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SOME IMPROVEMENTS ON GENERAL HILBERT INEQUALITY

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Abstract. In this paper, it is shown that some new improvements on general Hilbert and Hardy inequalities can be established by means of sharpening of Hölder's inequality and the positive definiteness of Gramm matrix. In particular, we give some strengthened results of classical Hilbert and Hardy inequalities.

1. Introduction

The Hilbert-type inequalities are of some significant weight inequalities which play an important role in analysis and its applications. Let us, firstly, repeat the well known Hilbert inequality and its equivalent form, which we usually call Hardy-Hilbert inequality, in integral version:

Theorem 1.1. *If f and $g \in L^2[0,\infty)$, then the following inequalities hold and are equivalent:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \le \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}$$

and

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^2 dy \le \pi^2 \int_0^\infty f^2(x)dx,$$

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where π and π^2 are the best possible constants.

The discrete analogue of Hilbert theorem is obtained by replacing the integral with the sum from 1 to ∞, and by replacing the functions f and g with non-negative real sequences.

In recent years lots of authors made some generalizations of this theorem. Let's mention some of them: Jichang, Yang, Yong, Peachey and Rassias.

In our paper [7], we have obtain some general results for estimating the integral

$$\int_\Omega \int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y),$$

where μ_1 and μ_2 are non-negative $\sigma-$finite measures. That is a content of the following

Theorem 1.2. *If $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$ and $K(x,y), f(x), g(y), \varphi(x), \psi(y)$ be nonnegative functions, then the following inequalities hold and are equivalent*

$$\begin{aligned}&\int_\Omega \int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\ &\le \left[\int_\Omega \varphi(x)^p F(x) f(x)^p d\mu_1(x)\right]^{\frac{1}{p}} \left[\int_\Omega \psi(y)^q G(y) g(y)^q d\mu_2(y)\right]^{\frac{1}{q}}\end{aligned} \tag{1.1}$$

and

$$\begin{aligned}&\int_\Omega G(y)^{1-p}\psi(y)^{-p}\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)\\ &\le \int_\Omega \varphi(x)^p F(x) f(x)^p d\mu_1(x)\end{aligned} \tag{1.2}$$

where $F(x)=\int_\Omega K(x,y)\psi(y)^{-p}d\mu_2(y)$ and $G(y)=\int_\Omega K(x,y)\varphi(x)^{-q}d\mu_1(x)$.

If $0<p<1$ then the reverse inequalities in (1.1) and (1.2) are valid as well as the inequality

$$\begin{aligned}&\int_\Omega F(x)^{1-q}\varphi(x)^{-q}\left[\int_\Omega K(x,y)g(y)d\mu_2(y)\right]^q d\mu_1(x)\\ &\le \int_\Omega \psi(y)^q G(y) g(y)^q d\mu_2(y)\end{aligned} \tag{1.3}$$

Then we have applied these results on some special choices of the kernel $K(x,y)$ and the functions $\varphi(x)$, $\psi(y)$ and obtained many inequalities which are generalizations of the previously mentioned authors (see [7]).

Some authors obtained notable improvement of Hilbert inequality by means of sharpening of Hölder's inequality. For example, in [9], one can find the following

Theorem 1.3. *If* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$, *then the following inequality is valid:*

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy\\ <&\frac{\pi}{\lambda\sin(\frac{\pi}{p})}(1-R(\overline{h}))^{\min\{\frac{1}{p},\frac{1}{q}\}}\\ \cdot&\left[\int_0^\infty x^{(p-1)(1-\lambda)}f(x)^p dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{(q-1)(1-\lambda)}g(y)^q dy\right]^{\frac{1}{q}}\end{aligned}$$

Obviously, the preceding inequality is the improvement of classical Hardy-Hilbert's inequality. A definition of the number $R(\overline{h})$ will be given in the next section, as well as the improvement of Hölder's inequality.

2. Preliminaries

Leping, Mingzhe and Weijan established in [10] an important improvement of Hölder's inequality, which will help us on extensions of Hilbert's and Hardy's inequalities.

Let f and g be the elements of an inner product space of measurable functions. Then the inner product is denoted by (f,g). The mentioned authors introduced a function defined by

$$S_r(f,u)=(f^{\frac{r}{2}},u)||f||_r{}^{-\frac{r}{2}},$$

where u is a parametric variable vector which is variable unit vector and $||f||_r=\sqrt[r]{(f^{\frac{r}{2}},f^{\frac{r}{2}})}$, the r-norm. Clearly, $S_r(f,u)=0$ when the vector u selected is orthogonal to $f^{\frac{r}{2}}$.

Mingzhe, Li, and Debnath established in [13], with the help of the positive definiteness of Gramm matrix, an important inequality of the form

$$(f,g)^2\le ||f||^2||g||^2-(||f||x-||g||y)^2=||f||^2||g||^2(1-r(h)), \tag{2.1}$$

where $r(h)=\left(\frac{y}{||f||}-\frac{x}{||g||}\right)^2$, $x=(g,h)$, $y=(f,h)$ with $||h||=1$ and $xy\ge 0$, where $||\quad||$ is a $2-$norm. The equality in (2.1) holds if and only if the vectors f and g are linearly dependent; or the vector h is linear combination of f and g with $xy=0$, $x\ne y$. The inequality (2.1) is a consequence of an earlier paper of Mitrović (see [16]).

So, the significant improvement of Hölder's inequality is given by

$$(f,g)<||f||_p||g||_q(1-R(h))^m, \tag{2.2}$$

where $R(h)=(S_p(f,h)-S_q(g,h))^2\ne 0$, $||h||=1$, $m=\min\{\frac{1}{p},\frac{1}{q}\}$ and $f^{\frac{p}{2}}$, $g^{\frac{q}{2}}$ and h are linearly independent.

We can, similarly as in [10], derive the improvement of reverse Hölder's inequality. Our inner product will be defined by $(f,g)=\int_\Omega K(x)f(x)g(x)d\mu(x)$.

Lemma 2.1. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $0<p<1$, *and* $K(x)$, $f(x)$, $g(x)$, $h(x)$ *be nonnegative functions such that* $f(x)^{\frac{p}{2}}$, $g(x)^{\frac{q}{2}}$ *and* $h(x)$ *are linearly independent. Then the following inequality holds*

$$(f,g)>||f||_p||g||_q(1-R(h))^{\frac{1}{q}}, \tag{2.3}$$

where $R(h)=(S_p(f,h)-S_q(g,h))^2\neq 0$, $||h||=1$.

