Nonlinear Funct. Anal. & Appl., Vol. 11, No. 3 (2006), pp. 421–436

CONVERGENCE THEOREMS OF THE ITERATIVE SCHEMES IN CONVEX METRIC SPACES

Ismat Beg, Mujahid Abbas and Jong Kyu Kim

Abstract. The purpose of this paper is to study the convergence problem of Mann and Ishikawa type iterative schemes of weakly contractive mapping in a complete convex metric space. We establish the results on invariant approximation for the mapping defined on a class of nonconvex sets in a convex metric space. Finally, we obtain the existence of common fixed points of two asymptotically nonexpansive mappings through the convergence of iteratively defined sequence in a uniformly convex metric space.

1. Introduction

The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric properties play a key role in metric fixed point problems, see for example [14] and references therein. These results mainly use convexity hypothesis and other geometric properties of Banach spaces. Takahashi [33] introduced the notion of convexity in metric spaces. Afterward-Beg-Abbas [4], Beg-Azam [5], Chang-Kim-Jin [8], Ciric [10], Ding [11], Guay-Singh-Whitfield [21] and many other authors have studied fixed point theorems in convex metric spaces.

Recently, Shimizu-Takahashi [29] introduced the concept of uniform convexity in a metric space, which was further exploited by Beg [3]. Ishikawa iteration scheme [22] and Mann iteration scheme [26] are two well known iteratively defined sequences which are generally used to solve the fixed point problems of different mappings. Applying fixed point theorems, useful results have been established in approximation theory ([1], [5], [31-32]). Meinardus

Received November 17, 2005. Revised May 29, 2006.

2000 Mathematics Subject Classification: 47H10, 47H09, 54H25.

Key words and phrases: Convex metric space, uniform convexity, common fixed point, asymptotically nonexpansive mapping, weakly contractive mapping, best approximation.

[27] was the first to employ fixed point theorem to prove the existence of an invariant approximation in a Banach space.

The aim of this paper is to study the convergence problem of Mann and Ishikawa type iterative schemes of weakly contractive mapping in a complete convex metric space. We establish the results on invariant approximation for the mapping defined on a class of nonconvex sets in a convex metric space. Finally, we obtain the existence of common fixed points of two asymptotically nonexpansive mappings through the convergence of iteratively defined sequence in a uniformly convex metric space. The results proved in this paper extend the corresponding results of Rhoades [28] and Al-Thagafi [1].

2. Preliminaries

In 1974, Ishikawa [22] introduced the iterative sequences generated by nonlinear mapping and proved the convergence theorems for the sequence:

Let K be a nonempty closed convex subset of a Banach space E and let $T : K \to K$ be a nonlinear pseudo-contractive mapping or accretive mapping. Recently concerning the problem of the Ishikawa iterative sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in K, \\ y_n = \beta_n x_n + (1-\beta_n)T(x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(y_n), \quad n \geq 0 \end{cases} \tag{2.1}$$

converging strongly to a fixed point of T or to a solution of the equation $Tx = f$ has been considered by many authors ([1], [3-7], [9], [12], [15], [17], [21], [23], [25], [34]), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying the certain control conditions.

In 1970, Takahashi [33] introduced the concept of convexity in a metric space and the properties of the space.

Definition 2.1. ([33]) Let (X,d) be a metric space. A mapping $W : X \times X \times [0,1] \longrightarrow X$ is said to be a *convex structure* on X if for each $(x,y,\lambda) \in X \times X \times [0,1]$ and $u \in X$,

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda) d(u,y).$$

A metric space X together with the convex structure W is called a *convex metric space*, denotes it by (X,d,W).

A nonempty subset K of the convex metric space X is said to be *convex* if $W(x,y,\lambda) \in K$ whenever $(x,y,\lambda) \in K \times K \times [0,1]$. Takahashi [33] has shown that open spheres $B(x,r)=\{y\in X : d(y,x)<r\}$ and closed spheres $B[x,r]=\{y\in X : d(y,x)\le r\}$ are convex.

Every normed space is convex metric space. But there are many examples of convex metric spaces which are not embedded in any normed space [33].

Remark 2.1. Every normed space is a convex metric space, where a convex structure $W(x,y,z;\alpha,\beta,\gamma)=\alpha x+\beta y+\gamma z$, for all $x,y,z\in X$ and $\alpha,\beta,\gamma\in I$ with $\alpha+\beta+\gamma=1$. In fact,

$$\begin{aligned} d(u,W(x,y,z;\alpha,\beta,\gamma)) &= \|u-(\alpha x+\beta y+\gamma z)\| \\ &\le \alpha\|u-x\|+\beta\|u-y\|+\gamma\|u-z\| \\ &= \alpha d(u,x)+\beta d(u,y)+\gamma d(u,z), \quad \forall\, u\in X. \end{aligned}$$

But there exists some convex metric spaces which can not be embedded into normed space.

