we know there is a $t_4 \in [0, 2\pi]$ such that $x'(t_4) = 0$. Hence, by Lemma 3, we have

$$\|x'\|_0 \leq \frac{1}{2} \int_0^{2\pi} \left| x''(t) \right| dt. \tag{50}$$

From (49) and (50) we see that

$$\|x'\|_0 \leq \eta_2 \|x'\|_0 + \frac{M_5}{2}. \tag{51}$$

It follows that

$$\|x'\|_0 \leq D_3, \tag{52}$$

where $D_3 = M_5/2(1-\eta_2)$. By (39) and (52), we get

$$\|x\|_0 \leq D_2, \tag{53}$$

where $D_2 = D + \pi D_3$. From (52) and (53), we see that there are positive numbers D_2 and D_3 such that (35) hold .We may now proceed as in the proof of Theorem 1 and complete our proof. □

Example. Consider a Rayleigh equation of the form (2) where $h(x) = x^3$ if $x \geqslant 0$ and $h(x) = x$ if $x < 0$,

$$\sigma(t) = \sin t,\ \tau(t) = \cos t,\ p(t) = \frac{\cos t}{5},$$

$$F(t,x) = (\frac{1+\cos t}{24\pi(1+\pi)})x + \exp\left(-x^2\right) - 1,$$

and

$$G(t,x) = \frac{\exp\left((\sin t)^2\right) h(x)}{13\pi(\pi+1)}.$$

It is then easy to verify that all the assumptions in Theorem 1 are satisfied with

$$K = 2, D = 1, r_1 = \frac{1}{12\pi(1+\pi)}, r_2 = \frac{1}{13\pi(1+\pi)}, r_3 = \frac{e}{13\pi(\pi+1)}.$$

Thus (2) has a 2π-periodic solution. Furthermore, this solution is nontrivial since $y(t) \equiv 0$ is not a solution of (2).

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