Proof. We start with the following identity

$$(f,g)=\int_\Omega K(x)\Big(f(x)^{\frac{p}{q}}g(x)\Big)f(x)^{1-\frac{p}{q}}d\mu(x).$$

Now, let $A=\frac{q}{2}$ and $B=\frac{q}{q-2}$. Obviously, $\frac{1}{A}+\frac{1}{B}=1$. So, if we apply reverse Hölder's inequality we obtain

$$\begin{aligned}(f,g)&\geqslant\left[\int_\Omega K(x)\Big(f(x)^{\frac{p}{q}}g(x)\Big)^A d\mu(x)\right]^{\frac{1}{A}}\left[\int_\Omega K(x)\Big(f(x)^{1-\frac{p}{q}}\Big)^B d\mu(x)\right]^{\frac{1}{B}}\\&=(f^{\frac{p}{2}},g^{\frac{q}{2}})||f||_p{}^{p(1-\frac{2}{q})}.\end{aligned} \tag{2.4}$$

If we replace f and g with $f^{\frac{p}{2}}$ and $g^{\frac{q}{2}}$ in (2.1) we obtain

$$(f^{\frac{p}{2}},g^{\frac{q}{2}})^2<||f||_p{}^p||g||_q{}^q(1-R(h)), \tag{2.5}$$

and substituing (2.5) in (2.4) we obtain (2.3). □

In the next we shall apply these results on our papers [6] and [7].

3. General case

Now we shall state and prove our general improvements, by using the results from the Section 2. First of all let's say that we suppose that all integrals and series converges and such types of conditions shall mostly be omitted. Further, all the functions that we will have in our results will be non-negative. Throughout this paper, the exponent m indicates $m=\min\{\frac{1}{p},\frac{1}{q}\}$, where p and q are conjugate exponents i.e. $\frac{1}{p}+\frac{1}{q}=1$. Also, the number $R(\overline{f},\overline{g},\overline{h})$ indicates $R(\overline{f},\overline{g},\overline{h})=(S_p(\overline{f},\overline{h})-S_q(\overline{g},\overline{h}))^2$, where $S_p(\overline{f},\overline{h})=(\overline{f}^{\frac{p}{2}},\overline{h})||\overline{f}||_p{}^{-\frac{p}{2}}$. Obviously, $S_p(\overline{f},\overline{h})$ depends on the inner product. For our general results, the inner product will be defined by

$$(\overline{f},\overline{g})=\int_\Omega\int_\Omega K(x,y)\overline{f}(x,y)\overline{g}(x,y)d\mu_1(x)d\mu_2(y), \tag{3.1}$$

so we have the following

Theorem 3.1. *If $p > 1$, then the following inequality holds*

$$\begin{aligned}&\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\&\le (1-R(\overline{f},\overline{g},\overline{h}))^m\\&\cdot\left[\int_\Omega \varphi(x)^p F(x)f(x)^p d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi(y)^q G(y)g(y)^q d\mu_2(y)\right]^{\frac{1}{q}},\end{aligned}\tag{3.2}$$

where the inner product is defined by (3.1), $F(x) = \int_\Omega \frac{K(x,y)}{\psi(y)^p}d\mu_2(y)$, $G(y) = \int_\Omega \frac{K(x,y)}{\varphi(x)^q}d\mu_1(x)$, $\overline{f}(x,y) = f(x)\frac{\varphi(x)}{\psi(y)}$, $\overline{g}(x,y) = g(y)\frac{\psi(y)}{\varphi(x)}$, and the function $\overline{h}(x,y)$ satisfy $\int_\Omega\int_\Omega K(x,y)\overline{h}(x,y)^2 d\mu_1(x)d\mu_2(y) = 1$. If $p < 1$ then the reverse inequality in (3.2) is valid.

Proof. We start with the following identity

$$\begin{aligned}&\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\=&\int_\Omega\int_\Omega K(x,y)\overline{f}(x,y)\overline{g}(x,y)d\mu_1(x)d\mu_2(y).\end{aligned}\tag{3.3}$$

Now, by applying the inequality (2.2) we obtain the inequality (3.2). We obtain the reverse inequalities in a similar way, by applying the Lemma 2.1. This completes the proof. □

Remark 3.2. *Equality in the previous theorem is possible if and only if the functions $\overline{f}(x,y)^{\frac{p}{2}}$, $\overline{g}(x,y)^{\frac{q}{2}}$, and $\overline{h}(x,y)$ are linearly dependent (see Section 2). Otherwise, inequalities in Theorem 3.1 are strict.*

If we put

$$\widetilde{g}(y) = G(y)^{1-p}\psi(y)^{-p}\left(\int_\Omega K(x,y)f(x)d\mu_1(x)\right)^{p-1},$$

instead of g in the Theorem 3.1, we obtain Hardy-Hilbert type inequality:

Theorem 3.3. *If $p > 1$ or $p < 0$ then the following inequality holds*

$$\begin{aligned}&\int_\Omega G(y)^{1-p}\psi(y)^{-p}\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)\\\le&(1-R(\overline{f},\overline{\widetilde{g}},\overline{h}))^{mp}\int_\Omega \varphi(x)^p F(x)f(x)^p d\mu_1(x),\end{aligned}\tag{3.4}$$

where the inner product is defined by (3.1), and the functions $F(x)$, $G(y)$, $\overline{f}(x,y)$, $\overline{\widetilde{g}}(x,y)$, $\overline{h}(x,y)$ are defined in the Theorem 3.1. If $0 < p < 1$ then the reverse inequality in (3.4) is valid.

Proof. Since $\frac{1}{p}+\frac{1}{q}=1$, we have, by using (3.2),

$$\begin{aligned}
&\int_\Omega G(y)^{1-p}\psi(y)^{-p}\Big(\int_\Omega K(x,y)f(x)d\mu_1(x)\Big)^p d\mu_2(y)\\
=&\int_\Omega\int_\Omega K(x,y)f(x)\widetilde{g}(y)d\mu_1(x)d\mu_2(y)\\
\leq&(1-R(\overline{f},\overline{\widetilde{g}},\overline{h}))^m\\
&\cdot\left[\int_\Omega \varphi(x)^pF(x)f(x)^pd\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi(y)^qG(y)\widetilde{g}(y)^qd\mu_2(y)\right]^{\frac{1}{q}}\\
=&\left[\int_\Omega \varphi(x)^pF(x)f(x)^pd\mu_1(x)\right]^{\frac{1}{p}}\\
&\cdot\left[\int_\Omega G(y)^{1-p}\psi(y)^{-p}\Big(\int_\Omega K(x,y)f(x)d\mu_1(x)\Big)^p d\mu_2(y)\right]^{\frac{1}{q}}
\end{aligned}$$

from where we have (3.4). We obtain the reverse inequalities in a similar way, by applying the Lemma 2.1. This completes the proof.

□

Remark 3.4. *Note that Theorems 3.1 and 3.3 are improvements of our general results in* [7]. *Also, we lost the equivalence of the inequalities (see Theorem 1.2). Further, the definitions of the functions $\widetilde{g}$ in Hardy-Hilbert type inequalities shall easily be seen from the inequalities and shall be omitted.*

The theorems in this section are improvements of our general results in [7].