Example 2.1. Let $X=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1>0,\ x_2>0,\ x_3>0\}$. For $x=(x_1,\ x_2,\ x_3)$, $y=(y_1,y_2,y_3)\in X$ and $\alpha,\beta,\gamma\in I$ with $\alpha+\beta+\gamma=1$, we define a mapping $W: X^3\times I^3\to X$ by

$$\begin{aligned} &W(x,y,z;\alpha,\beta,\gamma) \\ &= (\alpha x_1+\beta y_1+\gamma z_1,\ \alpha x_2+\beta y_2+\gamma z_2,\ \alpha x_3+\beta y_3+\gamma z_3) \end{aligned}$$

and define a metric $d: X\times X\to[0,\infty)$ by

$$d(x,y)=\Big|x_1y_1+x_2y_2+x_3y_3\Big|.$$

Then we can show that (X,d,W) is a convex metric space, but it is not a normed space.

Example 2.2. Let $Y=\{(x_1,x_2)\in\mathbb{R}^2 : x_1>0,\ x_2>0\}$. For each $x=(x_1,$ $x_2)$, $y=(y_1,y_2)\in Y$ and $\lambda\in I$. we define a mapping $W: Y^2\times I\to Y$ by

$$W(x,y;\lambda)=\Big(\lambda x_1+(1-\lambda)y_1,\frac{\lambda x_1x_2+(1-\lambda)y_1y_2}{\lambda x_1+(1-\lambda)y_1}\Big)$$

and define a metric $d: Y\times Y\to[0,\infty)$ by

$$d(x,y)=|x_1-y_1|+|x_1x_2-y_1y_2|.$$

Then we can show that (Y,d,W) is a convex metric space, but it is not a normed space.

Definition 2.2. Let X be a convex metric space.

(a) X is said to be *uniformly convex*, if for any $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon)$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$, give

$$d(z, W(x, y, \frac{1}{2})) \leq r(1 - \alpha(\varepsilon)) < r.$$

(b) X is said to be *strictly convex*, if for $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$, imply that $d(z, W(x, y, \frac{1}{2})) < r$.

It follows from above definitions that a uniformly convex metric space is strictly convex but converse does not hold in general. Uniformly convex normed spaces are uniformly convex metric spaces.

Remark 2.2. In this paper we assume that function α is increasing, continuous on $[0, 2)$ and $\alpha(0) = 0$, $\alpha(2) = 1$.

Definition 2.3. ([13]) Let K be a nonempty subset of a metric space X. A mapping $T : K \longrightarrow K$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}, k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n(x), T^n(y)) \leq k_n d(x, y)$$

for each x, y in K and each $n \in N$. If $k_n = 1$, for all $n \in N$, then T is known as a nonexpansive mapping.

A point $x \in X$ is called a *fixed point* of T if $T(x) = x$. We denote the set of fixed points of T by $F(T)$.

The asymptotically nonexpansive mapping was introduced by Goebel-Kirk [13], where it is shown that if K is a nonempty bounded closed convex subset of a uniformly convex Banach space and $T : K \longrightarrow K$ is asymptotically nonexpansive, then T has a fixed point and, moreover, the set $F(T)$ is closed and convex. These mappings have remained under the study of many authors ([3], [8], [24] and [30]).

Definition 2.4. ([2], [28]) Let K be a nonempty closed subset of a complete metric space (X, d). A mapping $T : K \longrightarrow K$ is said to be *weakly contractive* if for each $x, y \in K$,

$$d(T(x), T(y)) \leq d(x, y) - \Psi(d(x, y)),$$

where $\Psi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous and non-decreasing such that Ψ is positive on $(0, \infty), \Psi(0) = 0$, and $\lim_{t\to\infty} \Psi(t) = \infty$.

Definition 2.5. Let K be a nonempty closed convex subset of a convex metric space X. If there exists $y_0 \in K$ such that

$$d(x, y_0) = d(x, K) = \inf_{y \in K} d(x, y),$$

then y_0 is called a *best approximation to x out of* K. We denote by $P_K(x)$ the set of best approximation to x out of K.

Definition 2.6. Let K be a nonempty subset of a convex metric space X and $T : K \longrightarrow K$ be a mapping. K is said to be *T-regular* if and only if

$$W(x, T(x), \frac{1}{2}) \in K$$

for each $x \in K$.