4. Hardy type inequalities

The inequality (3.4) gives the improvement of so called Hardy-type inequality. In this section we shall obtain the improvements for some special choices of the kernel $K(x,y)$. If we put $K(x,y)=h(y)$ for $x\leq y$ and $K(x,y)=0$ for $x>y$ in Theorems 3.1 and 3.3, where $\Omega=[a,b]$, $0\leq a<b\leq\infty$, then our inner product is defined by

$$(\overline{f},\overline{g})=\int_a^b\int_a^y h(y)\overline{f}(x,y)\overline{g}(x,y)d\mu_1(x)d\mu_2(y), \tag{4.1}$$

so we obtain the following results:

Theorem 4.1. *If* $p > 1$ *then the following inequality holds*

$$\begin{aligned}
&\int_a^b \int_a^y h(y)f(x)g(y)d\mu_1(x)d\mu_2(y) \\
\leq &(1-R(\overline{f},\overline{g},\overline{h}))^m \left[\int_a^b \varphi(x)^p f(x)^p \Big(\int_x^b H(y)d\mu_2(y)\Big)d\mu_1(x)\right]^{\frac{1}{p}} \\
&\cdot \left[\int_a^b \psi(y)^q g(y)^q h(y)\Big(\int_a^y \varphi(x)^{-q}d\mu_1(x)\Big)d\mu_2(y)\right]^{\frac{1}{q}},
\end{aligned} \tag{4.2}$$

where the inner product is defined by (4.1), $H(y) = h(y)\psi(y)^{-p}$, *the functions* $\overline{f}(x,y)$, $\overline{g}(x,y)$ *are defined in the Theorem 3.1, and* $\int_a^b\int_a^y h(y)\overline{h}(x,y)^2 d\mu_1(x)d\mu_2(y) = 1$. *If* $p < 1$, *then the reverse inequality in (4.2) is valid.*

Theorem 4.2. *If* $p > 1$ *or* $p < 0$, *then the following inequality holds*

$$\begin{aligned}
&\int_a^b H(y)\left(\int_a^y \varphi(x)^{-q}d\mu_1(x)\right)^{1-p}\left(\int_a^y f(x)d\mu_1(x)\right)^p d\mu_2(y) \\
\leq &(1-R(\overline{f},\overline{\overline{g}},\overline{h}))^{mp}\int_a^b \varphi(x)^p f(x)^p\left(\int_x^b H(y)d\mu_2(y)\right)d\mu_1(x),
\end{aligned} \tag{4.3}$$

where the inner product is defined by (4.1), $H(y) = h(y)\psi(y)^{-p}$, *the functions* $\overline{f}(x,y)$, $\overline{\overline{g}}(x,y)$ *are defined in the Theorem 3.1, and* $\int_a^b\int_a^y h(y)\overline{h}(x,y)^2 d\mu_1(x)d\mu_2(y) = 1$. *If* $0 < p < 1$, *then the reverse inequality in (4.3) is valid.*

Similarly, we obtain the dual results by putting the kernel $K(x,y) = 0$ for $x \leq y$ and $K(x,y) = h(y)$ for $x > y$ in Theorems 3.1 and 3.3. The inner product is then defined by

$$(\overline{f},\overline{g}) = \int_a^b \int_y^b h(y)\overline{f}(x,y)\overline{g}(x,y)d\mu_1(x)d\mu_2(y), \tag{4.4}$$

and we obtain following two results:

Theorem 4.3. *If* $p > 1$, *then the following inequality holds*

$$\begin{aligned}
&\int_a^b \int_y^b h(y)f(x)g(y)d\mu_1(x)d\mu_2(y) \\
\leq &1-R(\overline{f},\overline{g},\overline{h}))^m \left[\int_a^b \varphi(x)^p f(x)^p \left(\int_a^x H(y)d\mu_2(y)\right)d\mu_1(x)\right]^{\frac{1}{p}} \\
&\cdot \left[\int_a^b \psi(y)^q g(y)^q h(y)\left(\int_y^b \varphi(x)^{-q}d\mu_1(x)\right)d\mu_2(y)\right]^{\frac{1}{q}},
\end{aligned} \tag{4.5}$$

where the inner product is defined by (4.4), $H(y) = h(y)\psi(y)^{-p}$, the functions $\overline{f}(x,y)$, $\overline{g}(x,y)$ are defined in the Theorem 3.1, and $\int_a^b \int_y^b h(y)\overline{h}(x,y)^2 d\mu_1(x)d\mu_2(y) = 1$. If $p < 1$, then the reverse inequality in (4.5) is satisfied.

Theorem 4.4. *If $p > 1$ or $p < 0$, then the following inequality holds*

$$\begin{aligned}
&\int_a^b H(y)\Big(\int_y^b \varphi(x)^{-q} d\mu_1(x)\Big)^{1-p}\Big(\int_y^b f(x)d\mu_1(x)\Big)^p d\mu_2(y) \\
\leq &(1 - R(\overline{f}, \overline{\overline{g}}, \overline{h}))^{mp} \int_a^b \varphi(x)^p f(x)^p \Big(\int_a^x H(y)d\mu_2(y)\Big)d\mu_1(x),
\end{aligned} \tag{4.6}$$

where the inner product is defined by (4.4), $H(y) = h(y)\psi(y)^{-p}$, the functions $\overline{f}(x,y)$, $\overline{\overline{g}}(x,y)$ are defined in the Theorem 3.1, and $\int_a^b \int_y^b h(y)\overline{h}(x,y)^2 d\mu_1(x)d\mu_2(y) = 1$. If $0 < p < 1$, then the reverse inequality in (4.6) is valid.

Further, we shall consider some special cases of previous theorems. Namely, if we put $h(y) = \frac{1}{y}$, $\varphi(x) = x^{A_1}$, $\psi(y) = y^{A_2}$ in the Theorems 4.1 and 4.2 we obtain

Corollary 4.5. *If $p > 1$, then the following inequality holds*

$$\begin{aligned}
&\int_a^b \int_a^y \frac{f(x)g(y)}{y} dxdy \\
<&\frac{|1 - qA_1|^{-\frac{1}{q}}}{|pA_2|^{\frac{1}{p}}}(1 - R(\overline{f}, \overline{g}, \overline{h}))^m \left[\int_a^b x^{p(A_1 - A_2)}\Big|1 - \Big(\frac{x}{b}\Big)^{pA_2}\Big| f(x)^p dx\right]^{\frac{1}{p}} \\
&\cdot \left[\int_a^b y^{q(A_2 - A_1)}\Big|1 - \Big(\frac{a}{y}\Big)^{1 - qA_1}\Big| g(y)^q dy\right]^{\frac{1}{q}},
\end{aligned} \tag{4.7}$$

where the inner product is defined by (4.1) with $h(y) = \frac{1}{y}$, A_1 and A_2 are arbitrary constants such that the integrals converges, and $\overline{h}(x,y) = \frac{2\sqrt{2y}e^{-x-y}}{(e^{-2a} - e^{-2b})^2}$. If $a = 0$ inequality holds under the condition $1 - qA_1 > 0$, and the case $b = \infty$ holds if $pA_2 > 0$. The reverse inequality is fulfilled if $p < 1$.

Here, we took the function $\overline{h}(x,y) = \frac{2\sqrt{2y}e^{-x-y}}{(e^{-2a} - e^{-2b})^2}$, since the function $\overline{h}(x,y)$ satisfy condition $\int_a^b \int_a^y \frac{1}{y}\overline{h}(x,y)^2 dxdy = 1$ (see Section 2).

Corollary 4.6. *If $p > 1$ or $p < 0$, then the following inequality holds*

$$\begin{aligned}&\int_a^b y^{p(A_1-A_2)-p}\Big|1-\Big(\frac{a}{y}\Big)^{1-qA_1}\Big|^{1-p}\Big(\int_a^y f(x)dx\Big)^p dy\\&<\frac{|1-qA_1|^{1-p}}{|pA_2|}(1-R(\overline{f},\overline{\overline{g}},\overline{h}))^{mp}\int_a^b x^{p(A_1-A_2)}\Big|1-\Big(\frac{x}{b}\Big)^{pA_2}\Big|f(x)^p dx\end{aligned}\tag{4.8}$$

where the inner product is defined by (4.1) with $h(y)=\frac{1}{y}$, A_1 and A_2 are arbitrary constants such that the integrals converges, and $\overline{h}(x,y)=\frac{2\sqrt{2y}e^{-x-y}}{(e^{-2a}-e^{-2b})^2}$. If $a=0$ inequality holds under the condition $1-qA_1>0$, and the case $b=\infty$ holds if $pA_2>0$. The reverse inequality is fulfilled if $0<p<1$.

Similarly, if the inner product is defined by (4.4), and $h(y)=\frac{1}{y}$, $\varphi(x)=x^{A_1}$, $\psi(y)=y^{A_2}$, one obtains dual results of those from the Corollaries 4.5 and 4.6. Here they are omitted.

Remark 4.7. *If $a=0$ and $b=\infty$ we see that we have additional conditions on the constants A_1 and A_2. For example, if $a=0$ and $b=\infty$ the inequality (4.8) become*

$$\begin{aligned}&\int_0^\infty y^{p(A_1-A_2)}\Big(\frac{1}{y}\int_0^y f(x)dx\Big)^p dy\\&<\frac{|1-qA_1|^{1-p}}{|pA_2|}(1-R(\overline{f},\overline{\overline{g}},\overline{h}))^{mp}\int_0^\infty x^{p(A_1-A_2)}f(x)^p dx\end{aligned}\tag{4.9}$$

if $p>1$, $A_1<\frac{1}{q}$, $A_2>0$ or $p<0$, $A_1<\frac{1}{q}$, $A_2<0$. The reverse in (4.9) holds if $0<p<1$, $A_1>\frac{1}{q}$, $A_2>0$.

In the previous remark we can take $\overline{h}(x,y)=2\sqrt{2y}e^{-x-y}$. Also, the inequality (4.9) is valid if the inner product is defined by (4.1). Note that the inequalities in the previous corollaries are strict (see Remark 3.2).

Remark 4.8. *If we put $A_1=\frac{1+\varepsilon}{pq}$ and $A_2=\frac{1+\varepsilon(1-q)}{pq}$ in the inequality (4.9), we obtain the inequality which is the generalization and also improvement of Kufner's paper* [8], *and for $\varepsilon=p-k$, from* [15].

In such a way we obtain improvements on many results about Hardy's inequality in recent years. So, let us discuss some more cases.

Remark 4.9. *If we put $a=0$, $A_1=\frac{p+1-k}{pq}$, $A_2=\frac{k-1}{p^2}$ in Corollary 4.6, the inequality (4.8) becomes*

$$\begin{aligned}&\int_0^b y^{-k}\Big(\int_0^y f(x)dx\Big)^p dy\\&<\Big(\frac{p}{k-1}\Big)^p(1-R(\overline{f},\overline{\overline{g}},\overline{h}))^m\int_0^b x^{p-k}\Big(1-\Big(\frac{x}{b}\Big)^{\frac{k-1}{p}}\Big)f(x)^p dx,\end{aligned}$$

if $1-qA_1>0$*, and this is improvement of the inequality from* [3].

Further, it is obvious that the inequalities $\left|1-\left(\frac{x}{b}\right)^{pA_2}\right| \le 1-\left(\frac{a}{b}\right)^{pA_2}$ *and* $\left|1-\left(\frac{a}{y}\right)^{1-qA_1}\right| \le 1-\left(\frac{a}{b}\right)^{1-qA_1}$ *are valid if* $1-qA_1>0$ *and* $pA_2>0$*, so from the inequality (4.8) we obtain*

$$\int_a^b y^{p(A_1-A_2)-p}\left(\int_a^y f(x)dx\right)^p dy < K^p \int_a^b x^{p(A_1-A_2)} f(x)^p dx,$$

where

$$K=\frac{|1-qA_1|^{-\frac{1}{q}}}{|pA_2|^{\frac{1}{p}}}(1-R(\overline{f},\overline{\overline{g}},\overline{h}))^m\left[1-\left(\frac{a}{b}\right)^{pA_2}\right]^{\frac{1}{p}}\left[1-\left(\frac{a}{b}\right)^{1-qA_1}\right]^{\frac{1}{q}},$$

Now, if $A_1=\frac{p+1-k}{pq}$ *and* $A_2=\frac{k-1}{p^2}$*, then the inequality becomes improvement of the result from the paper* [4].

5. Homogeneous functions

In this section we apply our main results on non-negative homogeneous functions. Recall that for homogeneous function of degree $-s$, $s>0$, equality $K(tx,ty)=t^{-s}K(x,y)$ is satisfied. We define the inner product by

$$(\overline{f},\overline{g})=\int_a^b\int_a^b K(x,y)\overline{f}(x,y)\overline{g}(x,y)d\mu_1(x)d\mu_2(y), \tag{5.1}$$

where $K(x,y)$ is the homogeneous function of degree $-s$ such that $k(\alpha)=\int_0^\infty K(1,u)u^{-\alpha}du<\infty$, $1-s<\alpha<1$, and obtain the following two results:

Theorem 5.1. *Let* $p>1$ *and* $K(x,y)$ *be homogeneous function of degree* $-s$*,* $s>0$*, strictly decreasing in both parameters* x *and* y*. Then the following inequality holds*

$$\begin{aligned}&\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy\\ &<(1-R(\overline{f},\overline{g},\overline{h}))^m\left[\int_a^b \big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f(x)^p dx\right]^{\frac{1}{p}}\\ &\quad\cdot\left[\int_a^b \big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)y^{1-s+q(A_2-A_1)}g(y)^q dy\right]^{\frac{1}{q}}\end{aligned} \tag{5.2}$$

for the inner product defined by (5.1) and for any $A_1\in(\frac{1-s}{q},\frac{1}{q})$*,* $A_2\in(\frac{1-s}{p},\frac{1}{p})$*. Further, the functions* $\overline{f}$*,* $\overline{g}$*,* $\overline{h}$ *are defined by* $\overline{f}(x,y)=f(x)\frac{x^{A_1}}{y^{A_2}}$*,* $\overline{g}(x,y)=$