Remark 2.3. If $\Psi(t) = (1-k)t$ for a constant $k \in (0,1)$, then the weakly contractive map becomes a contraction mapping and it has a unique fixed point by Banach contraction principle. Weakly contractive maps lie between those which satisfy Banach contraction principle [20] and contractive maps. Weakly contractive maps also satisfy the definition of Boyd-Wong [6]. It is also known that a complete metric space has a fixed point property for weakly contractive mappings. Moreover the fixed point in this case is also unique.

In 1988, Ding [8] proved the following fixed point theorem using Ishikawa type iterative scheme: Let K be a nonempty closed convex subset of a complete convex metric space X with convex structure W and let $T : K \to K$ be a quasi-contractive mapping [10], i.e., there exists a constant $q \in [0,1)$ such that for all $x, y \in K$,

$$d(Tx, Ty) \leq q \cdot \max\{d(x,y), d(x,T(x)), d(y,T(y)), d(x,T(y)), d(y,T(x))\}. \tag{2.2}$$

Suppose that $\{x_n\}$ is the iterative scheme defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = W(T(y_n), x_n, \alpha_n), \\ y_n = W(T(x_n), x_n, \beta_n), \quad n \geq 0, \end{cases} \tag{2.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0 \leq \alpha_n, \beta_n \leq 1$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges. Then $\{x_n\}$ converges to a unique fixed point of T in K.

In 2003, Chang-Kim [7] introduced the concept of new type iterative sequence $\{x_n\}$ with errors in a metric space. Let (E, d) be a convex metric space with a convex structure $W : E^3 \times I^3 \to E$ and $T : E \to E$ be a generalized quasi contractive mapping. Define the iterative scheme $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in E, \\ y_n = W(x_n, T(x_n), v_n; \xi_n, \eta_n, \delta_n), \\ x_{n+1} = W(x_n, T(y_n), u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0, \end{cases} \tag{2.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, $\{\eta_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\xi_n + \eta_n + \delta_n = 1$, $n = 0, 1, 2, \cdots$ and $\{u_n\}$, $\{v_n\}$ are sequences in E, then $\{x_n\}$ is called the *Ishikawa type iterative scheme with errors* of T.

Especially, if $\eta_n = 0$ and $\delta_n = 0$ for all $n \geq 0$, it follows from the definition of the convex structure W that $y_n = x_n$. Hence from (2.4) we have

$$x_{n+1} = W(x_n, T(x_n), u_n; \alpha_n, \beta_n, \gamma_n). \tag{2.5}$$

The sequence defined by (2.5) is called the *Mann type iterative scheme with errors* of T. Therefore, the Ishikawa iterative scheme (2.1) is a special cases of (2.4) with $\gamma_n = 0$, $\delta_n = 0$ and $u_n = v_n = 0$, for all $n \geq 0$ and also, (2.4) with $\delta_n = \gamma_n = u_n = v_n = 0$ for all $n \geq 0$ reduces to (2.3).

And also Chang-Kim [7] proved the following theorem for the Ishikawa type iterative scheme with errors (2.4): Let (E, d, W) be a complete convex metric space, $W : E^3 \times I^3 \to E$ be the convex structure of E, T be a generalized quasi-contractive mapping defined by

$$\begin{aligned} & d(T(x), T(y)) \\ & \leq \Phi\Big(\max\{d(x, y),\ d(x, T(x)),\ d(y, T(y)),\ d(x, T(y)),\ d(y, T(x))\}\Big), \end{aligned} \tag{2.6}$$

for all $x, y \in E$, and $\{x_n\}$ be the Ishikawa type iterative scheme with errors of T defined by (2.4). Then the scheme $\{x_n\}$ converges to a unique fixed point z of T in E.

After then, Kim et al.([16], [18], [19]) and Chang et al.[7] proved the convergence theorems of the iterative schemes for asymptotically quasi-nonexpansive mappings or asymptotically quasi-nonexpansive type mappings in convex metric spaces.

3. Iterative schemes for weakly contractive mapping

Now, we prove the convergence theorems of the Mann and Ishikawa type iterative schemes for weakly contractive mappings in convex metric spaces.

Theorem 3.1. *Let K be a nonempty closed convex subset of a complete convex metric space (X, d) and T be a weakly contractive self mapping on K. Let $\{x_n\}$ be the iterative scheme defined by*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = W(x_n, T(x_n), \alpha_n), \quad n \geq 0, \end{cases} \tag{3.1}$$

where $0 \leq \alpha_n \leq 1$ and $\sum \alpha_n = \infty$. Then, $\lim\limits_{n\to\infty} d(x_n, p) = 0$, where p is the unique fixed point of T.