$g(y)\frac{y^{A_2}}{x^{A_1}}$, $\overline{h}(x,y) = \frac{1}{\sqrt{K(x,y)}}\frac{2e^{-x-y}}{(e^{-2a}-e^{-2b})^2}$, *and*

$$\varphi_1(\alpha,x) = \left(\frac{a}{x}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du + \left(\frac{x}{b}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du,$$

$$\varphi_2(\alpha,y) = \left(\frac{a}{y}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du + \left(\frac{y}{b}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du.$$

If $b=\infty$ then the reverse in (5.2) is valid if $0<p<1$ and $K(x,y)$ is strictly decreasing in x and strictly increasing in y, for any $A_1 \in (\frac{1}{q}, \frac{1-s}{q})$ and $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$, or if $p<0$ and $K(x,y)$ is strictly increasing in x and strictly decreasing in y, for any $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1}{p}, \frac{1-s}{p})$.

Further, if $a=0$ then the reverse in (5.2) is valid if $0<p<1$ and $K(x,y)$ is strictly increasing in x and strictly decreasing in y for any $A_1 \in (\frac{1}{q}, \frac{1-s}{q})$ and $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$, or if $p<0$ and $K(x,y)$ is strictly decreasing in x and strictly increasing in y for any $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1}{p}, \frac{1-s}{p})$.

Proof. If we put $\varphi(x) = x^{A_1}$ and $\psi(y) = y^{A_2}$ in the Theorem 3.1, we obtain

$$\begin{aligned}&\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy\\ &<(1-R(\overline{f},\overline{g},\overline{h}))^m\left[\int_a^b f(x)^p x^{1-s+p(A_1-A_2)}\left(\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-pA_2}du\right)dx\right]^{\frac{1}{p}}\\ &\cdot\left[\int_a^b g(y)^q y^{1-s+q(A_2-A_1)}\left(\int_{\frac{y}{b}}^{\frac{y}{a}} K(1,u)u^{qA_1+s-2}du\right)dy\right]^{\frac{1}{q}}.\end{aligned}$$

Here, we used substitution $u=\dfrac{y}{x}$. Further, it can easily be shown (see [5]), that if $l(y) = y^{\alpha-1}\int_0^y K(1,u)u^{-\alpha}du$, $\alpha<1$ then

$$l'(y) = y^{\alpha-2}\int_0^y u^{1-\alpha}\frac{\partial K(1,u)}{\partial u}du. \tag{5.3}$$

Now, since

$$\begin{aligned}\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-pA_2}du &= \int_0^\infty K(1,u)u^{-pA_2}du - \int_0^{\frac{a}{x}} K(1,u)u^{-pA_2}du\\ &\quad - \int_0^{\frac{x}{b}} K(u,1)u^{pA_2+s-2}du,\end{aligned}$$

we obtain, by using (5.3), $\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-pA_2}du \leq k(pA_2) - \varphi_1(pA_2, x)$ and analogously $\int_{\frac{y}{b}}^{\frac{y}{a}} K(1,u)u^{qA_1+s-2}du \leq k(2-s-qA_1) - \varphi_2(2-s-qA_1, y)$, so the result follows from the Theorem 3.1 and monotony of the integral. $\square$

Note that the function $\overline{h}(x,y)$, defined in the previous theorem, satisfy condition $\int_a^b \int_a^b K(x,y)\overline{h}(x,y)^2 dxdy = 1$.

Remark 5.2. *Previous theorem is the improvement of our result in* [7]*. If the function $K(x,y)$ is symmetrical then $k(2-s-qA_1) = k(qA_1)$. So, if $\max\{\frac{1}{p}, \frac{1}{q}\} < s$, then we can put $A_1 = A_2 = \frac{1}{pq}$ in the Theorem 5.1 and obtain the improvement on* [5].

Theorem 5.3. *Let $p > 1$ and $K(x,y)$ be homogeneous function of degree $-s$, $s > 0$, strictly decreasing in both parameters x and y. Then the following inequality holds*

$$\begin{aligned}
&\int_a^b \big(k(2-s-qA_1) - \varphi_2(2-s-qA_1, y)\big)^{1-p} y^{(p-1)(s-1)+p(A_1-A_2)} \\
&\cdot \left(\int_a^b K(x,y)f(x)dx\right)^p dy \qquad (5.4)\\
<&(1 - R(\overline{f}, \overline{\overline{g}}, \overline{h}))^{mp} \int_a^b \big(k(pA_2) - \varphi_1(pA_2, x)\big) x^{1-s+p(A_1-A_2)} f(x)^p dx,
\end{aligned}$$

where the inner product is defined by (5.1), for any $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$, where the functions $\overline{f}(x,y)$, $\overline{\overline{g}}(x,y)$, $\overline{h}(x,y)$, $\varphi_1(\alpha, x)$, $\varphi_2(\alpha, y)$ are defined in the previous theorem.

If $b = \infty$ and $p < 0$, then the inequality (5.4) is also valid for any $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1}{p}, \frac{1-s}{p})$ if $K(x,y)$ is strictly increasing in x and strictly decreasing in y.

If $b = \infty$, then the reverse inequality in (5.4) is valid for any $A_1 \in (\frac{1}{q}, \frac{1-s}{q})$ and $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$, if $0 < p < 1$ and $K(x,y)$ is strictly decreasing in x and strictly increasing in y.

If $a = 0$ and $p < 0$, then the inequality (5.4) is also valid for any $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1}{p}, \frac{1-s}{p})$ if $K(x,y)$ is strictly decreasing in x and strictly increasing in y.

If $a = 0$, then the reverse inequality in (5.4) is valid for any $A_1 \in (\frac{1}{q}, \frac{1-s}{q})$ and $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$, if $0 < p < 1$ and $K(x,y)$ is strictly increasing in x and strictly decreasing in y.

If $a = 0$ and $b = \infty$, we obtain, from the Theorems 5.1 and 5.3, inequalities for arbitrary nonnegative homogeneous function of degree $-s$.

Corollary 5.4. *If $p > 1$ and $K(x,y)$ be homogeneous function of degree $-s$, then the following inequality holds*

$$\begin{aligned}&\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy\\ <&L(1-R(\overline{f},\overline{g},\overline{h}))^m\\ &\cdot\left[\int_0^\infty x^{1-s+p(A_1-A_2)}f(x)^p dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{1-s+q(A_2-A_1)}g(y)^q dy\right]^{\frac{1}{q}},\end{aligned}\tag{5.5}$$

where the inner product is defined by (5.1), with $a = 0$, $b = \infty$, for any $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$ where $L = k(pA_2)^{\frac{1}{p}} k(2-s-qA_1)^{\frac{1}{q}}$.
The reverse inequality is valid if $0 < p < 1$, for any $A_1 \in (\frac{1}{q}, \frac{1-s}{q})$ and $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$, or if $p < 0$, for any $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1}{p}, \frac{1-s}{p})$.