Proof. The iterative scheme $\{x_n\}$ is well defined, because $T(K) \subset K$, $x_0 \in K$ and K is convex. The existence of unique fixed point of T follows from [28]. Let $p \in F(T)$. Then, from the definition of weakly contractivity of T, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, T(x_n), \alpha_n), p) \\ &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(T(x_n), T(p)) \\ &\leq \alpha_n d(x_n, p) + (1 - \alpha_n)[d(x_n, p) - \Psi(d(x_n, p))] \\ &\leq d(x_n, p). \end{aligned}$$

So, $\{d(x_n, p)\}$ is a non-negative non-increasing real sequence. Hence it converges to the real number $q \geq 0$. Then we know that $q \leq d(x_n, p)$. Suppose that $q > 0$. Now, for any fixed positive integer n_0 we have,

$$\begin{aligned} \sum_{n=n_0}^{\infty} \alpha_n \Psi(q) &\leq \sum_{n=n_0}^{\infty} \alpha_n \Psi(d(x_n, p)) \\ &\leq \sum_{n=n_0}^{\infty} [d(x_n, p) - d(x_{n+1}, p)] \\ &\leq d(x_{n_0}, p), \end{aligned}$$

which contradicts to $\sum \alpha_n = \infty$. Hence, $q = 0$. This completes the proof. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a complete convex metric space (X, d) and T be a weakly contractive self-mapping on K. Let $\{x_n\}$ be the iterative scheme defined by*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = W(x_n, T(y_n), \alpha_n), \\ y_n = W(x_n, T(x_n), \beta_n), \quad n \geq 0, \end{cases} \tag{3.2}$$

where $0 \leq \alpha_n,\ \beta_n \leq 1,$ *and* $\sum \alpha_n \beta_n = \infty$. *Then* $\lim_{n\to\infty} d(x_n, p) = 0$, *where* p *is the unique fixed point of* T.

Proof. The existence of unique fixed point p (say) of T follows from [28]. The iterative schemes $\{x_n\}$ and $\{y_n\}$ are well defined, because $x_0 \in K, T(K) \subset K$ and K is convex subset of X. Then, from the definition of weakly contractivity of T, we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(W(x_n, T(y_n), \alpha_n), p) \\
&\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(T(y_n), T(p)) \\
&\leq \alpha_n d(x_n, p) + (1-\alpha_n)[d(y_n, p) - \Psi(d(y_n, p)) \\
&\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(W(x_n, T(x_n), \beta_n), p) \\
&\quad - (1-\alpha_n)\Psi(d(y_n, p)) \\
&\leq \alpha_n d(x_n, p) + (1-\alpha_n)[\beta_n d(x_n, p) + (1-\beta_n) d(T(x_n), T(p))] \\
&\quad - (1-\alpha_n)\Psi(d(y_n, p)) \\
&\leq \alpha_n d(x_n, p) + (1-\alpha_n)\beta_n d(x_n, p) + (1-\alpha_n)(1-\beta_n) d(x_n, p) \\
&\quad - (1-\alpha_n)(1-\beta_n)\Psi(d(x_n, p)) - (1-\alpha_n)\Psi(d(y_n, p)) \\
&\leq d(x_n, p) - (1-\alpha_n)(1-\beta_n)\Psi(d(x_n, p)) - (1-\alpha_n)\Psi(d(y_n, p)) \\
&\leq d(x_n, p).
\end{aligned}$$

So, $\{d(x_n, p)\}$ is a non-negative non-increasing real sequence. Hence it converges to the real number $q \geq 0$. Then we know that $q \leq d(x_n, p)$. Suppose that $q > 0$. Now, for any fixed positive integer n_0 we have,

$$\begin{aligned}
\sum_{n=n_0}^{\infty} (1-\alpha_n)(1-\beta_n)\Psi(q) &\leq \sum_{n=n_0}^{\infty} (1-\alpha_n)(1-\beta_n)\Psi(d(x_n, p)) \\
&\leq \sum_{n=n_0}^{\infty} d(x_n, p) - d(x_{n+1}, p) \\
&\leq d(x_{n_0}, p),
\end{aligned}$$

which contradicts to $\sum \alpha_n \beta_n = \infty$. Therefore, $q = 0$. This completes the proof. □

Remark 3.1. We can prove that the results of Theorems 3.1 and 3.2 hold for the iterative schemes with errors.

4. Fixed points and best approximation

In this section, we obtain the results on best approximation as a fixed points of nonexpansive type mappings in the setting of convex metric spaces.

Now we need the following useful lemma for the main theorems.

Lemma 4.1. *Let M be a subset of a strictly convex metric space X. If x, $y \in P_M(u)$ for $u \in X$, with $x \neq y$, then for $\lambda \in (0,1)$, $W(x,y,\lambda) \notin M$.*

Proof. If $W(x,y,\lambda) \in M$, then x, $y \in P_M(u)$. And so, we have $d(x,u) \leq d(u, W(x,y,\lambda))$, and $d(y,u) \leq d(u, W(x,y,\lambda))$. Since X is a strictly convex metric space, so we arrive at a contradiction. Hence, for $0 < \lambda < 1$, $W(x,y,\lambda) \notin M$. □

Theorem 4.1. *Let M be a subset of a strictly convex metric space X and $T : M \to M$ a mapping. If $P_M(u)$ is a nonempty T-regular set for any $u \in X$, then each point of $P_M(u)$ is a fixed point of T.*

Proof. Suppose that for some x in $P_M(u)$, $T(x) \neq x$. Then, from Lemma 4.1, $W(x, T(x), \frac{1}{2}) \notin M$. It implies that $W(x, T(x), \frac{1}{2}) \notin P_M(u)$. Since $P_M(u)$ is a T-regular set, $x = T(x)$. Thus each best M- approximation of u is a fixed point of T. □

Theorem 4.2. *Let M be a nonempty closed and T- regular subset of a strictly convex metric space X, and u be a point in M, where T is a compact mapping on M, and u be a point in M. Suppose that $d(T(x),u) \leq d(x,u)$ for all x in M. Then each x in M, which is best approximation to u, is a fixed point of T.*

Proof. Let $r = d(u, M)$. Then there is a minimizing sequence $\{y_n\}$ in M such that $\lim_{n\to\infty} d(u, y_n) = r$. Clearly $\{y_n\}$ is a bounded sequence. Since T is a compact map, $cl(\{T(y_n)\})$ is a compact subset of M and so $\{T(y_n)\}$ has a convergent subsequence $\{T(y_{n_k})\}$ with $\lim_{k\to\infty} T(y_{n_k}) = x$ in M. Now,

$$r \leq d(u,x) = \lim_{k\to\infty} d(u, T(y_{n_k})) \leq \lim_{k\to\infty} d(u, y_{n_k}) = \lim_{n\to\infty} d(u, y_n) = r.$$

Thus $x \in P_M(u)$. Now $y \in P_M(u)$ and $r \leq d(T(y),u) \leq d(y,u) = r$ imply that $T(y) \in P_M(u)$. Therefore $y \in P_M(u)$ gives $d(T(y),u) = r$. Now strict convexity of X implies $r \leq d(W(y,T(y),\frac{1}{2}),u) < r$. Thus $W(y,T(y),\frac{1}{2}) \in P_M(u)$.The result follows from Theorem 4.1. □

Theorem 4.3. *Let M be a nonempty closed and T- regular subset of a strictly convex metric space X, where T is a compact mapping. Let u be a fixed point of T in $X \setminus M$, and*

$$\begin{aligned} d(T(x),T(y)) \leq & \alpha d(x,y) + \beta(d(x,T(x)) + d(y,T(y))) \\ & + \gamma(d(x,T(y)) + d(y,T(x))), \end{aligned}$$

for all $x, y \in X$, where α, β and γ are real numbers with $\alpha + 2\beta + \gamma \leq 1$. Then each best approximation in M to u is a fixed point of T.

Proof. For $x \in X$, we have

$$\begin{aligned} d(T(x),T(u)) &\leq \alpha d(x,u) + \beta(d(x,T(x)) + d(u,T(u))) + \gamma(d(x,T(u) \\ &\quad + d(u,T(x)) \\ &= \alpha d(x,u) + \beta(d(x,T(x))) + \gamma(d(x,u) + d(T(u),T(x))) \\ &\leq \alpha d(x,u) + \beta(d(x,u) + d(T(u),T(x))) + \gamma(d(x,u) \\ &\quad + d(T(u),T(x))) \\ &= (\alpha + \beta + \gamma) d(x,u) + (\beta + \gamma) d(T(u),T(x)). \end{aligned}$$

Thus $d(T(x),T(u)) \leq (\dfrac{\alpha+\beta+\gamma}{1-\beta-\gamma}) d(x,u)$, and so, $d(T(x),T(u)) \leq d(x,u)$. The result follows from Theorem 4.2. □

5. Iterative schemes of two asymptotically nonexpansive mappings

In this section, we proved the convergence theorem of the modified Ishikawa type iterative schemes for a pair of two asymptotically nonexpansive mappings defined on the nonempty bounded convex subset of a convex metric space.

Let K be a nonempty convex subset of a convex metric space and $S, T : K \longrightarrow K$ be two asymptotically nonexpansive mappings. Define the modified Ishikawa type iterative schemes $\{x_n\}$ as follows.

$$\begin{cases} x_1 \in K, \\ x_{n+1} = W(S^n(y_n), x_n, \alpha_n), \\ y_n = W(T^n(x_n), x_n, \beta_n), \qquad n \geq 0, \end{cases} \tag{5.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[0,1]$ for $n = 1, 2, \cdots$.