Corollary 5.5. *If $p > 1$ and $K(x,y)$ be homogeneous function of degree $-s$, then the following inequality holds*

$$\begin{aligned}&\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left(\int_0^\infty K(x,y)f(x)dx\right)^p\\ <&L^p(1-R(\overline{f},\overline{\overline{g}},\overline{h}))^{mp}\int_0^\infty x^{1-s+p(A_1-A_2)}f(x)^p dx\end{aligned}\tag{5.6}$$

where the inner product is defined by (5.1), with $a = 0$, $b = \infty$, for any $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$. If $p < 0$, the inequality holds too, for any $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1}{p}, \frac{1-s}{p})$. If $0 < p < 1$, then the reverse inequality holds for any $A_1 \in (\frac{1}{q}, \frac{1-s}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$.

Remark 5.6. *If $K(x,y) = (x+y)^{-s}$, then the constant L becomes $L = B(1-A_2p, s-1+A_2p)^{\frac{1}{p}} B(1-A_1q, s-1+A_1q)^{\frac{1}{q}}$, where B is a Beta function. So, we obtain improvements of papers* [2] *and* [6].

Now, we shall make some generalizations of the Theorems 5.1 and 5.3. If we use substitution $u = x + \lambda$ and $v = y + \lambda$ and the inner product

$$(\overline{f},\overline{g}) = \int_a^b\int_a^b K(x+\lambda, y+\lambda)\overline{f}(x,y)\overline{g}(x,y)dxdy \tag{5.7}$$

we have

Theorem 5.7. *Let $p > 1$ and $K(x,y)$ be homogeneous function of degree $-s$, $s > 0$, strictly decreasing in both parameters x and y. Then the following*

inequality holds

$$\begin{aligned}&\int_a^b\int_a^b K(x+\lambda,y+\lambda)f(x)g(y)dxdy\\&<(1-R(\overline{f},\overline{g},\overline{h}))^m\\&\cdot\left[\int_a^b\big(k(pA_2)-\psi_1(pA_2,x,\lambda)\big)(x+\lambda)^{1-s+p(A_1-A_2)}f(x)^p dx\right]^{\frac{1}{p}}\\&\left[\int_a^b\big(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\lambda)\big)(y+\lambda)^{1-s+q(A_2-A_1)}g(y)^q dy\right]^{\frac{1}{q}}\end{aligned} \tag{5.8}$$

if the inner product is defined by (5.7), for any parameters $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$, *where* $\overline{f}(x,y) = f(x)\frac{(x+\lambda)^{A_1}}{(y+\lambda)^{A_2}}$, $\overline{g}(x,y) = g(y)\frac{(y+\lambda)^{A_2}}{(x+\lambda)^{A_1}}$, $\overline{h}(x,y) = \frac{1}{\sqrt{K(x+\lambda,y+\lambda)}}\frac{2e^{-x-y}}{(e^{-2a}-e^{-2b})^2}$, *and*

$$\begin{aligned}\psi_1(\alpha,x,\lambda)=&\left(\frac{a+\lambda}{x+\lambda}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du\\&+\left(\frac{x+\lambda}{b+\lambda}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du,\end{aligned}$$

$$\begin{aligned}\psi_2(\alpha,y,\lambda)=&\left(\frac{a+\lambda}{y+\lambda}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du\\&+\left(\frac{y+\lambda}{b+\lambda}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du.\end{aligned}$$

If $b=\infty$ *the reverse inequality holds under the same conditions as in the Theorem 5.1.*

Remark 5.8. *If the function* $K(x,y)$ *from the Theorem 5.7 is symmetrical and* $0 < 1-\frac{2\lambda}{p} < s$, $0 < 1-\frac{2\lambda}{q} < s$ *then, by putting* $A_1 = A_2 = \frac{2\lambda}{pq}$ *in the theorem we obtain improvement on the results of Jichang and Rassias* ([5]).

Theorem 5.9. *Let* $p>1$ *and* $K(x,y)$ *be homogeneous function of degree* $-s$, $s>0$, *strictly decreasing in both parameters* x *and* y. *Then the following*

inequality holds

$$\begin{aligned}&\int_a^b \big(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\lambda)\big)^{1-p}(y+\lambda)^{(p-1)(s-1)+p(A_1-A_2)}\\&\cdot\Big(\int_a^b K(x+\lambda,y+\lambda)f(x)dx\Big)^p dy\\&<(1-R(\overline{f},\overline{\overline{g}},\overline{h}))^{mp}\int_a^b\big(k(pA_2)-\psi_1(pA_2,x,\lambda)\big)(x+\lambda)^{1-s+p(A_1-A_2)}f(x)^p dx\end{aligned} \tag{5.9}$$

if the inner product is defined by (5.7), for any $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, *where the functions* $\overline{f}(x,y)$, $\overline{\overline{g}}(x,y)$, $\overline{h}(x,y)$, $\psi_1(\alpha,x,\lambda)$, $\psi_2(\alpha,y,\lambda)$ *are defined in the previous theorem. The case* $p<1$ *and* $b=\infty$ *is treated in the same way as in the Theorem 5.3.*

Remark 5.10. *If* $a=0$, $b=\infty$, $K(x,y)=(x+y)^{-s}$, $\lambda=\frac{1}{2}$, $A_1=A_2=\frac{2-s}{pq}$ *and* $s>2-\min\{p,q\}$, *then the inequality (5.8) reads*

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y+1)^s}dxdy\\&<(1-R(\overline{f},\overline{g},\overline{h}))^m\\&\cdot\left[\int_0^\infty\Big(k_\lambda(p)-(2x+1)^{\frac{2-s}{q}-1}\int_0^1\frac{u^{\frac{s-2}{q}}}{(1+u)^s}du\Big)(x+\frac{1}{2})^{1-s}f(x)^p dx\right]^{\frac{1}{p}}\\&\cdot\left[\int_0^\infty\Big(k_\lambda(p)-(2y+1)^{\frac{2-s}{p}-1}\int_0^1\frac{u^{\frac{s-2}{p}}}{(1+u)^s}du\Big)(y+\frac{1}{2})^{1-s}g(y)^q dy\right]^{\frac{1}{q}},\end{aligned}$$

where $k_\lambda(p)=B\left(\frac{p+s-2}{p},\frac{q+s-2}{q}\right)$. *That inequality is the result from the paper* [11].

Another way of generalizing Theorems 5.1 and 5.3 arises from the substitution $u=Ax^\alpha$ and $v=By^\beta$, where $A,B,\alpha,\beta>0$. More precisely, if

$$(\overline{f},\overline{g})=\int_a^b\int_a^b K(Ax^\alpha,By^\beta)\overline{f}(x,y)\overline{g}(x,y)dxdy \tag{5.10}$$

we obtain such generalizations:

Theorem 5.11. *Let* $p>1$ *and* $K(x,y)$ *be homogeneous function of degree* $-s$, $s>0$, *strictly decreasing in both parameters* x *and* y. *Then the following*

inequality holds

$$
\begin{aligned}
&\int_a^b\int_a^b K(Ax^\alpha, By^\beta)f(x)g(y)dxdy \\
&< M(1-R(\overline{f},\overline{g},\overline{h}))^m \\
&\cdot\left[\int_a^b \big(k(pA_2)-\zeta_1(pA_2,x)\big)x^{\alpha(1-s)+\alpha p(A_1-A_2)-(\alpha-1)(p-1)}f(x)^p dx\right]^{\frac{1}{p}} \\
&\cdot\left[\int_a^b \big(k(2-s-qA_1)-\zeta_2(2-s-qA_1,y)\big)y^{\beta(1-s)+\beta q(A_2-A_1)-(\beta-1)(q-1)}g(y)^q dy\right]^{\frac{1}{q}},
\end{aligned}
\tag{5.11}
$$

if the inner product is defined by (5.10), for any parameters $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, *where* $\overline{f}(x,y)=f(x)\frac{x^{\alpha A_1+\frac{1-\alpha}{q}}}{y^{\beta A_2+\frac{1-\beta}{p}}}$, $\overline{g}(x,y)=g(y)\frac{y^{\beta A_2+\frac{1-\beta}{p}}}{x^{\alpha A_1+\frac{1-\alpha}{q}}}$, $\overline{h}(x,y)=\frac{1}{\sqrt{K(Ax^\alpha,By^\beta)}}\frac{2e^{-x-y}}{(e^{-2a}-e^{-2b})^2}$, *with the constant* M *defined by* $M=\alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}A^{\frac{2-s}{p}+A_1-A_2-1}B^{\frac{2-s}{q}+A_2-A_1-1}$ *and*

$$
\begin{aligned}
\zeta_1(\gamma,x)=&\left(\frac{a}{x}\right)^{\alpha(1-\gamma)}\int_0^{\frac{B}{A}a^{\beta-\alpha}}K(1,u)u^{-\gamma}du \\
&+\left(\frac{x}{b}\right)^{\alpha(s+\gamma-1)}\int_0^{\frac{A}{B}b^{\alpha-\beta}}K(u,1)u^{s+\gamma-2}du,
\end{aligned}
$$

$$
\begin{aligned}
\zeta_2(\gamma,y)=&\left(\frac{a}{y}\right)^{\beta(s+\gamma-1)}\int_0^{\frac{A}{B}a^{\alpha-\beta}}K(u,1)u^{s+\gamma-2}du \\
&+\left(\frac{y}{b}\right)^{\beta(1-\gamma)}\int_0^{\frac{B}{A}b^{\beta-\alpha}}K(1,u)u^{-\gamma}du.
\end{aligned}
$$

If $a=0$ *or* $b=\infty$ *the reverse inequality holds under the same conditions as in the Theorem 5.1.*

Theorem 5.12. *Let* $p>1$ *and* $K(x,y)$ *be homogeneous function of degree* $-s$, $s>0$, *strictly decreasing in both parameters* x *and* y. *Then the following*

inequality holds

$$\begin{aligned}&\int_a^b \big(k(2-s-qA_1)-\zeta_2(2-s-qA_1,y)\big)^{1-p} y^{\beta(p-1)(s-1)+\beta p(A_1-A_2)+\beta-1}\\ &\cdot\Big(\int_a^b K(Ax^\alpha,By^\beta)f(x)dx\Big)^p dy < M^p(1-R(\overline{f},\overline{\overline{g}},\overline{h}))^{mp}\\ &\cdot\int_a^b \big(k(pA_2)-\zeta_1(pA_2,x)\big)x^{\alpha(1-s)+\alpha p(A_1-A_2)-(\alpha-1)(p-1)}f(x)^p dx\end{aligned} \tag{5.12}$$

if the inner product is defined by (5.10), for any $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, *where* $\overline{f}(x,y)$, $\overline{\overline{g}}(x,y)$,$\overline{h}(x,y)$, M, $\zeta_1(\gamma,x)$, $\zeta_2(\gamma,y)$ *are defined in the previous theorem. The cases* $p<1$ *and* $a=0$ *or* $b=\infty$ *are treated in the same way as in the Theorem 5.3.*

If $a=0$ and $b=\infty$ we have the inequalities for arbitrary nonnegative homogeneous function of degree $-s$, what follows from the Theorems 5.11 and 5.12.

Theorem 5.13. *Let* $p>1$ *and* $K(x,y)$ *be homogeneous function of degree* $-s$. *Then the following inequality holds*

$$\begin{aligned}&\int_0^\infty\int_0^\infty K(Ax^\alpha,By^\beta)f(x)g(y)dxdy\\ &<N(1-R(\overline{f},\overline{g},\overline{h}))^m\left[\int_0^\infty x^{\alpha(1-s)+\alpha p(A_1-A_2)-(\alpha-1)(p-1)}f(x)^p dx\right]^{\frac{1}{p}}\\ &\cdot\left[\int_0^\infty y^{\beta(1-s)+\beta q(A_2-A_1)-(\beta-1)(q-1)}g(y)^q dy\right]^{\frac{1}{q}},\end{aligned} \tag{5.13}$$

if the inner product is defined by (5.10), with $a=0$, $b=\infty$, $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$ *and* $N=L\cdot M$, *where* L *is defined in Corollary 5.4 and* M *in Theorem 5.11. The reverse inequality, when* $p<1$, *holds under the same conditions as in the Corollary 5.4.*

Remark 5.14. *If* $K(x,y)=(x+y)^{-s}$, $A_1=A_2=\frac{2-s}{pq}$, $\alpha=\beta=1$ *and* $s>2-\min\{p,q\}$, *then the inequality (5.13) becomes*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(Ax+By)^s}dxdy < k_{A,B}\left[\int_0^\infty x^{1-s}f(x)^p dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{1-s}g(y)^q dy\right]^{\frac{1}{q}}$$

where $k_{A,B}=B(\frac{p+s-2}{p},\frac{q+s-2}{q})(1-R(\overline{f},\overline{g},\overline{h}))^m A^{\frac{2-s-p}{p}}B^{\frac{2-s-q}{q}}$, *what is the result from* [12]. *Further, if we put* $K(x,y)=(x+y)^{-1}$, $A=B=1$, $\alpha=\beta=\lambda$, $A_1=A_2=\frac{1}{pq}$ *in the Theorem 5.13, we obtain Theorem 1.3 from the Introduction.*

Theorem 5.15. *Let $p > 1$ and $K(x,y)$ be homogeneous function of degree $-s$. Then the following inequality holds*

$$\begin{aligned} &\int_0^\infty y^{\beta(p-1)(s-1)+\beta p(A_1-A_2)+\beta-1}\left(\int_0^\infty K(Ax^\alpha, By^\beta)f(x)dx\right)^p dy \\ &< N^p(1-R(\overline{f},\overline{\overline{g}},\overline{h}))^{mp}\int_0^\infty x^{\alpha(1-s)+\alpha p(A_1-A_2)-(\alpha-1)(p-1)}f(x)^p dx, \end{aligned} \tag{5.14}$$

if the inner product is defined by (5.10), with $a = 0$, $b = \infty$, where $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$ and N is defined in the Theorem 5.13. The case $p < 1$ is treated in the same way as in the Corollary 5.5.