We can easily prove the following lemma from the definition of uniform convexity.

Lemma 5.1. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a uniformly convex metric space X such that for some $z \in X$, and $r \geq 0$,*

$$\limsup_{n\to\infty} d(x_n, z) \leq r, \quad \limsup_{n\to\infty} d(y_n, z) \leq r \ \text{and} \ \lim_{n\to\infty} d(z, W(x_n, y_n, \lambda)) = r.$$

Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 5.2. *Let K be a nonempty closed bounded and convex subset of a uniformly convex complete metric space X. Let $S, T : K \longrightarrow K$ be two mappings satisfying*

$$d(S^n(x), S^n(y)) \leq k_n d(x, y)$$

and

$$d(T^n(x), T^n(y)) \leq k_n d(x, y)$$

for $n = 1, 2\cdots$, where $\{k_n\}$ is the sequence of numbers with $1 \leq k_n < \infty$, for each n, and $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be as in (5.1). That is,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = W(S^n(y_n), x_n, \alpha_n), \\ y_n = W(T^n(x_n), x_n, \beta_n), \qquad n \geq 0, \end{cases}$$

where, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0, 1)$. If S and T have common fixed points, then

$$\lim_{n\to\infty} d(S(x_n), x_n) = \lim_{n\to\infty} d(T(x_n), x_n) = 0.$$

Proof. Let p be a common fixed point of S and T. Then we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(S^n(y_n), x_n, \alpha_n), p) \\ &\leq \alpha_n d(p, S^n(y_n)) + (1-\alpha_n) d(p, x_n) \\ &\leq \alpha_n k_n d(p, y_n) + (1-\alpha_n) d(p, x_n) \\ &= \alpha_n k_n d(p, W(T^n(x_n), x_n, \beta_n)) + (1-\alpha_n) d(p, x_n) \\ &\leq \alpha_n k_n [\beta_n d(p, T^n(x_n)) + (1-\beta_n) d(p, x_n)] + (1-\alpha_n) d(p, x_n) \\ &\leq \alpha_n k_n [\beta_n k_n d(p, x_n) + (1-\beta_n) d(p, x_n)] + (1-\alpha_n) d(p, x_n) \\ &= u_n d(p, x_n), \end{aligned}$$

where $u_n = \alpha_n k_n[\beta_n + (1-\beta_n)] + (1-\alpha_n)$. Therefore, we have

$$d(x_{n+1}, p) \le \left(\prod_{i=n}^{n+m-1} u_i \right) d(x_n, p)$$

for all $m, n = 1, 2, \cdots$. Also, $\sum\limits_{n=1}^{\infty} (u_n - 1) < \infty$ and $\lim\limits_{n\to\infty} \prod\limits_{i=1}^{n+m-1} u_i = 1$. Hence, $\lim\limits_{n\to\infty} d(x_n, p)$ exists. Now, if $\lim\limits_{n\to\infty} d(x_n, p) = 0$, then the result immediately follows. Suppose $\lim\limits_{n\to\infty} d(x_n, p) = c$, and $c > 0$. Since,

$$d(T^n(x_n), p) \le k_n d(x_n, p) \quad \text{for} \quad n = 1, 2, \cdots ,$$

we have

$$\limsup_{n\to\infty} d(T^n(x_n), p) \le c.$$

Also,

$$\begin{aligned} d(y_n, p) &= d(W(T^n(x_n), x_n, \beta_n), p) \\ &\le \beta_n d(T^n(x_n), p) + (1-\beta_n) d(x_n, p) \\ &\le \beta_n k_n d(x_n, p) + (1-\beta_n) d(x_n, p) \\ &\le d(x_n, p) + \beta_n (k_n - 1) d(x_n, p) \\ &\le d(x_n, p) + (1-\varepsilon)(k_n - 1) d(x_n, p). \end{aligned}$$

Hence,

$$\limsup_{n\to\infty} d(y_n, p) \le c.$$

Since $d(S^n(y_n), p) \le k_n d(y_n, p)$, $\limsup_{n\to\infty} d(S^n(y_n), p) \le c$, because $k_n \to 1$, as, $n \to \infty$ and, $\limsup\limits_{n\to\infty} d(y_n, p) \le c$. Also, we have

$$d(x_{n+1}, p) = d(W(S^n(y_n), x_n, \alpha_n), p) = c.$$

From Lemma 5.1, we have

$$\lim_{n\to\infty} d(S^n(y_n), x_n) = 0.$$

On the other hand, we have

$$\begin{aligned} d(x_n, p) &\le d(x_n, S^n(y_n)) + d(S^n(y_n), p) \\ &\le d(x_n, S^n(y_n)) + k_n d(y_n, p), \end{aligned}$$