Finally, we give the results in discrete case. We define the inner product by

$$(\{a_{m,n}\}, \{b_{m,n}\}) = \sum_{m=1}^\infty\sum_{n=1}^\infty K(Am^\alpha, Bn^\beta)a_{m,n}b_{m,n}, \tag{5.15}$$

where $A, B, \alpha, \beta > 0$ and $\{a_{m,n}\}, \{b_{m,n}\}$ are non-negative sequences.

Theorem 5.16. *Let $\{a_n\}$ and $\{b_n\}$ be nonnegative real sequences and $K(x,y)$ be homogeneous function of degree $-s$ strictly decreasing in both parameters x and y and $p > 1$. Then the following inequality holds*

$$\begin{aligned} &\sum_{m=1}^\infty\sum_{n=1}^\infty K(Am^\alpha, Bn^\beta)a_m b_n \\ &< N(1-R(\overline{a},\overline{b},\overline{h}))^m\left[\sum_{m=1}^\infty m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}{a_m}^p\right]^{\frac{1}{p}} \\ &\cdot\left[\sum_{n=1}^\infty n^{\beta(1-s)+\beta q(A_2-A_1)+(q-1)(1-\beta)}{b_n}^q\right]^{\frac{1}{q}}, \end{aligned} \tag{5.16}$$

if the inner product is defined by (5.15), for any $A_1 \in \left(\max\{\frac{1-s}{q}, \frac{\alpha-1}{\alpha q}\}, \frac{1}{q}\right)$, $A_2 \in \left(\max\{\frac{1-s}{p}, \frac{\beta-1}{\beta p}\}, \frac{1}{p}\right)$, with the sequences defined by $\overline{a}_{m,n} = a_m\frac{m^{\alpha A_1+\frac{1-\alpha}{q}}}{n^{\beta A_2+\frac{1-\beta}{p}}}$, $\overline{b}_{m,n} = b_n\frac{n^{\beta A_2+\frac{1-\beta}{p}}}{m^{\alpha A_1+\frac{1-\alpha}{q}}}$, $\overline{h}_{m,n} = \frac{1}{\sqrt{K(Am^\alpha,Bn^\beta)}}\frac{6}{mn\pi^2}$ and N defined in the Theorem 5.13.

Proof. Let's put $\varphi(Am^\alpha) = (Am^\alpha)^{A_1+\frac{1}{q\alpha}-\frac{1}{q}}$ and $\psi(Bn^\beta) = (Bn^\beta)^{A_2+\frac{1}{p\beta}-\frac{1}{p}}$ in Theorem 3.1. Since $qA_1 + \frac{1}{\alpha} - 1 \geq 0$ and $pA_2 + \frac{1}{\beta} - 1 \geq 0$, the functions $F(Am^\alpha) = \sum_{n=1}^\infty \frac{K(Am^\alpha,Bn^\beta)}{(Bn^\beta)^{pA_2+\frac{1}{\beta}-1}}$ and $G(Bn^\beta) = \sum_{n=1}^\infty \frac{K(Am^\alpha,Bn^\beta)}{(Ax^\alpha)^{qA_1+\frac{1}{\alpha}-1}}$ are strictly decreasing, wherefrom one obtains following estimates: $F(Am^\alpha) \leq$

$\int_0^\infty \frac{K(Am^\alpha,By^\beta)}{(By^\beta)^{pA_2+\frac{1}{\beta}-1}}dy$ and $G(Bn^\beta) \le \int_0^\infty \frac{K(Ax^\alpha,Bn^\beta)}{(Ax^\alpha)^{qA_1+\frac{1}{\alpha}-1}}dx$. So the result follows from the Theorem 3.1. □

Similarly, we have:

Theorem 5.17. *Let $\{a_n\}$ and $\{b_n\}$ be nonnegative real sequences and $K(x,y)$ be homogeneous function of degree $-s$ strictly decreasing in both parameters x and y, and $p>1$. Then the following inequality holds*

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{\beta(s-1)(p-1)+p\beta(A_1-A_2)+\beta-1}\left(\sum_{m=1}^{\infty} K(Am^\alpha,Bn^\beta)a_m\right)^p\\ &<N^p(1-R(\overline{a},\overline{\overline{b}},\overline{h}))^{mp}\sum_{m=1}^{\infty} m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}{a_m}^p,\end{aligned}\tag{5.17}$$

if the inner product is defined by (5.15), for any $A_1 \in \left(\max\{\frac{1-s}{q},\frac{\alpha-1}{\alpha q}\},\frac{1}{q}\right)$, $A_2 \in \left(\max\{\frac{1-s}{p},\frac{\beta-1}{\beta p}\},\frac{1}{p}\right)$, where the sequences $\overline{h}$, $\overline{a}$, $\overline{\overline{b}}$ are defined in the Theorem 5.16 and N in the Theorem 5.13.

Remark 5.18. *If one put $K(x,y)=(x+y)^{-1}$, $A=B=1$, $\alpha=\beta=\lambda$, $A_1=A_2=\frac{1}{pq}$ and $\lambda\le\min\{p,q\}$ in the Theorems 5.16 and 5.17, one obtains discrete analogue of the Theorem 1.3 i.e. following inequalities:*

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda}\\ &<\frac{\pi}{\lambda\sin(\frac{\pi}{p})}(1-R(\overline{a},\overline{b},\overline{h}))^m\left[\sum_{m=1}^{\infty} m^{(p-1)(1-\lambda)}{a_m}^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)}{b_n}^q\right]^{\frac{1}{q}}\end{aligned}$$

and

$$\sum_{n=1}^{\infty} n^{\lambda-1}\Big(\sum_{m=1}^{\infty}\frac{a_m}{m^\lambda+n^\lambda}\Big)^p<\frac{\pi^p}{\lambda^p\sin(\frac{\pi}{p})^p}(1-R(\overline{a},\overline{\overline{b}},\overline{h}))^{mp}\sum_{m=1}^{\infty} m^{(p-1)(1-\lambda)}{a_m}^p.$$

The first inequality is the result from [9]*. If we add condition $s\le 2$ in Remark 5.10 we obtain discrete analogues of those inequalities.*

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