this implies,

$$c \le \liminf_{n\to\infty} d(y_n, p).$$

Thus, $\lim_{n\to\infty} d(y_n, p) = c$, that is, $\lim_{n\to\infty} d(W(T^n(x_n), x_n, \beta_n), p) = c$. By Lemma 5.1, we get,

$$\lim_{n\to\infty} d(x_n, T^n(x_n)) = 0.$$

Now,

$$\begin{aligned} d(S^n(x_n), x_n) &\le d(S^n(x_n), S^n(y_n)) + d(S^n(y_n), x_n) \\ &\le k_n d(x_n, y_n) + d(S^n(y_n), x_n) \\ &= k_n d(x_n, W(T^n(x_n), x_n, \beta_n)) + d(S^n(y_n), x_n) \\ &\le k_n \beta_n d(x_n, T^n(x_n)) + d(S^n(y_n), x_n) \\ &\le k_n(1-\varepsilon) d(x_n, T^n(x_n)) + d(S^n(y_n), x_n) \end{aligned}$$

So, we have, $\lim_{n\to\infty} d(S^n(x_n), x_n) = 0$. Hence,

$$\lim_{n\to\infty} d(S(x_n), x_n) = 0 = \lim_{n\to\infty} d(T(x_n), x_n).$$

This completes the proof. □

Theorem 5.1. *Let K be a nonempty compact and convex subset of a uniformly convex complete metric space X. Let $S, T : K \longrightarrow K$ be two mappings satisfying*

$$d(S^n(x), S^n(y)) \le k_n d(x, y)$$

and

$$d(T^n(x), T^n(y)) \le k_n d(x, y)$$

for $n = 1, 2 \cdots$, where $\{k_n\}$ is the sequence of numbers with $1 \le k_n < \infty$, for each n, and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be as in (5.1). That is,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = W(S^n(y_n), x_n, \alpha_n), \\ y_n = W(T^n(x_n), x_n, \beta_n), \qquad n \ge 0, \end{cases}$$

where, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0, 1)$. Let p be a common fixed point of S and T. Then, $\lim_{n\to\infty} d(x_n, p) = 0$.

That is, the iterative scheme $\{x_n\}$ is convergent to a common fixed point p of S and T.

Proof. From Lemma 5.2, we know that

$$\lim_{n\to\infty} d(S(x_n), x_n) = 0 = \lim_{n\to\infty} d(T(x_n), x_n).$$

Since K is compact, we have subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \to q$ in K. Continuity of T and S imply, $T(x_{n_k}) \to T(q)$ and $S(x_{n_k}) \to S(q)$, as $k \to \infty$. Therefore, we have $d(S(q), q) = 0 = d(T(q), q)$. Since, $\lim_{n\to\infty} d(x_n, p)$ exists, if p is the common fixed point of S and T, then $p = q$. This completes the proof. $\square$

References

1. M. A. Al-Thagafi, *Common fixed points and best approximation*, J. Approx. Theory **85** (1996), 318-320.
2. Ya. I. Alber and S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces, new results in operator theory, Advances and Appl. (ed. by I. Gohberg and Yu. Lyubich)*, Birkhauser Verlag, Basel 98, 1997, pp. 7-22.
3. I. Beg, *An iteration scheme for asymptotically nonexpansive mappings on uniformly convex metric spaces*, Nonlinear Analysis Forum **6(1)** (2001), 27-34.
4. I. Beg and M. Abbas, *Fixed point theorems for weakly inward multivlued maps on a convex metric space*, Demonstratio Mathematica **39(1)** (2006), 149-160.
5. I. Beg and A. Azam, *Common fixed points for commuting and compatible maps*, Discussiones Mathematicae, Differential inclusions **16** (1996), 121-135.
6. D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458-464.
7. S. S. Chang and J. K. Kim, *Convergence theorems of the Ishikawa type Iterative sequences with errors for generalized quasi-contractive mappings in convex metric spaces*, Applied Math. Letter **16** (2003), 535-542.
8. S. S. Chang, J. K. Kim and D. S. Jin, *Iterative sequences with errors for asymptotically quasi nonexpansive mappings in convex metric spaces*, Archives Inequ. and Appl. **2(4)** (2004), 365-374.
9. S. S. Chang, J. K. Kim and S. M. Kang, *Approximating fixed points of asymptotically quasi-nonexpansive type mappings by the Ishikawa iterative sequences with mixed errors*, Dynamic Systems and Appl. **13** (2004), 179–186.
10. L. Ciric, *On some discontinuous fixed point theorems in convex metric spaces*, Czech. Math. J. **43, 188** (1993), 319-326.
11. X. P. Ding, *Iteration processes for nonlinear mappings in convex metric spaces*, J. Math. Anal. Appl. **132** (1988), 114-122.
12. M. K. Ghosh and L. Debnath, *Convergence of Ishikawa iterative of quasi-nonexpa-nsive mappings*, J. Math. Anal. Appl. **207** (1997), 96–103.
13. K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35, 1** (1972), 171-174.

14. K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Stud. Adv. Math., 28, Cambridge University Press,, London, 1990.
15. J. K. Kim, K. H. Kim and K. S. Kim, *Convergence theorems of modified three-step iterative sequences with mixed errors for asymptotically quasi-nonexpansive mappings in Banach spaces*, PanAmerican Math. Jour. **14** (2004), 45–54.
16. J. K. Kim, K. H. Kim and K. S. Kim, *Three-step iterative sequences with errors for asymptotically quasi-nonexpansive mappings in convex metric spaces*, Nonlinear Analysis and Convex Analysis, Research Institute for Mathematical Sciences Kyoto University, Kyoto, Japan, **1365** (2004), 156–165.
17. J. K. Kim, Z. Liu and S. M. Kang, *Almost stability of Ishikawa iterative schemes with errors for $\phi-$strongly quasi-accretive and $\phi-$hemicontractive operators*, Commun. Kor. Math. Soc., **19(2)** (2004), 267–281.
18. J. K. Kim, K. H. Kim and K. S. Kim, *Convergence theorems of implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces*, Nonlinear Analysis and Convex Analysis, Research Institute for Mathematical Sciences Kyoto University, Kyoto, Japan, **1484** (2006), 40-51.
19. J. K. Kim, K. S. Kim and Y. M. Nam, *Convergence and stability of iterative processes for a pair of simultaneously asymptotically quasi-nonexpansive type mappings in convex metric spaces*, to appear in Jour. of Comput. Anal. and Appl. (2007).
20. D. Göhde, *Zum Prinzip der Kontraktiven abbildung*, Math. Nachr. **30** (1995), 251-258.
21. M. D. Guay, K. L. Singh and J. H. M. Whitfield, *Fixed point theorems for nonexpansive mappings in convex metric spaces*, vol. 80, Proceedings, Conference on Nonlinear Analysis, Marcel Dekker Inc., New York, 1982, pp. 179-189.
22. S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1953), 506-510.
23. J. Li, J. K. Kim and N. J. Huang, *Iterative schemes for a pair of simultaneously asymptotically quasi-nonexpansive type mappings in Banach spaces*, Taiwanese J. of Math. **10(6)** (2006), 1419-1429.
24. Q. H. Liu, *Iterative sequence for asymptotically quasinonexpansive mappings*, J. Math. Anal. Appl. **256** (2001), 1-7.
25. Q. H. Liu, *Iteration sequences for asymptotically quasi-nonexpansive mappings with error member of uniformly convex Banach spaces*, J. Math. Anal. Appl. **266** (2002), 468–471.
26. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **20** (1969), 458-464.
27. G. Meinardus, *Invarianz bei linearn Approximation*, Arch. Rational Mech. Anal. **14** (1963), 301-303.
28. B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Analysis **47** (2001), 2683-2693.
29. T. Shimizu and W. Takahashi, *Fixed point theorems in certain convex metric spaces*, Math. Japon. **37** (1992), 855-859.
30. J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mapings*, Bull, Austral. Math. Soc. **43** (1991), 153-159.
31. S. P. Singh, *Application of a fixed point theorem to approximation theory*, J. Approx. Theory **25** (1979), 88-89.
32. S. P. Singh, *Some results on best approximation in locally convex spaces*, J. Approx. Theory **28** (1980), 329-332.

33. W. Takahashi, *A convexity in metric spaces and nonexpansive mapping I*, Kodai Math. Sem. Rep. **22** (1970), 142-149.
34. K. K. Tan and H. K. Xu, *Approximating fixed point of nonexpansive mappings by the Ishikawa iterative process*, J. Math. Anal. Appl. **178** (1993), 301–308.

Ismat Beg
Centre for Advanced Studies in Mathematics and
Department of Mathematics,
Lahore University of Management Sciences,
54792-Lahore, Pakistan.
E-mail address: ibeg@lums.edu.pk

Mujahid Abbas
Centre for Advanced Studies in Mathematics and
Department of Mathematics,
Lahore University of Management Sciences,
54792-Lahore, Pakistan.
E-mail address: mujahid@lums.edu.pk

Jong Kyu Kim
Department of Mathematics,
Kyungnam University
Masan, Kyungnam, 631-701, Korea
E-mail address: jongkyuk@kyungnam.ac.kr